MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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1. What is the value of the product

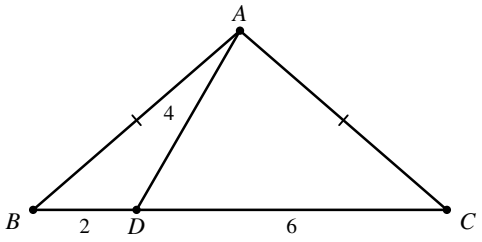
$$\left(\log_2 3 + \log_4 9 + \log_8 27\right)\left(\log_9 2 + \log_{27} 4\right)?$$

(A) $\frac{7}{6}$ **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** $\frac{7}{2}$

2. Exactly one real number x satisfies $\arcsin x + \arcsin(2x) = \dfrac{\pi}{2}$, where arcsin takes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. What is x?

(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{5}}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\sqrt{3}}{3}$ **(E)** $\frac{2\sqrt{5}}{5}$

3. Triangle ABC has $AB = AC$. Point D lies on side $\overline{BC}$ with $BD = 2$ and $DC = 6$, and the segment AD has length 4. What is AB?



(A) $2\sqrt{5}$ **(B)** $2\sqrt{6}$ **(C)** $2\sqrt{7}$ **(D)** $4\sqrt{2}$ **(E)** 6

4. Three fair six-sided dice are rolled. The three numbers shown are listed in increasing order, with repeated numbers kept (for example, a roll of $5, 2, 5$ gives the list $2, 5, 5$). What is the probability that the middle number in the list is 4?

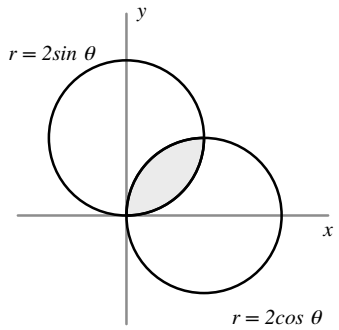
(A) $\frac{1}{6}$ **(B)** $\frac{37}{216}$ **(C)** $\frac{7}{36}$ **(D)** $\frac{17}{72}$ **(E)** $\frac{13}{54}$

5. In triangle ABC, $BC = 7$, $CA = 8$ and $\angle BAC = 60^\circ$. These conditions do not determine the triangle uniquely. What is the sum of the areas of all triangles (up to congruence) that satisfy them?

(A) $6\sqrt{3}$ **(B)** $10\sqrt{3}$ **(C)** $14\sqrt{3}$ **(D)** $16\sqrt{3}$ **(E)** $20\sqrt{3}$

6. In polar coordinates, consider the curves $r = 2\cos\theta$ and $r = 2\sin\theta$. Each curve is a closed curve in the plane. What is the area of the region that lies inside both curves?

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(A) $\frac{\pi}{4} - \frac{1}{2}$ **(B)** $\frac{\pi}{2} - 1$ **(C)** $\frac{\pi}{4}$ **(D)** $\pi - 2$ **(E)** $\frac{\pi}{2}$

7. How many positive divisors of 20^{26} end in exactly one zero when written in base ten? In other words, how many of them are multiples of 10 but not multiples of 100?

(A) 77 **(B)** 78 **(C)** 79 **(D)** 104 **(E)** 1352

8. A regular tetrahedron has all six edges of length 6. Choose two opposite edges (edges that share no vertex). A plane is parallel to both of these edges and lies at the same distance from each of them. What is the area of the cross-section in which this plane cuts the tetrahedron?

(A) $\frac{9\sqrt{3}}{4}$ **(B)** $\frac{9\sqrt{2}}{2}$ **(C)** 9 **(D)** $6\sqrt{3}$ **(E)** 12

9. An urn holds 5 red balls and 5 blue balls. All ten balls are drawn one at a time, at random and without replacement, and their colours are written down in order. A *change* is a place where two consecutive draws have different colours. For example, the sequence RRBRBBBRRB has 5 changes. What is the expected number of changes?

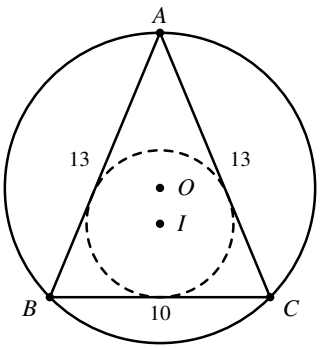
(A) $\frac{5}{2}$ **(B)** 4 **(C)** $\frac{40}{9}$ **(D)** $\frac{9}{2}$ **(E)** 5

10. The number $N = 1 + 3 + 3^2 + 3^3 + \cdots + 3^{2025}$ is written out in full in base ten. What two-digit number do its last two digits form?

(A) 14 **(B)** 28 **(C)** 29 **(D)** 64 **(E)** 93

11. Triangle ABC has $AB = AC = 13$ and $BC = 10$. Let I be the centre of the circle inscribed in the triangle, and let O be the centre of the circle passing through its three vertices. What is the distance OI?

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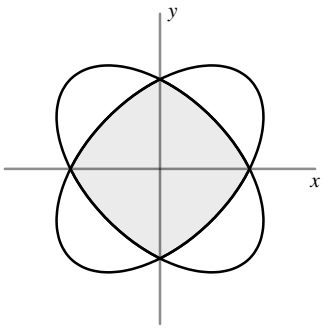


(A) $\frac{7}{8}$ **(B)** $\frac{13}{8}$ **(C)** 2 **(D)** $\frac{89}{24}$ **(E)** $\frac{119}{24}$

12. A sequence is defined by $a_1 = 0$ and $a_{n+1} = \dfrac{a_n + 4}{a_n + 1}$ for $n \geq 1$. (One can check that no term equals -1, so every term is defined.) What is a_{12}?

