MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. An infinite geometric series of real numbers has sum 12. If every term of the series is squared, the new series has sum 48. What is the sum of the series obtained by cubing every term of the original series?

(A) 144 **(B)** 192 **(C)** 216 **(D)** 243 **(E)** $\frac{1728}{7}$

2. The sines of the three angles of a triangle are in the ratio $3 : 5 : 7$, and the triangle has area $60\sqrt{3}$. What is the radius of the circle that passes through all three vertices of the triangle?

(A) $\frac{28\sqrt{3}}{3}$ **(B)** 20 **(C)** $14\sqrt{3}$ **(D)** 28 **(E)** $\frac{56\sqrt{3}}{3}$

3. Four fair six-sided dice, numbered 1 through 6, are rolled and the four numbers showing are multiplied together. What is the probability that the product is divisible by 12?

(A) $\frac{1}{2}$ **(B)** $\frac{671}{1296}$ **(C)** $\frac{695}{1296}$ **(D)** $\frac{743}{1296}$ **(E)** $\frac{791}{1296}$

4. Two circles have radii 17 and 10. They cross at two points, and those two points are 16 units apart. What is the sum of all possible values of the distance between the centres of the two circles?

(A) 9 **(B)** 21 **(C)** 24 **(D)** 30 **(E)** 42

5. For exactly one real number k, the cubic equation $x^3 - kx + 16 = 0$ has a repeated root. What is the value of k?

(A) 12 **(B)** 16 **(C)** 20 **(D)** 24 **(E)** 48

6. What is the smallest positive integer n for which the product $n(n+1)(n+2)$ of three consecutive integers is divisible by 1000?

(A) 123 **(B)** 124 **(C)** 125 **(D)** 248 **(E)** 373

7. Write $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{20} = \frac{p}{q}$ as a fraction in lowest terms. The least common multiple L of $1, 2, 3, \ldots, 20$ is a common denominator for all twenty terms, but after reducing, q may be smaller than L. What is $\frac{L}{q}$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 15 **(E)** 45

8. A fair spinner has four equal sectors coloured red, green, blue and yellow. It is spun repeatedly, and the spinning stops as soon as a colour comes up that has already come up before. What is the expected number of spins, counting the last one?

(A) $\frac{71}{32}$ **(B)** $\frac{5}{2}$ **(C)** 3 **(D)** $\frac{25}{8}$ **(E)** $\frac{103}{32}$

9. How many ordered pairs (x, y) of integers with $0 \le x < 360$ and $0 \le y < 360$ satisfy $\sin x^\circ + \sin y^\circ = \sin(x + y)^\circ$?

(A) 719 **(B)** 1077 **(C)** 1078 **(D)** 1079 **(E)** 1080

10. Let $M = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Let n be the smallest positive integer for which M^n is a multiple of the identity matrix, that is, $M^n = cI$ for some real number c. What is c?

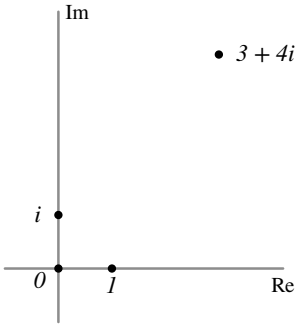
(A) -729 **(B)** -27 **(C)** -9 **(D)** 27 **(E)** 729

11. In three-dimensional space, a sphere is tangent to all three coordinate planes ($x = 0$, $y = 0$ and $z = 0$)

and passes through the point $(2, 3, 7)$. There are exactly two such spheres. What is the distance between their centres?

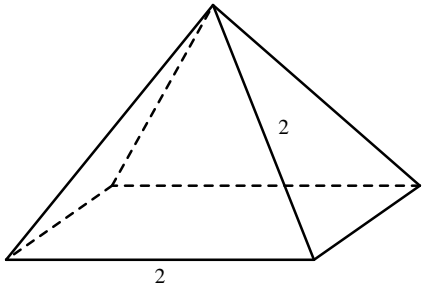
(A) $2\sqrt{15}$ **(B)** $4\sqrt{5}$ **(C)** $6\sqrt{3}$ **(D)** 12 **(E)** $6\sqrt{5}$

12. For a complex number z, let $S(z) = |z| + |z-1| + |z-i| + |z-(3+4i)|$, the sum of the distances from z to the four points 0, 1, i and $3+4i$ in the complex plane. What is the smallest possible value of $S(z)$?



(A) $\sqrt{2}+4$ **(B)** 6 **(C)** $\sqrt{2}+5$ **(D)** $2\sqrt{2}+4$ **(E)** 7

13. A pyramid has a square base, and all eight of its edges (the four base edges and the four slanted edges to the apex) have length 2. Two of its triangular faces share a slanted edge. What is the cosine of the dihedral angle between these two faces, measured inside the pyramid?



(A) $-\frac{\sqrt{3}}{3}$ **(B)** $-\frac{1}{2}$ **(C)** $-\frac{1}{3}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

14. A geometric sequence of real numbers has a nonzero first term. Its first, third and seventh terms, in that order, form an arithmetic progression whose common difference is not zero. What is the ratio of the seventh term to the first term?

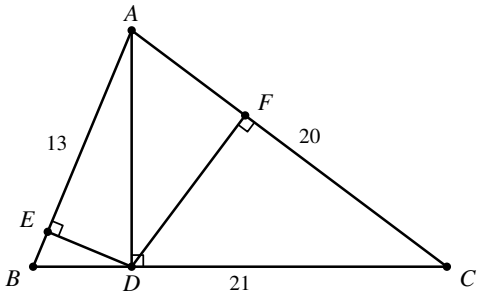
(A) $\sqrt{5}-2$ **(B)** $2-\sqrt{3}$ **(C)** $\frac{3-\sqrt{5}}{2}$ **(D)** $\frac{\sqrt{5}-1}{2}$ **(E)** 1

15. Find the minimum value of $\sin^4 x + \cos^4 x + \sec^4 x + \csc^4 x$ over all real x for which every term is defined.

(A) 4 **(B)** 8 **(C)** $\frac{17}{2}$ **(D)** 9 **(E)** 10

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16. Triangle ABC has $AB = 13$, $AC = 20$ and $BC = 21$. Let D be the foot of the altitude from A to $\overline{BC}$. Let E and F be the feet of the perpendiculars from D to $\overline{AB}$ and to $\overline{AC}$, respectively. What is the length EF?



(A) $\frac{756}{65}$ **(B)** 12 **(C)** $\frac{260}{21}$ **(D)** $\frac{63}{5}$ **(E)** $\frac{3024}{169}$

17. A circle of radius 2 has its centre on the positive x-axis. The circle lies inside the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and touches the ellipse at exactly two points. How far is the centre of the circle from the origin?

(A) $\frac{8}{3}$ **(B)** $\frac{4\sqrt{5}}{3}$ **(C)** 3 **(D)** $\frac{16}{5}$ **(E)** $\frac{4\sqrt{7}}{3}$

18. Three different integers are chosen at random from $\{1, 2, 3, \ldots, 20\}$, with all $\binom{20}{3}$ choices equally likely. What is the probability that the three numbers are the side lengths of a triangle with positive area?

(A) $\frac{29}{76}$ **(B)** $\frac{37}{95}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{33}{76}$ **(E)** $\frac{35}{76}$

19. An angle θ satisfies $\cos\theta = \frac{1}{3}$. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n} = 1 + \frac{\cos\theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + \cdots$.

(A) $\frac{8}{11}$ **(B)** $\frac{5}{6}$ **(C)** $\frac{10}{11}$ **(D)** $\frac{11}{10}$ **(E)** $\frac{6}{5}$

20. A function f from $\{1, 2, 3, 4, 5, 6\}$ to itself is chosen uniformly at random from all 6^6 such functions. What is the expected number of elements x in $\{1, 2, 3, 4, 5, 6\}$ with $f(f(x)) = x$?

(A) $\frac{5}{6}$ **(B)** 1 **(C)** $\frac{7}{6}$ **(D)** $\frac{11}{6}$ **(E)** $\frac{8}{3}$

21. Triangle ABC has a right angle at A, with $AB = 6$ and $AC = 8$. A point P moves along side $\overline{BC}$, strictly between B and C. Let O_1 be the circumcentre of triangle ABP and O_2 the circumcentre of triangle ACP. What is the smallest possible value of O_1O_2?

(A) $\frac{24}{5}$ **(B)** 5 **(C)** $\frac{25}{4}$ **(D)** 7 **(E)** 10

22. For each positive integer n, compare the power 2^n with the exponent n itself, modulo 125. For how many integers n with $1 \le n \le 1000$ is $2^n - n$ divisible by 125?

(A) 4 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 40

23. Find the sum of $n + p$ over all pairs (n, p), where n is a positive integer and p is a prime, for which $n^2 + p^n$ is a perfect square.

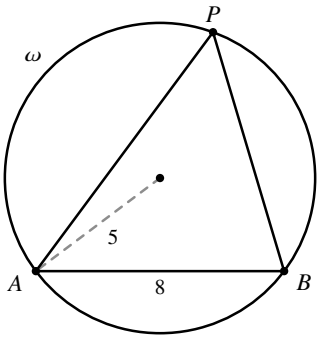
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(A) 6 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 18

24. The equation $(z-1)^{12} = z^{12}$ has exactly 11 complex solutions $z_1, z_2, \ldots, z_{11}$. What is the value of $|z_1|^2 + |z_2|^2 + \cdots + |z_{11}|^2$?

(A) $\frac{11}{4}$ **(B)** $\frac{11}{2}$ **(C)** $\frac{77}{12}$ **(D)** $\frac{143}{12}$ **(E)** $\frac{143}{3}$

25. Points A and B lie on a circle ω of radius 5, with $AB = 8$. For every point P on ω other than A and B, mark four points: the incentre of triangle PAB and its three excentres (the centres of the three circles that are tangent to all three lines PA, PB, AB and lie outside the triangle). What is the total length of the set of all points that get marked, over all choices of P?



(A) $4\sqrt{5}\,\pi$ **(B)** $6\sqrt{5}\,\pi$ **(C)** $8\sqrt{5}\,\pi$ **(D)** $12\sqrt{5}\,\pi$ **(E)** $16\sqrt{5}\,\pi$

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Answer Key — AMC 12 Practice Contest 7

1. (E) $\frac{1728}{7}$

2. (A) $\frac{28\sqrt{3}}{3}$

3. (E) $\frac{791}{1296}$

4. (D) 30

5. (A) 12

6. (B) 124

7. (D) 15

8. (E) $\frac{103}{32}$

9. (C) 1078

10. (B) -27

11. (A) $2\sqrt{15}$

12. (C) $\sqrt{2}+5$

13. (C) $-\frac{1}{3}$

14. (A) $\sqrt{5}-2$

15. (C) $\frac{17}{2}$

16. (A) $\frac{756}{65}$

17. (B) $\frac{4\sqrt{5}}{3}$

18. (E) $\frac{35}{76}$

19. (C) $\frac{10}{11}$

20. (D) $\frac{11}{6}$

21. (B) 5

22. (B) 8

23. (E) 18

24. (D) $\frac{143}{12}$

25. (D) $12\sqrt{5}\,\pi$

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Solutions — AMC 12 Practice Contest 7

1. Answer: (E) $\frac{1728}{7}$. The idea. Write the series as $a+ar+ar^2+\cdots$ with $|r|<1$. The squared series is again geometric, with first term a^2 and ratio r^2. Dividing the two sums uses the factorisation $1-r^2=(1-r)(1+r)$ and leaves a very simple equation for r.

Work it out. We have $\frac{a}{1-r}=12$ and $\frac{a^2}{1-r^2}=48$. Dividing the second equation by the first gives $\frac{a}{1+r}=4$. So $a=12(1-r)=4(1+r)$, which gives $12-12r=4+4r$, so $r=\frac{1}{2}$ and $a=6$. The series of cubes is geometric with first term $a^3=216$ and ratio $r^3=\frac{1}{8}$. Its sum is $\frac{216}{1-\frac{1}{8}}=\frac{216\cdot 8}{7}=\frac{1728}{7}$.

Check. The original series is $6+3+\frac{3}{2}+\cdots=12$ and the squares are $36+9+\frac{9}{4}+\cdots=\frac{36}{3/4}=48$, as required. The cubes are $216+27+\frac{27}{8}+\cdots$, which is a little more than $216+27=243$. That fits $\frac{1728}{7}\approx 246.9$. Writing 216 means stopping at the first term, and 243 means stopping after two.

2. Answer: (A) $\frac{28\sqrt{3}}{3}$. The idea. By the law of sines, $a:b:c=\sin A:\sin B:\sin C$. So the sides themselves are in the ratio $3:5:7$. A 3–5–7 triangle has a 120° angle, and that fixes both the scale and the circumradius.