(A) $\frac{3^{11}+1}{3^{11}-1}$ **(B)** $\frac{2(3^{11}-1)}{3^{11}+1}$ **(C)** $\frac{2(3^{12}-1)}{3^{12}+1}$ **(D)** $\frac{2(3^{12}+1)}{3^{12}-1}$ **(E)** $\frac{2(3^{11}+1)}{3^{11}-1}$

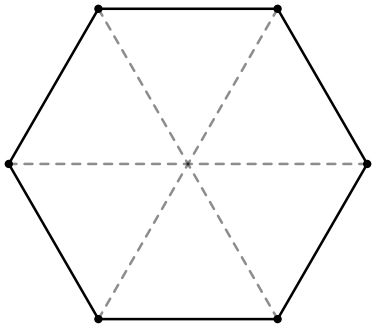
13. Let R be the set of all points (x, y) in the coordinate plane that satisfy both
$x^2 + xy + y^2 \leq 3$ and $x^2 - xy + y^2 \leq 3$.
What is the area of R?



(A) $\pi\sqrt{3}$ **(B)** 2π **(C)** $\frac{4\sqrt{3}}{3}\pi$ **(D)** $\frac{8\pi}{3}$ **(E)** $2\pi\sqrt{3}$

14. Each of the six vertices of a regular hexagon is coloured red, white or blue. Two vertices that are next to each other along the hexagon must get different colours, and so must two vertices that are directly opposite each other. How many colourings are possible? (The hexagon does not move, so colourings that differ by a rotation count as different.)

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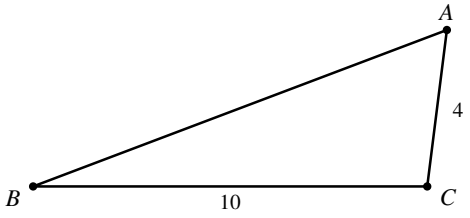


(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 66

15. For a positive integer n, let $s_2(n)$ be the sum of the digits of n written in base 2, and let $s_4(n)$ be the sum of the digits of n written in base 4. For example, $11 = 1011_2 = 23_4$, so $s_2(11) = 3$ and $s_4(11) = 5$. For how many integers n with $1 \le n \le 300$ is $s_2(n) = s_4(n)$?

(A) 15 **(B)** 16 **(C)** 23 **(D)** 24 **(E)** 31

16. Triangle ABC has $BC = 10$ and $CA = 4$, and its angle at A is exactly three times its angle at B, that is, $\angle A = 3\angle B$. What is the length AB?

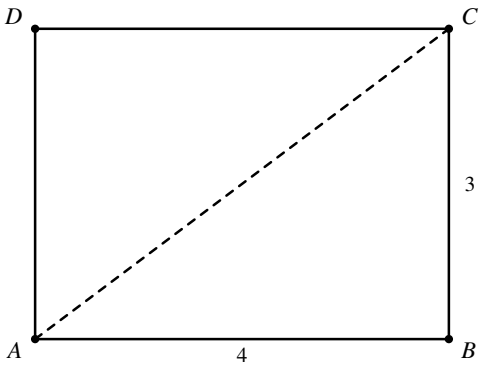


(A) $3\sqrt{14}$ **(B)** $2\sqrt{35}$ **(C)** 12 **(D)** $\frac{7\sqrt{14}}{2}$ **(E)** $4\sqrt{14}$

17. Real numbers $x_1, x_2, \ldots, x_{100}$ satisfy $|x_i| \le \frac{1}{10}$ for every i, together with
$x_1 + x_2 + \cdots + x_{100} = 0$ and $|x_1| + |x_2| + \cdots + |x_{100}| = 1$.
What is the largest possible value of $x_1 + 2x_2 + 3x_3 + \cdots + 100x_{100}$?

(A) 45 **(B)** $\frac{95}{2}$ **(C)** 48 **(D)** 49 **(E)** $\frac{99}{2}$

18. Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. The paper is folded along the diagonal $\overline{AC}$: triangle ACD stays where it is, while triangle ABC swings about the line AC until B is exactly 3 units from D. Now A, B, C and D are the vertices of a tetrahedron. What is the volume of this tetrahedron?



(A) $\frac{28}{15}$ **(B)** $\frac{4\sqrt{11}}{3}$ **(C)** $\frac{3\sqrt{39}}{4}$ **(D)** $\frac{24}{5}$ **(E)** $\frac{8\sqrt{11}}{3}$

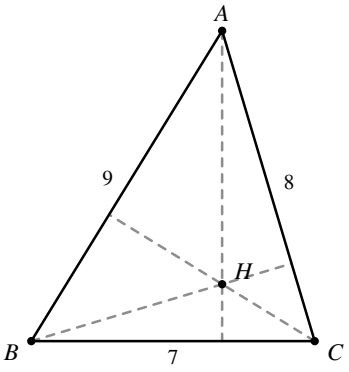
19. A robot starts at the origin of the coordinate plane and makes 12 straight moves. For $k = 1, 2, \ldots, 12$, its k-th move has length k and points in the direction making an angle of $30k^\circ$ counterclockwise from the positive x-axis. How far from the origin is the robot after its twelfth move?

(A) $6(\sqrt{6} - \sqrt{2})$ **(B)** 12 **(C)** $12\sqrt{3}$ **(D)** $6(\sqrt{6} + \sqrt{2})$ **(E)** 24

20. A permutation σ of $\{1, 2, \ldots, 10\}$ is chosen uniformly at random. Write σ^6 for the permutation obtained by applying σ six times in a row. What is the expected number of $i \in \{1, 2, \ldots, 10\}$ with $\sigma^6(i) = i$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

21. Triangle ABC has $BC = 7$, $CA = 8$ and $AB = 9$, and H is its orthocenter. Let O_A, O_B and O_C be the circumcenters of triangles HBC, HCA and HAB respectively. What is the area of triangle $O_AO_BO_C$?