Work it out. Let the sides be $3k, 5k, 7k$, and let C be the angle opposite $7k$. Then $\cos C=\frac{9+25-49}{2\cdot 3\cdot 5}=-\frac{1}{2}$, so $C=120^\circ$ and $\sin C=\frac{\sqrt{3}}{2}$. The area is $\frac{1}{2}(3k)(5k)\sin C=\frac{15\sqrt{3}}{4}k^2$. Setting this equal to $60\sqrt{3}$ gives $k^2=16$, so $k=4$ and the sides are $12, 20, 28$. By the extended law of sines, $2R=\frac{c}{\sin C}=\frac{28}{\sqrt{3}/2}=\frac{56}{\sqrt{3}}$. So $R=\frac{28}{\sqrt{3}}=\frac{28\sqrt{3}}{3}$.

Check. Use $R=\frac{abc}{4K}$ instead: $\frac{12\cdot 20\cdot 28}{4\cdot 60\sqrt{3}}=\frac{6720}{240\sqrt{3}}=\frac{28}{\sqrt{3}}$, the same value. Because the triangle is obtuse, R must exceed half the longest side, 14. Indeed $\frac{28\sqrt{3}}{3}\approx 16.2$.

3. Answer: (E) $\frac{791}{1296}$. The idea. Since $12=2^2\cdot 3$, the product fails exactly when it has no factor 3 or has at most one factor 2. Count these two bad events and use inclusion–exclusion for the overlap.

Work it out. There are $6^4=1296$ outcomes.

No factor 3: every die shows $1, 2, 4, 5$, giving $4^4=256$ outcomes.

At most one factor 2: the faces $1, 3, 5$ carry no 2, the faces 2 and 6 carry exactly one, and 4 carries two. Either all four dice are odd, which gives $3^4=81$ outcomes, or exactly one die shows 2 or 6 and the rest are odd, which gives $4\cdot 2\cdot 3^3=216$ outcomes. That is 297 in total.

Both: here the faces must come from $\{1, 2, 4, 5\}$ with at most one factor 2. Either all dice are in $\{1, 5\}$, which gives $2^4=16$ outcomes, or exactly one die shows 2 and the rest are in $\{1, 5\}$, which gives $4\cdot 2^3=32$ outcomes. That is 48 in total.

So the number of bad outcomes is $256+297-48=505$. The good outcomes number $1296-505=791$, and the probability is $\frac{791}{1296}$.

Check. Counting the complement of only one bad event gives a wrong answer. Leaving out the overlap subtracts too much and gives $\frac{1296-553}{1296}=\frac{743}{1296}$. Adding the overlap instead of subtracting it gives $\frac{1296-601}{1296}=\frac{695}{1296}$. A computer count of all 1296 rolls confirms 791.

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4. Answer: (D) 30**. The idea.** The line of centres is the perpendicular bisector of the common chord. So each centre, the midpoint M of the chord and one crossing point form a right triangle, which gives the distance from each centre to M. The two centres can lie on opposite sides of the chord or on the same side, and both pictures really happen.

Work it out. Let the crossing points be P and Q, with midpoint M, so $PM = 8$. For the big circle with centre O_1, $O_1M = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$. For the small circle with centre O_2, $O_2M = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$. Both centres lie on the line through M perpendicular to PQ. If they are on opposite sides of the chord, $O_1O_2 = 15 + 6 = 21$. If they are on the same side, $O_1O_2 = 15 - 6 = 9$. The sum of the possible distances is $21 + 9 = 30$.

Check. Two circles with radii 17 and 10 cross exactly when the distance d between centres satisfies $17 - 10 < d < 17 + 10$, that is $7 < d < 27$. Both 9 and 21 qualify, so both configurations exist. For $d = 9$, put $O_1 = (0, 0)$ and $O_2 = (9, 0)$: subtracting $x^2+y^2 = 289$ and $(x-9)^2+y^2 = 100$ gives $18x - 81 = 189$, so $x = 15$ and $y = \pm 8$, a chord of length 16. Answering 21 alone forgets the same-side picture, where the small circle's centre sits between the chord and the big circle's centre.

5. Answer: (A) 12**. The idea.** A repeated root r of a polynomial p is also a root of p'. Solving $p(r) = 0$ and $p'(r) = 0$ together removes k.

Work it out. Here $p'(x) = 3x^2 - k$, so $k = 3r^2$. Substituting into $r^3 - kr + 16 = 0$ gives $r^3 - 3r^3 + 16 = 0$, so $r^3 = 8$. A non-real repeated root would come with its conjugate as a second repeated root, which is too many roots for a cubic. So r is real, $r = 2$ and $k = 3 \cdot 4 = 12$.

Check. $x^3 - 12x + 16 = (x - 2)^2(x + 4)$. Expanding, $(x^2 - 4x + 4)(x + 4) = x^3 - 12x + 16$. So 2 is a double root and -4 is the third root. The discriminant of $x^3 + px + q$ is $-4p^3 - 27q^2$, and it is 0 here: $4k^3 = 27 \cdot 256$, so $k^3 = 1728$ and $k = 12$.

6. Answer: (B) 124**. The idea.** We need 2^3 and 5^3 to divide the product. At most one of three consecutive integers is a multiple of 5, so all three 5s must come from a single factor.

Work it out. One of $n, n + 1, n + 2$ must be a multiple of 125, so $n + 2 \geq 125$ and $n \geq 123$. For $n = 123$ the product is $123 \cdot 124 \cdot 125$. Here 123 and 125 are odd and $124 = 4 \cdot 31$, so there are only two factors of 2. That fails. For $n = 124$ the product is $124 \cdot 125 \cdot 126$. Here $124 = 4 \cdot 31$ gives two factors of 2 and $126 = 2 \cdot 63$ gives a third, so $2^3 \cdot 5^3$ divides the product. The answer is $n = 124$.

Check. The product is $124 \cdot 125 \cdot 126 = 1953000$, which is a multiple of 1000. The trap is 123: it gets the 125 but needs a second even number to reach 2^3. A computer search from $n = 1$ upward stops at 124.

7. Answer: (D) 15**. The idea.** Here $L = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Check each prime r separately: how many factors of r survive in the denominator? If exactly one term has the top power of r in its denominator, that power cannot cancel, because the other terms together have a smaller power of r in their denominators. When several terms share the top power, add those terms first and see what is left.

Work it out. *Prime 2*: only $\frac{1}{16}$ has 2^4 in its denominator, so $2^4 \mid q$. *Primes 11, 13, 17, 19*: only $\frac{1}{11}$ has an 11, and so on (their doubles $22, 26, \ldots$ are beyond 20), so each divides q. *Prime 7*: $\frac{1}{7} + \frac{1}{14} = \frac{3}{14}$ still has a 7, so $7 \mid q$.

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Prime 3: the terms with 9 in the denominator are $\frac{1}{9}+\frac{1}{18}=\frac{3}{18}=\frac{1}{6}$, which has only one 3. The terms with exactly one 3, together with this $\frac{1}{6}$, sum to $\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}\right)+\frac{1}{6}=\frac{1}{3}\cdot\frac{49}{20}$. Since $3\nmid 49$, this has exactly one 3 in its denominator, and all other terms have none. So exactly 3^1 divides q, and one factor 3 is lost.

Prime 5: $\frac{1}{5}+\frac{1}{10}+\frac{1}{15}+\frac{1}{20}=\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=\frac{1}{5}\cdot\frac{25}{12}=\frac{5}{12}$. The 5 disappears completely, so $5\nmid q$.

So $q=\frac{L}{3\cdot 5}$ and $\frac{L}{q}=15$.

Check. Exact arithmetic gives $1+\frac{1}{2}+\cdots+\frac{1}{20}=\frac{55835135}{15519504}$, and $15\cdot 15519504=232792560=L$. Answering 1 assumes nothing cancels. Answering 3 or 5 catches only one of the two cancellations.

8. Answer: (E) $\frac{103}{32}$**. The idea.** Let N be the number of spins. Instead of finding each probability $P(N=k)$, use the tail-sum formula $E[N]=P(N>0)+P(N>1)+P(N>2)+\cdots$. The event $N>k$ is easy to describe: the first k spins show k different colours.

Work it out. The first k spins are all different with probability $\frac{4\cdot 3\cdots(4-k+1)}{4^k}$. So $P(N>0)=1$, $P(N>1)=1$, $P(N>2)=\frac{4\cdot 3}{16}=\frac{3}{4}$, $P(N>3)=\frac{4\cdot 3\cdot 2}{64}=\frac{3}{8}$, $P(N>4)=\frac{24}{256}=\frac{3}{32}$, and $P(N>5)=0$ because five spins cannot show five different colours. Therefore $E[N]=1+1+\frac{3}{4}+\frac{3}{8}+\frac{3}{32}=\frac{32+32+24+12+3}{32}=\frac{103}{32}$.

Check. Compute the distribution directly: $P(N=2)=\frac{1}{4}$, $P(N=3)=\frac{3}{4}\cdot\frac{2}{4}=\frac{3}{8}$, $P(N=4)=\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}=\frac{9}{32}$ and $P(N=5)=\frac{3}{32}$. These add to $\frac{8+12+9+3}{32}=1$, and $2\cdot\frac{8}{32}+3\cdot\frac{12}{32}+4\cdot\frac{9}{32}+5\cdot\frac{3}{32}=\frac{16+36+36+15}{32}=\frac{103}{32}$. Not counting the last spin gives $\frac{71}{32}$, and $\frac{103}{32}\approx 3.2$ sits sensibly between the least possible value 2 and the greatest possible value 5.

9. Answer: (C) 1078**. The idea.** Turn the equation into a product. Sum-to-product on the left and a double-angle formula on the right share the factor $\sin\frac{x+y}{2}$. What is left is a difference of cosines, which is again a product. The equation becomes $4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{x+y}{2}=0$.

Work it out. We have $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ and $\sin(x+y)=2\sin\frac{x+y}{2}\cos\frac{x+y}{2}$. Their difference is $2\sin\frac{x+y}{2}\left(\cos\frac{x-y}{2}-\cos\frac{x+y}{2}\right)=4\sin\frac{x+y}{2}\sin\frac{x}{2}\sin\frac{y}{2}$. Working in degrees throughout: for $0\le x<360$, $\sin\frac{x}{2}=0$ only when $x=0$. Likewise $\sin\frac{y}{2}=0$ only when $y=0$. Since $0\le x+y<720$, $\sin\frac{x+y}{2}=0$ only when $x+y=0$ or $x+y=360$.

Now count the union. The line $x=0$ gives 360 pairs, and so does $y=0$. They share $(0,0)$, so together they give 719 pairs. The condition $x+y=360$ gives the pairs $(1,359),\ldots,(359,1)$, which is 359 pairs, none with a zero coordinate. (The case $x+y=0$ is just $(0,0)$.) The total is $719+359=1078$.

Check. Test $(100,260)$: $\sin 100^\circ+\sin 260^\circ=\sin 100^\circ-\sin 80^\circ=0=\sin 360^\circ$. It works. Test $(90,90)$: $1+1\neq\sin 180^\circ=0$, and indeed no factor vanishes. The answer 719 misses the family $x+y=360$, and 1079 counts $(0,0)$ twice. A computer check of all 129600 pairs finds 1078.

10. Answer: (B) -27**. The idea.** A 2×2 matrix satisfies its own characteristic equation (Cayley–Hamilton). Here that equation has complex roots of the form $\sqrt{3}\,e^{\pm i\pi/6}$, which says that M acts like "stretch by $\sqrt{3}$ and turn by 30°". A power of it is a multiple of I exactly when the total turn is a multiple of 180°.

Work it out. The characteristic polynomial is $\lambda^2-(2+1)\lambda+(2\cdot 1+1)=\lambda^2-3\lambda+3$, with roots $\lambda=\frac{3\pm i\sqrt{3}}{2}=\sqrt{3}\,e^{\pm i\pi/6}$. These are distinct, so $M^n=cI$ exactly when $\lambda_1^n=\lambda_2^n$. The two

numbers are complex conjugates, so they are equal exactly when $\lambda_1^n = 3^{n/2}e^{in\pi/6}$ is real, that is, when $6 \mid n$. The smallest such n is 6, and then $c = 3^3 e^{i\pi} = -27$.

Check. By hand: $M^2 = 3M - 3I$, so $M^3 = 3M^2 - 3M = 6M - 9I = \begin{pmatrix} 3 & -6 \\ 6 & -3 \end{pmatrix}$, and $M^6 = (6M - 9I)^2 = 36M^2 - 108M + 81I = 36(3M - 3I) - 108M + 81I = -27I$. The powers $M^1, \dots, M^5$ are not diagonal (for instance M^3 above). The choice 729 is M^{12}, the first power equal to a *positive* multiple of I.

11. Answer: (A) $2\sqrt{15}$**. The idea.** A sphere tangent to a plane lies entirely on one side of it. Since the sphere contains $(2, 3, 7)$, it lies in the first octant, and a sphere of radius r there has centre (r, r, r). Both centres therefore lie on the line $x = y = z$, and their distance is $\sqrt{3}$ times the difference of the radii.