(A) $12\sqrt{5}$ **(B)** $18\sqrt{5}$ **(C)** $24\sqrt{5}$ **(D)** $36\sqrt{5}$ **(E)** $48\sqrt{5}$

22. A circular pond has 100 lily pads, numbered $0, 1, 2, \ldots, 99$ clockwise. For each $k = 1, 2, \ldots, 100$, a frog starts on pad 0 and jumps again and again, each jump carrying it k pads clockwise (so a jump of 100 pads lands it back where it started). Let $V(k)$ be the number of different pads the frog ever lands on, including pad 0. Find

$V(1) + V(2) + \cdots + V(100)$.

(A) 520 **(B)** 4000 **(C)** 5730 **(D)** 5731 **(E)** 10000

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23. Integers b and c are chosen with $1 \le b \le 100$ and $1 \le c \le 1000$. The quadratic $x^2 + bx + c$ has two complex roots r and s. For how many of these pairs (b, c) are the roots r and s different, yet $r^n = s^n$ for some positive integer n?

(A) 31 **(B)** 40 **(C)** 49 **(D)** 53 **(E)** 71

24. Let $\theta = \dfrac{\pi}{2027}$. Compute

$$\sum_{k=1}^{2026} \tan(k\theta)\,\tan(2k\theta).$$

(A) -4054 **(B)** -4052 **(C)** -2027 **(D)** -2026 **(E)** 0

25. A token starts at 0 on the number line and makes 10 moves. Each move is $+1$ or -1, each with probability $\frac{1}{2}$, independently of the other moves. Let M be the largest position the token occupies at any moment, counting its starting position (so $M \ge 0$). What is the expected value of M?

(A) $\frac{1067}{512}$ **(B)** $\frac{565}{256}$ **(C)** $\frac{4751}{2048}$ **(D)** $\frac{315}{128}$ **(E)** $\frac{1579}{512}$

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Answer Key — AMC 12 Practice Contest 6

1. (E) $\frac{7}{2}$

2. (B) $\frac{\sqrt{5}}{5}$

3. (C) $2\sqrt{7}$

4. (E) $\frac{13}{54}$

5. (D) $16\sqrt{3}$

6. (B) $\frac{\pi}{2} - 1$

7. (A) 77

8. (C) 9

9. (E) 5

10. (D) 64

11. (B) $\frac{13}{8}$

12. (E) $\frac{2(3^{11}+1)}{3^{11}-1}$

13. (C) $\frac{4\sqrt{3}}{3}\pi$

14. (C) 42

15. (C) 23

16. (A) $3\sqrt{14}$

17. (B) $\frac{95}{2}$

18. (B) $\frac{4\sqrt{11}}{3}$

19. (D) $6(\sqrt{6}+\sqrt{2})$

20. (D) 4

21. (A) $12\sqrt{5}$

22. (D) 5731

23. (E) 71

24. (A) -4054

25. (A) $\frac{1067}{512}$

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Solutions — AMC 12 Practice Contest 6

1. Answer: (E) $\frac{7}{2}$. The idea. Rewrite every logarithm in terms of just two numbers, $\log_2 3$ and $\log_3 2$, using $\log_{a^m} b^n = \frac{n}{m}\log_a b$. Those two numbers are reciprocals, so their product is 1.

Work it out. In the first bracket, $\log_4 9 = \log_{2^2} 3^2 = \log_2 3$ and $\log_8 27 = \log_{2^3} 3^3 = \log_2 3$, so the first bracket is $3\log_2 3$. In the second bracket, $\log_9 2 = \log_{3^2} 2 = \frac{1}{2}\log_3 2$ and $\log_{27} 4 = \log_{3^3} 2^2 = \frac{2}{3}\log_3 2$, so the second bracket is $\left(\frac{1}{2}+\frac{2}{3}\right)\log_3 2 = \frac{7}{6}\log_3 2$. The product is $3\cdot\frac{7}{6}\cdot\log_2 3\cdot\log_3 2 = \frac{7}{2}\cdot 1 = \frac{7}{2}$.

Check. Numerically $\log_2 3 \approx 1.585$, so the first bracket is about 4.755; $\log_3 2 \approx 0.631$, so the second is about 0.736. Their product is about 3.50. The choice $\frac{7}{6}$ keeps only the second bracket's coefficient, 3 keeps only the first's, and $\frac{5}{2}$ comes from dropping the exponent of $4 = 2^2$ in $\log_{27} 4$.

2. Answer: (B) $\frac{\sqrt{5}}{5}$. The idea. Write $\alpha = \arcsin x$. The equation says $\arcsin(2x) = \frac{\pi}{2} - \alpha$, and taking the sine of both sides turns this into an algebraic equation: $2x = \sin(\frac{\pi}{2} - \alpha) = \cos\alpha$. Since $\cos\alpha$ is never negative for α in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\cos\alpha = \sqrt{1-x^2}$.

Work it out. We need $2x = \sqrt{1-x^2}$. The right side is at least 0, so $x \geq 0$. Squaring gives $4x^2 = 1 - x^2$, so $x^2 = \frac{1}{5}$ and $x = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$. This is less than $\frac{1}{2}$, so $2x \leq 1$ and $\arcsin(2x)$ is defined. Going back: $\alpha = \arcsin\frac{1}{\sqrt{5}}$ lies in $(0, \frac{\pi}{2})$, and $\frac{\pi}{2} - \alpha$ is also in $(0, \frac{\pi}{2})$ with sine $\cos\alpha = \frac{2}{\sqrt{5}} = 2x$. So $\arcsin(2x) = \frac{\pi}{2} - \alpha$ exactly, and $x = \frac{\sqrt{5}}{5}$ works.

Check. Think of a right triangle with legs 1 and 2 and hypotenuse $\sqrt{5}$: its two acute angles add to $\frac{\pi}{2}$, and their sines are $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$, which are x and $2x$. The choice $\frac{2\sqrt{5}}{5}$ is $2x$ rather than x, and it is too large, since then $2x > 1$.