Work it out. The condition that the sphere passes through the point is $(r-2)^2 + (r-3)^2 + (r-7)^2 = r^2$. This expands to $3r^2 - 24r + 62 = r^2$, which simplifies to $r^2 - 12r + 31 = 0$. So $r = 6 \pm \sqrt{5}$, and both radii are positive. The centres are (r_1, r_1, r_1) and (r_2, r_2, r_2), so their distance is $\sqrt{3}\,|r_1 - r_2| = \sqrt{3} \cdot 2\sqrt{5} = 2\sqrt{15}$.

Check. From Vieta, $(r_1 - r_2)^2 = (r_1 + r_2)^2 - 4r_1r_2 = 144 - 124 = 20$, so $|r_1 - r_2| = 2\sqrt{5}$. That matches. The factor from the diagonal direction is $\sqrt{3}$, not 3; multiplying by 3 gives $6\sqrt{5}$. Both roots, about 3.76 and 8.24, are positive, so both spheres really exist; a centre in any other octant would put the sphere on the wrong side of a coordinate plane from $(2, 3, 7)$.

12. Answer: (C) $\sqrt{2} + 5$**. The idea.** Pair the four points so that each pair is joined by a diagonal of the convex quadrilateral they form. For each pair, the triangle inequality says the two distances add to at least the length of that diagonal. Both bounds become equalities at the point where the diagonals cross.

Work it out. Going around the plane, the points $0, 1, 3 + 4i, i$ form a convex quadrilateral. Its diagonals join 0 to $3 + 4i$ and 1 to i. By the triangle inequality, $|z| + |z - (3 + 4i)| \geq |3 + 4i| = 5$ and $|z - 1| + |z - i| \geq |1 - i| = \sqrt{2}$. So $S(z) \geq 5 + \sqrt{2}$. Equality needs z on both segments. The diagonal from 0 is the line $y = \frac{4}{3}x$, and the diagonal from 1 to i is $x + y = 1$. They meet at $z = \frac{3}{7} + \frac{4}{7}i$, which lies inside both segments. So the minimum is $5 + \sqrt{2}$.

Check. At $z = \frac{3}{7} + \frac{4}{7}i$ we have $|z| = \frac{5}{7}$ and $|z - (3 + 4i)| = \frac{30}{7}$, which sum to 5. Also $|z - 1| = \frac{4\sqrt{2}}{7}$ and $|z - i| = \frac{3\sqrt{2}}{7}$, which sum to $\sqrt{2}$. A nearby guess such as $z = \frac{1}{2} + \frac{1}{2}i$ gives about 6.42, which is larger than $5 + \sqrt{2} \approx 6.41$. Pairing the points the wrong way, for example 0 with 1, gives the weaker bound $1 + |i - (3 + 4i)| = 1 + 3\sqrt{2}$, and no point attains it.

13. Answer: (C) $-\frac{1}{3}$**. The idea.** A dihedral angle is measured in a plane perpendicular to the shared edge. Each lateral face is an equilateral triangle, so the perpendicular from a base vertex to the far slanted edge lands at that edge's midpoint. The two perpendiculars and a base diagonal form a triangle, and the law of cosines gives the angle.

Work it out. Call the apex E and the base $ABCD$, and look at faces EAB and EBC, which share $\overline{EB}$. Both faces are equilateral with side 2, so the medians from A and from C to side $\overline{EB}$ both meet $\overline{EB}$ at its midpoint M and are perpendicular to it, each of length $\sqrt{3}$. Hence $\angle AMC$ is the dihedral angle. In $\triangle AMC$ we have $AM = CM = \sqrt{3}$ and $AC = 2\sqrt{2}$ (a diagonal of the square base), so $\cos \angle AMC = \dfrac{3 + 3 - 8}{2 \cdot \sqrt{3} \cdot \sqrt{3}} = \dfrac{-2}{6} = -\dfrac{1}{3}$.

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Check. The angle is obtuse, as it must be: the pyramid is half of a regular octahedron, whose dihedral angle is $\arccos(-1/3) \approx 109.5°$, and the two faces here are adjacent faces of that octahedron. The value $+\frac{1}{3}$ is the angle between the outward normals, a common sign slip.

14. Answer: (A) $\sqrt{5}-2$. The idea. With first term g and ratio r, the three terms are g, gr^2, gr^6, so everything depends on $t = r^2 \geq 0$. The arithmetic condition is a cubic in t with an obvious root, and the wanted ratio t^3 can be simplified with the cubic itself instead of being computed from a radical.

Work it out. The middle term of an arithmetic progression is the average of the outer two, so $2gt = g + gt^3$, that is, $t^3 - 2t + 1 = 0$. This factors as $(t-1)(t^2+t-1) = 0$. The root $t = 1$ makes all three terms equal to g, a progression with common difference 0, which is excluded. Of the roots of $t^2 + t - 1 = 0$, only $t = \frac{\sqrt{5}-1}{2}$ is nonnegative. The ratio of the seventh term to the first is $r^6 = t^3$, and from the cubic, $t^3 = 2t - 1 = \sqrt{5} - 2$.

Check. With $t \approx 0.618$ the terms are proportional to $1, 0.618, 0.236$, and the gaps 0.382 and 0.382 agree, so they are in arithmetic progression. The choice $\frac{\sqrt{5}-1}{2}$ is r^2 (the third term over the first) and $\frac{3-\sqrt{5}}{2}$ is r^4; the choice 1 comes from the excluded root $t = 1$.

15. Answer: (C) $\frac{17}{2}$. The idea. Everything depends only on $p = \sin^2 x \cos^2 x$, which ranges over $(0, \frac{1}{4}]$. Rewrite the expression in terms of p and see which way it moves.

Work it out. Since $\sin^2 x + \cos^2 x = 1$, we get $\sin^4 x + \cos^4 x = 1 - 2p$. Also $\sec^4 x + \csc^4 x = \dfrac{1}{\cos^4 x} + \dfrac{1}{\sin^4 x} = \dfrac{\sin^4 x + \cos^4 x}{\sin^4 x \cos^4 x} = \dfrac{1-2p}{p^2}$. So the expression equals $(1-2p)\left(1 + \dfrac{1}{p^2}\right)$. As p increases on $(0, \frac{1}{4}]$, both factors decrease and both stay positive, so the product is smallest at $p = \frac{1}{4}$, that is at $\sin^2 x = \cos^2 x = \frac{1}{2}$ (for example $x = 45°$). There the value is $\frac{1}{2} \cdot (1 + 16) = \frac{17}{2}$.

Check. At $x = 45°$: $\sin^4 x = \cos^4 x = \frac{1}{4}$ and $\sec^4 x = \csc^4 x = 4$, total 8.5. Pairing $\sin^4 x + \csc^4 x \geq 2$ with AM–GM gives only the bound 4, which is never attained because $\sin^4 x = 1$ forces $\cos x = 0$.

16. Answer: (A) $\frac{756}{65}$. The idea. The right angles at E and F both look at the segment AD, so A, E, D, F lie on the circle with diameter AD. In that circle the chord EF subtends the angle $\angle EAF = \angle A$, so the extended law of sines gives $EF = AD \sin A$.

Work it out. Heron's formula with $s = 27$ gives the area $K = \sqrt{27 \cdot 14 \cdot 7 \cdot 6} = 126$. Then $AD = \frac{2K}{BC} = \frac{252}{21} = 12$ and $\sin A = \frac{2K}{AB \cdot AC} = \frac{252}{260} = \frac{63}{65}$. Therefore $EF = AD \sin A = 12 \cdot \frac{63}{65} = \frac{756}{65}$.

Check. In the right triangles ADB and ADC, $AE = \frac{AD^2}{AB} = \frac{144}{13}$ and $AF = \frac{AD^2}{AC} = \frac{36}{5}$. Since $AE \cdot AB = AF \cdot AC$, triangle AEF is similar to triangle ACB with ratio $\frac{AE}{AC} = \frac{144/13}{20} = \frac{36}{65}$, so $EF = 21 \cdot \frac{36}{65} = \frac{756}{65}$ again. Stopping at AD gives 12, and dividing by $\sin A$ instead of multiplying gives $\frac{260}{21}$.

17. Answer: (B) $\frac{4\sqrt{5}}{3}$. The idea. With centre $C = (c, 0)$, the circle lies inside the ellipse exactly when every point of the ellipse is at least 2 from C, and it touches the ellipse at the points where that distance equals 2. So study the distance from C to a point of the ellipse as a function of x. Touching at exactly two points means the nearest points are a symmetric pair $(x_0, \pm y_0)$ off the axis, not the single vertex $(5, 0)$.

Work it out. The centre must be inside the ellipse, so $0 < c < 5$. For a point (x, y) of the ellipse, $y^2 = 9 - \frac{9}{25}x^2$, so its squared distance from C is $D(x) = (x-c)^2 + 9 - \frac{9}{25}x^2 = \frac{16}{25}x^2 - 2cx + c^2 + 9$, $\quad -5 \le x \le 5$. This is a parabola in x with its lowest point at $x_0 = \frac{25c}{16}$. If $c \ge \frac{16}{5}$, then $x_0 \ge 5$, so on $[-5, 5]$ the minimum is at $x = 5$ only: the circle could touch the ellipse only at the one point $(5, 0)$. So $c < \frac{16}{5}$, the minimum is at x_0 inside $(-5, 5)$, and it is reached at the two points $(x_0, \pm y_0)$ with $y_0 \neq 0$. The minimum value is $D(x_0) = c^2 + 9 - \frac{16}{25} \cdot \frac{625c^2}{256} = 9 - \frac{9c^2}{16}$. The circle touches exactly when this equals $2^2 = 4$: $\frac{9c^2}{16} = 5$, so $c^2 = \frac{80}{9}$ and $c = \frac{4\sqrt{5}}{3} \approx 2.98$, which is indeed less than $\frac{16}{5}$.

Check. At $c = \frac{4\sqrt{5}}{3}$ the touching points have $x_0 = \frac{25\sqrt{5}}{12}$ and $y_0^2 = 9 - \frac{9}{25} \cdot \frac{3125}{144} = \frac{19}{16}$, and $(x_0 - c)^2 + y_0^2 = \left(\frac{9\sqrt{5}}{12}\right)^2 + \frac{19}{16} = \frac{405+171}{144} = 4$. The tempting answer 3 puts the circle against the vertex $(5, 0)$, but then the nearest distance is $\sqrt{9 - \frac{81}{16}} = \frac{\sqrt{63}}{4} < 2$, so that circle pokes out of the ellipse near $x = \frac{75}{16}$. The value $\frac{16}{5}$ is only the threshold where the nearest point reaches the vertex, and $\frac{4\sqrt{7}}{3}$ comes from setting $9 - \frac{9c^2}{16}$ equal to 2 instead of 4.

18. Answer: (E) $\frac{35}{76}$. The idea. Sort the numbers as $a < b < c$. Only one triangle inequality can fail, $a + b > c$, so count, for each largest value c, the pairs below it with a large enough sum.

Work it out. Fix c and b with $b < c$. We need $c - b < a < b$, so a runs from $c - b + 1$ to $b - 1$: that is $2b - c - 1$ choices, positive only when $b > \frac{c+1}{2}$. Summing over b gives $\left\lfloor \frac{(c-2)^2}{4} \right\rfloor$ pairs. For example $c = 7$: $b = 6$ gives 4, $b = 5$ gives 2, $b = 4$ gives 0, total $6 = \lfloor 25/4 \rfloor$. For $c = 3, 4, \ldots, 20$ the counts are $0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81$, and these add to 525. So the probability is $\dfrac{525}{1140} = \dfrac{35}{76}$.

Check. The degenerate triples with $a + b = c$ number 90 (for each c there are $\lfloor (c-1)/2 \rfloor$ of them). Subtracting them a second time would give $\frac{435}{1140} = \frac{29}{76}$, and stopping the sum at $c = 19$ gives $\frac{444}{1140} = \frac{37}{95}$. A direct enumeration of all 1140 triples confirms 525.

19. Answer: (C) $\frac{10}{11}$. The idea. $\cos(n\theta)$ is the real part of z^n with $z = e^{i\theta}$, so the series is the real part of a geometric series with ratio $z/2$, which has modulus $\frac{1}{2}$ and so converges.

Work it out. $\displaystyle\sum_{n \ge 0} \left(\frac{z}{2}\right)^n = \frac{1}{1 - z/2} = \frac{2}{2 - z}$. Multiply by the conjugate: $\dfrac{2(2 - \bar{z})}{|2 - z|^2}$. Its real part is $\dfrac{2(2 - \cos\theta)}{(2 - \cos\theta)^2 + \sin^2\theta} = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$. With $\cos\theta = \frac{1}{3}$: $\dfrac{4 - 2/3}{5 - 4/3} = \dfrac{10/3}{11/3} = \dfrac{10}{11}$. The sign of $\sin\theta$ never enters.