3. Answer: (C) $2\sqrt{7}$. The idea. Drop the altitude from A. In an isosceles triangle it lands at the midpoint M of $\overline{BC}$, and both AB and AD are hypotenuses of right triangles sharing the leg AM. Subtracting the two Pythagorean equations removes the unknown height: $AB^2 - AD^2 = BM^2 - DM^2 = (BM - DM)(BM + DM) = BD \cdot DC$.

Work it out. $BC = 8$, so $BM = 4$ and D is 2 from B, hence $DM = 2$. The height satisfies $AM^2 = AD^2 - DM^2 = 16 - 4 = 12$. Then $AB^2 = AM^2 + BM^2 = 12 + 16 = 28$, and $AB = \sqrt{28} = 2\sqrt{7}$.

Check. The shortcut $AB^2 = AD^2 + BD \cdot DC = 16 + 12 = 28$ agrees. It is Stewart's theorem with $b = c$, or the power of the point D with respect to the circle centred at A through B and C. The choice $2\sqrt{5}$ comes from $AD^2 + BD^2$, treating $\angle ADB$ as a right angle, which it is not.

4. Answer: (E) $\frac{13}{54}$. The idea. Count the $6^3 = 216$ equally likely ordered rolls by how many 4s appear. The middle number is 4 exactly when at least one die shows 4, at most one die is below 4, and at most one die is above 4.

Work it out. *Exactly one* 4*:* the other two dice must be one below 4 (a value from $1, 2, 3$) and one above 4 (a value from $5, 6$). Choose which die shows 4 (3 ways), which of the other two is the low one (2 ways), and the values ($3 \cdot 2$): $3 \cdot 2 \cdot 6 = 36$. *Exactly two* 4*s:* the third die can be anything except 4, in any position: $3 \cdot 5 = 15$. *Three* 4*s:* 1 roll. The total is $36 + 15 + 1 = 52$, and $\frac{52}{216} = \frac{13}{54}$.

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Check. The middle number is at most k exactly when at least two dice are at most k. With $p = \frac{k}{6}$ this has probability $3p^2(1-p) + p^3 = 3p^2 - 2p^3$. For $k = 4$ it is $\frac{20}{27}$ and for $k = 3$ it is $\frac{1}{2}$, and the difference is $\frac{40-27}{54} = \frac{13}{54}$. The choice $\frac{1}{6}$ counts only the 36 rolls with one 4, $\frac{17}{72}$ forgets the roll $4, 4, 4$, and $\frac{37}{216}$ is the chance that the *largest* number is 4.

5. Answer: (D) $16\sqrt{3}$**. The idea.** Two sides and a non-included angle (SSA) can give two triangles. The clean way to find both is the law of cosines at the known angle A, with the unknown side $c = AB$. This gives a quadratic in c, and every positive root is a genuine triangle, since the law of cosines builds the angle A into the equation.

Work it out. The law of cosines gives $BC^2 = CA^2 + AB^2 - 2 \cdot CA \cdot AB \cos 60°$, so $49 = 64 + c^2 - 8c$. That is $c^2 - 8c + 15 = 0$, so $c = 3$ or $c = 5$. Both are positive, and the side lists $7, 8, 3$ and $7, 8, 5$ both satisfy the triangle inequality. The area is $\frac{1}{2} \cdot CA \cdot AB \sin 60° = \frac{1}{2} \cdot 8 \cdot c \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}\, c$, which gives $6\sqrt{3}$ and $10\sqrt{3}$. The sum is $16\sqrt{3}$.

Check. Heron's formula for $3, 7, 8$: $s = 9$, area $= \sqrt{9 \cdot 6 \cdot 2 \cdot 1} = \sqrt{108} = 6\sqrt{3}$. For $5, 7, 8$: $s = 10$, area $= \sqrt{10 \cdot 5 \cdot 3 \cdot 2} = \sqrt{300} = 10\sqrt{3}$. The choices $6\sqrt{3}$ and $10\sqrt{3}$ each keep only one of the two triangles.

6. Answer: (B) $\frac{\pi}{2} - 1$**. The idea.** Change to Cartesian coordinates. Multiplying $r = 2\cos\theta$ by r gives $x^2 + y^2 = 2x$, which is the circle of radius 1 centred at $(1, 0)$. In the same way $r = 2\sin\theta$ is the circle of radius 1 centred at $(0, 1)$. The overlap of two circles is a lens, which splits into two circular segments.

Work it out. The circles meet where $2x = 2y$ and $x^2 + y^2 = 2x$, which gives $(0, 0)$ and $(1, 1)$. The common chord from $(0, 0)$ to $(1, 1)$ has length $\sqrt{2}$. For the circle centred at $(1, 0)$, the radii to the two ends of the chord point along $(-1, 0)$ and $(0, 1)$, so they are perpendicular and the chord cuts off a 90° segment. That segment has area $\frac{\pi}{4} - \frac{1}{2}$ (a quarter disc minus a right triangle with legs 1 and 1). By symmetry the other circle contributes the same area. The lens has area $2\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{2} - 1$.

Check. In polar form, the boundary nearer the origin is $r = 2\sin\theta$ for $0 \le \theta \le \frac{\pi}{4}$ and $r = 2\cos\theta$ for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$. So the area is $2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta = 2\int_0^{\pi/4} (1 - \cos 2\theta)\, d\theta = 2\left(\frac{\pi}{4} - \frac{1}{2}\right)$, the same value. The choice $\frac{\pi}{4} - \frac{1}{2}$ counts only one segment.

7. Answer: (A) 77**. The idea.** Since $20^{26} = 2^{52} \cdot 5^{26}$, every divisor has the form $2^a 5^b$ with $0 \le a \le 52$ and $0 \le b \le 26$. The number of trailing zeros of $2^a 5^b$ is $\min(a, b)$. We want $\min(a, b) = 1$.