Check. The terms $c_n = \cos n\theta$ obey $c_{n+1} = \frac{2}{3}c_n - c_{n-1}$: $1, \frac{1}{3}, -\frac{7}{9}, -\frac{23}{27}, \ldots$ The partial sums $1, 1.1667, 0.9722, 0.8657, \ldots$ settle near $0.909 = \frac{10}{11}$. The trap $\frac{6}{5}$ comes from treating $\cos(n\theta)$ as $\cos^n\theta$.

20. Answer: (D) $\frac{11}{6}$. The idea. Linearity of expectation: the answer is 6 times the probability that one particular x satisfies $f(f(x)) = x$. For one x, split on whether $f(x)$ is x itself or some other element.

Work it out. Fix x. If $f(x) = x$ (probability $\frac{1}{6}$), then $f(f(x)) = f(x) = x$ automatically. If $f(x) = y \neq x$ (probability $\frac{5}{6}$), we need $f(y) = x$; since $f(y)$ is independent of $f(x)$, this has

probability $\frac{1}{6}$. So $P\big(f(f(x)) = x\big) = \frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{11}{36}$, and the expected count is $6 \cdot \frac{11}{36} = \frac{11}{6}$.

Check. The count is (fixed points) + (elements in 2-cycles). The expected number of fixed points is 1. The expected number of 2-cycles $\{x, y\}$ is $\binom{6}{2} \cdot \frac{1}{36} = \frac{5}{12}$, each contributing 2 elements, for $\frac{5}{6}$. Total $1 + \frac{5}{6} = \frac{11}{6}$. An enumeration of all 46656 functions agrees.

21. Answer: (B) 5. **The idea.** Triangle AO_1O_2 always has the same shape as ABC, and its size is controlled by the angle at P. So O_1O_2 is BC times a factor that is smallest when $AP \perp BC$.

Work it out. Let $\varphi = \angle APB$, so $\angle APC = 180^\circ - \varphi$ and both angles have sine $\sin\varphi$. By the extended law of sines, $AO_1 = \dfrac{AB}{2\sin\varphi}$ and $AO_2 = \dfrac{AC}{2\sin\varphi}$, so $AO_1 : AO_2 = AB : AC$. Next, the angle between AO_1 and AB is $|90^\circ - \angle APB|$ (the angle between a chord and the radius to its endpoint), and the angle between AO_2 and AC is $|90^\circ - \angle APC|$, the same size. One rotates away from AB toward the inside of angle BAC exactly when the other rotates away from AC toward the outside, so $\angle O_1AO_2 = \angle BAC$. By SAS similarity, $\triangle AO_1O_2 \sim \triangle ABC$ with ratio $\dfrac{1}{2\sin\varphi}$, and hence $O_1O_2 = \dfrac{BC}{2\sin\varphi} \geq \dfrac{BC}{2} = 5$, with equality when $\varphi = 90^\circ$, i.e. P is the foot of the altitude from A (which lies strictly inside $\overline{BC}$).

Check. When $AP \perp BC$, the circles have diameters AB and AC, so O_1 and O_2 are the midpoints of $\overline{AB}$ and $\overline{AC}$, and $O_1O_2 = \frac{1}{2}BC = 5$ by the midline theorem. The value $\frac{24}{5}$ is the length of that altitude, a different segment.

22. Answer: (B) 8. **The idea.** $2^n \bmod 125$ depends only on $n \bmod 100$ (Euler: $2^{100} \equiv 1$), while $n \bmod 125$ is n itself. Combining the two moduli with the Chinese Remainder Theorem needs a compatibility condition modulo $\gcd(100, 125) = 25$, and that condition is the same problem one level down. So climb a ladder $5 \to 25 \to 125$.

Work it out. *Modulo 5.* $2^n \bmod 5$ depends on $n \bmod 4$ (cycle $1, 2, 4, 3$). By CRT, $n \bmod 20$ is a free pair ($n \bmod 4$, $n \bmod 5$), and for each value of $n \bmod 4$ exactly one value of $n \bmod 5$ equals $2^n \bmod 5$. So $5 \mid 2^n - n$ for exactly 4 classes of n modulo 20. *Modulo 25.* $2^n \bmod 25$ depends on $n \bmod 20$. Fix $r = n \bmod 20$; we need $n \equiv 2^r \pmod{25}$ and $n \equiv r \pmod{20}$. These are compatible exactly when they agree modulo $\gcd(20, 25) = 5$, i.e. $2^r \equiv r \pmod 5$, which is true for exactly the 4 classes above, and then n is unique modulo $\operatorname{lcm}(20, 25) = 100$. So $25 \mid 2^n - n$ for exactly 4 classes modulo 100. *Modulo 125.* $2^n \bmod 125$ depends on $n \bmod 100$. Fix $r = n \bmod 100$; we need $n \equiv 2^r \pmod{125}$ and $n \equiv r \pmod{100}$, compatible exactly when $2^r \equiv r \pmod{25}$: the 4 classes just found. Each gives a unique class modulo $\operatorname{lcm}(100, 125) = 500$. So there are 4 classes modulo 500, and $1 \leq n \leq 1000$ contains exactly $2 \cdot 4 = 8$ such n.

Check. The four smallest are $236, 283, 297, 434$. For instance 2^{236}: since $236 \equiv 36 \pmod{100}$, $2^{236} \equiv 2^{36} \pmod{125}$, and $2^{36} = 68719476736 \equiv 111 \equiv 236 \pmod{125}$. A direct search finds exactly $236, 283, 297, 434, 736, 783, 797, 934$. Stopping at modulus 25 would give 4 classes per 100, or 40.

23. Answer: (E) 18. **The idea.** Write $n^2 + p^n = m^2$, so $(m - n)(m + n) = p^n$. Both factors are powers of p whose exponents add to n, while their difference is only $2n$. Powers of p that differ by so little must be tiny, and an exponential-versus-linear bound finishes the job.