Work it out. Count the pairs with $\min(a, b) \ge 1$ and subtract those with $\min(a, b) \ge 2$. There are $52 \cdot 26 = 1352$ pairs with $a, b \ge 1$, and $51 \cdot 25 = 1275$ pairs with $a, b \ge 2$. The difference is $1352 - 1275 = 77$.

Check. Count directly: $\min(a, b) = 1$ means $a = 1$ and $b \ge 1$ (26 pairs), or $b = 1$ and $a \ge 1$ (52 pairs). The pair $a = b = 1$ (the divisor 10 itself) is in both lists, so the total is $26 + 52 - 1 = 77$. The choice 78 counts 10 twice, and 1352 counts every multiple of 10, forgetting to exclude the multiples of 100.

8. Answer: (C) 9**. The idea.** Call the opposite edges $\overline{AB}$ and $\overline{CD}$. A plane halfway between them cuts each of the other four edges (AC, AD, BC, BD) at its midpoint. The cross-section is

therefore the quadrilateral of those four midpoints. Its sides are midsegments, so they are parallel to AB or to CD and half as long. The last ingredient is that opposite edges of a regular tetrahedron are perpendicular.

Work it out. Let P, Q, R, S be the midpoints of AC, CB, BD, DA. In triangle ABC, $PQ \parallel AB$ and $PQ = 3$. In triangle ABD, $RS \parallel AB$ and $RS = 3$. Likewise QR and SP are parallel to CD with length 3. So $PQRS$ is a rhombus of side 3 whose sides are parallel to AB and CD. Now $AB \perp CD$: if M is the midpoint of AB, then $CM \perp AB$ and $DM \perp AB$ (medians of equilateral triangles), so AB is perpendicular to the plane CDM and hence to CD. A rhombus with perpendicular sides is a square, so the area is $3^2 = 9$.

Check. Put the tetrahedron on alternate corners of a cube: $A, B = (\pm k, \pm k, k)$ and $C, D = (k, -k, -k), (-k, k, -k)$ with $2\sqrt{2}\,k = 6$. The halfway plane is $z = 0$, and the four midpoints are $(\pm k, 0, 0)$ and $(0, \pm k, 0)$. They form a square with diagonals $2k = 3\sqrt{2}$, so the area is $\frac{1}{2}(3\sqrt{2})^2 = 9$. The choice $\frac{9\sqrt{3}}{4}$ is the area of a face cut at half scale, which is what a plane parallel to a *face* would give.

9. Answer: (E) 5. The idea. Use linearity of expectation over the 9 adjacent pairs of positions $(1, 2), (2, 3), \ldots, (9, 10)$. The expected number of changes is the sum over these pairs of the probability that the pair has different colours. By symmetry, every pair of positions has the same probability.

Work it out. Any two given positions hold a uniformly random pair of the 10 balls. The pair has different colours with probability $\dfrac{5 \cdot 5}{\binom{10}{2}} = \dfrac{25}{45} = \dfrac{5}{9}$. So the expected number of changes is $9 \cdot \frac{5}{9} = 5$.

Check. A small case: with 1 red and 1 blue ball there is always exactly 1 change, and the formula gives $1 \cdot \dfrac{1 \cdot 1}{\binom{2}{2}} = 1$. With 2 red and 2 blue, the 6 sequences RRBB, RBRB, RBBR, BBRR, BRBR, BRRB have 1, 3, 2, 1, 3, 2 changes, averaging 2, and the formula gives $3 \cdot \dfrac{4}{6} = 2$. If the draws were *with* replacement, each pair would differ with probability $\frac{1}{2}$ and the answer would be $\frac{9}{2}$. Drawing without replacement makes changes slightly more likely.

10. Answer: (D) 64. The idea. The geometric-series formula gives $N = \frac{3^{2026}-1}{2}$. To know N modulo 100 we must know $3^{2026} - 1$ modulo 200, not just modulo 100: halving a number known only modulo 100 leaves two possibilities that differ by 50.

Work it out. Modulo 200: $3^{10} = 59049 \equiv 49$, so $3^{20} \equiv 49^2 = 2401 \equiv 1 \pmod{200}$. Since $2026 = 20 \cdot 101 + 6$, we get $3^{2026} \equiv 3^6 = 729 \equiv 129 \pmod{200}$. So $3^{2026} - 1 = 200m + 128$ for some integer m, and $N = 100m + 64$. The last two digits of N form 64.

Check. The same pattern shows up in small cases: $1 + 3 + \cdots + 3^5 = 364$ and $\frac{3^6-1}{2} = \frac{728}{2} = 364$, with $3^6 = 729 \equiv 129 \pmod{200}$ as used above. Working only modulo 100 gives $3^{2026} \equiv 29$ and the tempting but wrong halving $\frac{29-1}{2} = 14$; the choice 93 adds one term too many, 3^{2026}.

11. Answer: (B) $\frac{13}{8}$. The idea. By symmetry both centres lie on the altitude from A, which meets BC at its midpoint M. So it is enough to find how high above M each centre sits: the incentre is at height r (the inradius), and the circumcentre is at distance R (the circumradius) below A.

Work it out. With $BM = 5$ and $AB = 13$, the altitude is $AM = 12$, so the area is $K = \frac{1}{2} \cdot 10 \cdot 12 = 60$ and the semiperimeter is $s = 18$. The inradius is $r = \frac{K}{s} = \frac{10}{3}$, so I is $\frac{10}{3} = \frac{80}{24}$ above M. The

circumradius is $R = \frac{abc}{4K} = \frac{13\cdot13\cdot10}{240} = \frac{169}{24}$, so O is $12 - \frac{169}{24} = \frac{119}{24}$ above M. Both are on the same side of M, so $OI = \frac{119}{24} - \frac{80}{24} = \frac{39}{24} = \frac{13}{8}$.

Check. Euler's formula $OI^2 = R^2 - 2Rr$ agrees: $\frac{169}{24}\left(\frac{169}{24} - \frac{160}{24}\right) = \frac{169\cdot9}{576}$, whose square root is $\frac{39}{24} = \frac{13}{8}$. The choice $\frac{119}{24}$ is the height of O alone, and $\frac{89}{24} = R - r$ comes from subtracting the two radii directly.

12. Answer: (E) $\frac{2(3^{11}+1)}{3^{11}-1}$. The idea. The map $f(a) = \dfrac{a+4}{a+1}$ has fixed points where $a^2 + a = a + 4$, that is, $a = \pm2$. For such a map, the quantity $q = \dfrac{a-2}{a+2}$, which measures position relative to the two fixed points, is multiplied by the same constant at every step. That turns the recurrence into a geometric sequence.

Work it out. If $b = f(a)$, then $b-2 = \dfrac{a+4-2a-2}{a+1} = \dfrac{-(a-2)}{a+1}$ and $b+2 = \dfrac{3a+6}{a+1} = \dfrac{3(a+2)}{a+1}$. So $\dfrac{b-2}{b+2} = -\dfrac{1}{3}\cdot\dfrac{a-2}{a+2}$. With $q_n = \dfrac{a_n-2}{a_n+2}$ we have $q_1 = \dfrac{-2}{2} = -1$ and $q_n = -\left(-\frac{1}{3}\right)^{n-1}$. So $q_{12} = -\left(-\frac{1}{3}\right)^{11} = \dfrac{1}{3^{11}}$. Solving $q = \dfrac{a-2}{a+2}$ for a gives $a = \dfrac{2(1+q)}{1-q}$, so $a_{12} = \dfrac{2(1+3^{-11})}{1-3^{-11}} = \dfrac{2(3^{11}+1)}{3^{11}-1}$.

Check. The formula gives $q_2 = \frac{1}{3}$ and $a_2 = \dfrac{2\cdot\frac{4}{3}}{\frac{2}{3}} = 4$, which matches $a_2 = \frac{0+4}{0+1}$. It gives $q_3 = -\frac{1}{9}$ and $a_3 = \dfrac{2\cdot\frac{8}{9}}{\frac{10}{9}} = \frac{8}{5}$, which matches $\frac{4+4}{4+1}$. The terms alternate above and below 2: $a_n > 2$ exactly when $q_n > 0$, which happens for even n. That rules out $\frac{2(3^{11}-1)}{3^{11}+1} < 2$, the answer you get with the sign of q_1 flipped. The choices with 3^{12} are off by one step.

13. Answer: (C) $\frac{4\sqrt{3}}{3}\pi$. The idea. The xy terms say both curves are ellipses tilted at 45°. Rotating the axes by 45° removes the xy terms, and then the two ellipses turn out to be the same ellipse turned through 90°. By symmetry the region splits into eight congruent pieces, and one piece is a sector of a single ellipse, whose area a stretch turns into a sector of a circle.

Work it out. Put $u = \frac{x+y}{\sqrt{2}}$ and $v = \frac{x-y}{\sqrt{2}}$, a rotation that preserves area. Then $x^2 + y^2 = u^2 + v^2$ and $xy = \frac{u^2-v^2}{2}$, so the conditions become

$\frac{3}{2}u^2 + \frac{1}{2}v^2 \le 3$, that is $\frac{u^2}{2} + \frac{v^2}{6} \le 1$, and $\frac{u^2}{6} + \frac{v^2}{2} \le 1$.

The region is symmetric under $u \to -u$, $v \to -v$ and $u \leftrightarrow v$, so it consists of 8 congruent copies of its part with $0 \le v \le u$. There the first condition is the stronger one, because $\left(\frac{u^2}{2} + \frac{v^2}{6}\right) - \left(\frac{u^2}{6} + \frac{v^2}{2}\right) = \frac{u^2-v^2}{3} \ge 0$. So that piece is the part of the ellipse $E : \frac{u^2}{2} + \frac{v^2}{6} \le 1$ between the rays $v = 0$ and $v = u$ (with $u \ge 0$).

Stretch by $U = \frac{u}{\sqrt{2}}$, $V = \frac{v}{\sqrt{6}}$. This turns E into the unit disk and divides areas by $\sqrt{2}\cdot\sqrt{6} = 2\sqrt{3}$. The ray $v = u$ becomes $\sqrt{6}\,V = \sqrt{2}\,U$, that is $V = \frac{U}{\sqrt{3}}$, the ray at 30°. So the piece becomes a 30° sector of the unit disk, of area $\frac{\pi}{12}$, and the original piece has area $2\sqrt{3}\cdot\frac{\pi}{12} = \frac{\sqrt{3}\,\pi}{6}$.

The whole region has area $8\cdot\frac{\sqrt{3}\,\pi}{6} = \frac{4\sqrt{3}}{3}\pi \approx 7.26$.

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Check. The answer must lie between the area of the circle $u^2+v^2 \le 2$ that fits inside both ellipses ($2\pi \approx 6.28$) and the area of one ellipse ($\pi\sqrt{2}\sqrt{6} = 2\sqrt{3}\,\pi \approx 10.88$). It does, and it is exactly $\frac{2}{3}$ of one ellipse: in the stretched picture the ellipse is a full 360° disk while the region keeps eight 30° sectors, 240° of it.

14. Answer: (C) 42**. The idea.** Number the vertices 1 to 6 around the hexagon. Each vertex must differ from its two neighbours and from the opposite vertex, and those three vertices are exactly the three vertices of the other parity: vertex 1 is linked to 2, 6 and 4. So every odd vertex is linked to every even vertex, and the rule simply says that no colour may be used on both an odd vertex and an even vertex.