Work it out. Since p is prime, $m - n = p^a$ and $m + n = p^b$ with $0 \leq a < b$ and $a + b = n$. Subtracting gives $p^b - p^a = 2n = 2(a + b)$. Put $k = b - a \geq 1$, so the condition is $p^a(p^k - 1) =$

$4a+2k$. $(*)$ Case p odd, $a=0$: $(*)$ reads $p^k-1=2k$. For $k\ge 2$, $p^k-1\ge 3^k-1>2k$. So $k=1$, $p=3$, $n=a+b=1$: $1+3=4$. Pair $(1,3)$. Case p odd, $a\ge 1$: since $3^a\ge 3a$ and $3^k-1\ge 2k$, $p^a(p^k-1)\ge 3a\cdot 2k=6ak$, and $6ak-(4a+2k)=2a(3k-2)-2k\ge 2(3k-2)-2k=4k-4\ge 0$. Equality throughout forces $p=3$, $a=1$, $k=1$, and that case does satisfy $(*)$: $3\cdot 2=6$. So $b=2$, $n=3$: $9+27=36$. Pair $(3,3)$. Case $p=2$: here $a\ge 1$, since $m+n=2^b$ is even and $m-n$ has the same parity. For $k\ge 3$ the left side of $(*)$ minus the right side is at least $2(2^k-1)-4-2k>0$ at $a=1$ and grows by $2^a(2^k-1)-4>0$ each time a increases, so there is no solution. For $k=2$: $3\cdot 2^a=4a+4$ holds only at $a=2$ ($12=12$; the difference $3\cdot 2^a-4a-4$ is $-2,0,8$ at $a=1,2,3$ and increasing from then on). This gives $b=4$, $n=6$: $36+64=100$. For $k=1$: $2^a=4a+2$ has no solution ($2^a-4a-2=-4,-8,-6,-2$ for $a=1,\dots,4$, then 10 at $a=5$ and increasing). Pair $(6,2)$. The pairs are $(1,3)$, $(3,3)$, $(6,2)$, and the sum is $4+6+8=18$.

Check. A computer search over all primes below 2000 and all $n<70$ finds exactly these three pairs. Forgetting the $a=0$ case loses $(1,3)$ and gives 14; handling only $p=2$ gives 8, and 6 is the sum of the n for $p=2$ alone.

24. Answer: (D) $\frac{143}{12}$. The idea. Taking absolute values, every solution has $|z-1|=|z|$: it is equally far from 0 and 1, so it lies on the vertical line $\operatorname{Re} z=\frac12$. On that line the conjugate is $\overline{z}=1-z$, so $|z|^2=z\overline{z}=z-z^2$, a polynomial in z. Its sum over the roots comes from Vieta's formulas.

Work it out. Expanding, $(z-1)^{12}-z^{12}=-12z^{11}+66z^{10}-220z^9+\cdots$, a polynomial of degree 11 whose roots are $z_1,\dots,z_{11}$. Dividing by -12, the elementary symmetric sums are $e_1=\frac{66}{12}=\frac{11}{2}$ and $e_2=\frac{220}{12}=\frac{55}{3}$. So $\sum z_k=\frac{11}{2}$ and $\sum z_k^2=e_1^2-2e_2=\frac{121}{4}-\frac{110}{3}=-\frac{77}{12}$. Therefore

$$\sum |z_k|^2=\sum z_k-\sum z_k^2=\frac{11}{2}+\frac{77}{12}=\frac{143}{12}.$$

Check. The solutions are $z=\frac{1}{1-\omega}$ for the twelfth roots of unity $\omega\neq 1$, so $|z|^2=\frac{1}{|1-\omega|^2}=\frac{1}{4\sin^2(\pi k/12)}$ for $k=1,\dots,11$. For $k=6$ this is $\frac14$ (the real root $z=\frac12$), and the known sum $\sum_{k=1}^{n-1}\csc^2\frac{k\pi}{n}=\frac{n^2-1}{3}$ gives $\frac14\cdot\frac{143}{3}=\frac{143}{12}$. The choice $\frac{11}{4}$ keeps only the real parts, and $\frac{11}{2}$ is $\sum z_k$ alone.

25. Answer: (D) $12\sqrt{5}\,\pi$. The idea. Although the triangle changes, all four centres live on just two fixed circles: the circles through A and B centred at the midpoints of the two arcs AB of ω (the incentre–excentre lemma, applied at every vertex). Then show that both circles are covered completely.

Work it out. Let O be the centre of ω; the distance from O to AB is $\sqrt{25-16}=3$. Let M and N be the midpoints of the two arcs AB, with M on the minor arc (distance $5-3=2$ from line AB) and N on the major arc (distance $5+3=8$). Let Γ_M be the circle with centre M through A and B, of radius $\sqrt{4^2+2^2}=2\sqrt5$, and Γ_N the circle with centre N through A and B, of radius $\sqrt{4^2+8^2}=4\sqrt5$. Fix P, with incentre I and excentres J_P,J_A,J_B (opposite P, A, B). The internal and external bisectors at a vertex are perpendicular, so $\angle IAJ_P=\angle IBJ_P=90^\circ$ and $\angle J_AAJ_B=\angle J_ABJ_B=90^\circ$ (at A, J_A is on the internal bisector and J_B on the external one; at B the other way round). So A,B,I,J_P lie on the circle with diameter IJ_P, and A,B,J_A,J_B on the circle with diameter J_AJ_B. The centre of the first circle is on the perpendicular bisector of AB and on line PI, the internal bisector of $\angle APB$, which meets ω again at the midpoint of the arc AB not containing P. The centre of the second is on the perpendicular bisector of AB and on line J_AJ_B, the external bisector at P, which meets ω again at the midpoint of the arc containing P. So

for P on the major arc, $I, J_P \in \Gamma_M$ and $J_A, J_B \in \Gamma_N$; for P on the minor arc the roles of M and N swap. Every marked point is on Γ_M or Γ_N. Conversely, take a point X of Γ_M and draw line MX. Unless that line is MA, MB or the tangent to ω at M, it meets ω at a second point $P \neq A, B$, and it is a bisector of $\angle APB$ (internal if P is on the major arc, external if on the minor arc). In either case the two points where that bisector meets Γ_M are two of the marked points for this P, and X is one of them. So Γ_M is covered except for finitely many points, and likewise Γ_N. The marked set is the two circles minus finitely many points, of total length $2\pi\left(2\sqrt{5}+4\sqrt{5}\right) = 12\sqrt{5}\,\pi$.

Check. Take $P = N$: triangle PAB is isosceles with $PA = PB = 4\sqrt{5}$ and height 8, area 32, semiperimeter $4+4\sqrt{5}$, so the inradius is $\frac{32}{4+4\sqrt{5}} = 2\sqrt{5}-2$. The incentre is $2\sqrt{5}-2$ above AB and M is 2 below it, so $MI = 2\sqrt{5}$, as claimed. Tracking only the incentres gives two arcs, not whole circles; forgetting the excentres opposite A and B, or the circle Γ_N, leads to $4\sqrt{5}\,\pi$ or $8\sqrt{5}\,\pi$.

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