Work it out. Colour the odd vertices $1, 3, 5$ first, then the even vertices must use only colours not used on the odd ones. If the odd vertices use exactly one colour (3 ways), the even vertices may use the other two colours freely: $2^3 = 8$ ways, giving 24. If they use exactly two colours ($\binom{3}{2}(2^3-2) = 18$ ways), the even vertices must all get the third colour: 1 way, giving 18. If they use all three colours, nothing is left for the even vertices. The total is $24 + 18 = 42$.

Check. By the symmetry between the two colour classes, the count is also "choose disjoint nonempty palettes": both triples use one colour each in $3 \cdot 2 = 6$ ways, one triple uses a single colour and the other uses both remaining colours in $2 \cdot 3 \cdot 6 = 36$ ways (which triple, which colour, and $2^3 - 2 = 6$ ways to use both of the others), and $6 + 36 = 42$. Forgetting the opposite pairs would give the 66 proper colourings of a plain hexagon.

15. Answer: (C) 23**. The idea.** Base 4 is base 2 read two digits at a time: each base-4 digit d becomes a block of two binary digits. So both digit sums can be computed digit by digit of the base-4 form, and we can compare them block by block.

Work it out. The base-4 digits $0, 1, 2, 3$ become the binary blocks $00, 01, 10, 11$, whose digit sums are $0, 1, 1, 2$. So a base-4 digit d contributes d to $s_4(n)$ and $0, 1, 1, 2$ respectively to $s_2(n)$, and the excess $d -$ (binary sum) is $0, 0, 1, 1$. Every excess is ≥ 0, so $s_4(n) - s_2(n) \ge 0$, with equality exactly when every base-4 digit of n is 0 or 1.

Now count such n from 1 to 300. Since $4^4 = 256 \le 300 < 4^5$, n has at most 5 base-4 digits.

With at most 4 digits: each of the 4 places is 0 or 1, not all 0: $2^4 - 1 = 15$ numbers, all below 256.

With 5 digits $1abcd_4 = 256 + 64a + 16b + 4c + d$: we need this ≤ 300, so $a = 0$ (as $256 + 64 = 320 > 300$), while $b, c, d \in \{0, 1\}$ are free (the largest is $256 + 21 = 277$). That is 8 numbers.

Total: $15 + 8 = 23$.

Check. The example $11 = 23_4$ has digits 2 and 3, excesses $1+1 = 2$, and indeed $s_4 - s_2 = 5 - 3 = 2$. And $5 = 11_4 = 101_2$ has $s_2 = s_4 = 2$, as predicted.

16. Answer: (A) $3\sqrt{14}$**. The idea.** The law of sines turns the side ratio into a ratio of sines, $\frac{BC}{CA} = \frac{\sin 3B}{\sin B}$, and the triple-angle formula makes that ratio a function of $\sin^2 B$ alone. Once B is known, the third angle is $180° - 4B$, and its sine gives AB.

Work it out. Write $b = CA = 4$ and $a = BC = 10$. Since $\sin 3B = 3\sin B - 4\sin^3 B$, the law of sines gives $\frac{a}{b} = 3 - 4\sin^2 B = \frac{10}{4}$, so $\sin^2 B = \frac{1}{8}$, $\cos^2 B = \frac{7}{8}$ and $\cos 2B = 1 - 2\sin^2 B = \frac{3}{4}$. (Here $B < 45°$, since $A + B = 4B < 180°$, so $\cos B > 0$.) The angle at C is $180° - 4B$, so $\sin C = \sin 4B = 4\sin B \cos B \cos 2B$, and

$$AB = \frac{b \sin C}{\sin B} = 4 \cdot 4 \cos B \cos 2B = 16 \cdot \sqrt{\tfrac{7}{8}} \cdot \tfrac{3}{4} = 12\sqrt{\tfrac{7}{8}} = 3\sqrt{14}.$$

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Check. By the law of cosines, $\cos C = \frac{10^2+4^2-(3\sqrt{14})^2}{2\cdot 10\cdot 4} = \frac{116-126}{80} = -\frac{1}{8}$, and indeed $\cos C = -\cos 4B = -(2\cos^2 2B - 1) = -\left(\frac{9}{8} - 1\right) = -\frac{1}{8}$. The choice $4\sqrt{14}$ drops the factor $\cos 2B$, and $\frac{7\sqrt{14}}{2}$ uses the wrong double-angle formula $\cos 2B = 1 - \sin^2 B = \frac{7}{8}$.

17. Answer: (B) $\frac{95}{2}$. The idea. Split the numbers into their positive and negative parts. The two conditions say the positive parts total $\frac{1}{2}$ and the negative parts total $-\frac{1}{2}$. To make the weighted sum large, the positive mass should sit on the largest weights and the negative mass on the smallest ones, but the cap $\frac{1}{10}$ forces each mass to be spread over at least 5 indices.

Work it out. Let P be the sum of the positive x_i and N the sum of the absolute values of the negative ones. Then $P - N = 0$ and $P + N = 1$, so $P = N = \frac{1}{2}$.

Positive part: write each weight as $i = 95 + (i - 95)$. Then

$\sum_{x_i>0} i\,x_i = 95P + \sum_{x_i>0}(i-95)x_i \le \frac{95}{2} + \sum_{i=96}^{100}(i-95)\cdot\frac{1}{10} = \frac{95}{2} + \frac{15}{10} = 49,$

since a term with $i \le 95$ contributes at most 0 to the second sum and a term with $i > 95$ at most $(i-95)\cdot\frac{1}{10}$.

Negative part: similarly, with $i = 6 - (6 - i)$,

$\sum_{x_i<0} i\,|x_i| = 6N - \sum_{x_i<0}(6-i)|x_i| \ge 3 - \sum_{i=1}^{5}(6-i)\cdot\frac{1}{10} = 3 - \frac{15}{10} = \frac{3}{2}.$

So $\sum i\,x_i \le 49 - \frac{3}{2} = \frac{95}{2}$. Equality holds for $x_{96} = \cdots = x_{100} = \frac{1}{10}$, $x_1 = \cdots = x_5 = -\frac{1}{10}$ and all others 0: the sum is $\frac{96+97+98+99+100}{10} - \frac{1+2+3+4+5}{10} = 49 - \frac{3}{2} = \frac{95}{2}$.

Check. Without the cap the whole positive mass would go on x_{100} and the negative mass on x_1, giving $50 - \frac{1}{2} = \frac{99}{2}$; the cap moves the average weight of the positive mass down from 100 to 98 and that of the negative mass up from 1 to 3, each change costing $\frac{1}{2}\cdot 2 = 1$, so the answer should be $\frac{99}{2} - 2 = \frac{95}{2}$, as found.

18. Answer: (B) $\frac{4\sqrt{11}}{3}$. The idea. During the fold, B moves on a circle around the line AC: its foot on AC and its distance from AC never change. So describe B and D by their feet on AC and their distances from AC; then BD pins down the angle between the two triangles (the fold angle), and the volume follows from that angle.

Work it out. In the flat rectangle, $AC = 5$. The right triangles ABC and ADC each have legs 4 and 3, so B and D are both at distance $\frac{4\cdot 3}{5} = \frac{12}{5}$ from AC. Their feet are different points, though: the foot F of D is at $AF = \frac{AD^2}{AC} = \frac{9}{5}$ from A, while the foot G of B is at $AG = \frac{AB^2}{AC} = \frac{16}{5}$ from A. So $FG = \frac{7}{5}$.

Let θ be the angle between the segments $\overline{FD}$ and $\overline{GB}$ (both perpendicular to AC), i.e. the angle between the two triangles along AC. Split $\overrightarrow{DB}$ into a part along AC, of length $\frac{7}{5}$, and a part perpendicular to AC, which is the third side of a triangle with two sides $\frac{12}{5}$ and angle θ between them. By the Pythagorean theorem and the law of cosines,

$BD^2 = \left(\frac{7}{5}\right)^2 + 2\left(\frac{12}{5}\right)^2(1 - \cos\theta) = \frac{49+288(1-\cos\theta)}{25}.$

(Flat, $\theta = 180^\circ$ gives $BD^2 = \frac{625}{25} = 25$, the rectangle's other diagonal.) Setting $BD^2 = 9$: $288(1 - \cos\theta) = 176$, so $\cos\theta = \frac{7}{18}$ and $\sin\theta = \frac{\sqrt{324-49}}{18} = \frac{5\sqrt{11}}{18}$.

Take triangle ACD, of area 6, as the base. The height of B above its plane is $GB\sin\theta = \frac{12}{5}\cdot\frac{5\sqrt{11}}{18} = \frac{2\sqrt{11}}{3}$. So the volume is

$V = \frac{1}{3} \cdot 6 \cdot \frac{2\sqrt{11}}{3} = \frac{4\sqrt{11}}{3} \approx 4.42$.

Check. Coordinates: $A = (0,0,0), C = (5,0,0), D = \left(\frac{9}{5}, \frac{12}{5}, 0\right)$ and $B = \left(\frac{16}{5}, \frac{12}{5}\cos\theta, \frac{12}{5}\sin\theta\right) = \left(\frac{16}{5}, \frac{14}{15}, \frac{2\sqrt{11}}{3}\right)$. Then $AB^2 = \frac{2304+196+1100}{225} = 16$, $CB^2 = \left(\frac{9}{5}\right)^2 + \left(\frac{14}{15}\right)^2 + \frac{44}{9} = \frac{729+196+1100}{225} = 9$, and $DB^2 = \left(\frac{7}{5}\right)^2 + \left(\frac{22}{15}\right)^2 + \frac{44}{9} = \frac{441+484+1100}{225} = 9$, as required. The volume is at most $\frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$ (the right-angle fold), and $\frac{4\sqrt{11}}{3} < \frac{24}{5}$. Forgetting that the two feet differ (dropping the $\frac{7}{5}$) gives $\frac{3\sqrt{39}}{4}$ instead.

19. Answer: (D) $6(\sqrt{6}+\sqrt{2})$**. The idea.** As complex numbers the moves are $k\omega^k$ with $\omega = e^{2\pi i/12}$, so the final position is $S = \sum_{k=1}^{12} k\omega^k$, an arithmetic-geometric sum. Multiplying by ω and subtracting collapses it, just as for a geometric series.

Work it out. $S - \omega S = \sum_{k=1}^{12} k\omega^k - \sum_{k=2}^{13}(k-1)\omega^k = \sum_{k=1}^{12} \omega^k - 12\omega^{13}$.

The twelve 12th roots of unity sum to 0, and $\omega^{13} = \omega$, so $(1-\omega)S = -12\omega$ and $|S| = \frac{12}{|1-\omega|}$.

For a unit vector at angle 30°, $|1-\omega| = 2\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{2}$. Hence

$|S| = \frac{24}{\sqrt{6}-\sqrt{2}} = \frac{24(\sqrt{6}+\sqrt{2})}{4} = 6(\sqrt{6}+\sqrt{2}) \approx 23.18$.

Check. Pair move k with move $k+6$: they point in opposite directions, so together they amount to a move of length 6 in the direction $30(k+6)°$. So the path is equivalent to six moves of length 6 at $210°, 240°, \ldots, 360°$, whose total length is $6\,|\omega^7 + \cdots + \omega^{12}| = 6 \cdot \frac{|1-\omega^6|}{|1-\omega|} = \frac{12}{|1-\omega|}$, the same value.

20. Answer: (D) 4**. The idea.** Follow $i, \sigma(i), \sigma^2(i), \ldots$: it runs around the cycle of σ containing i, and returns to i after exactly L steps, where L is that cycle's length. So $\sigma^6(i) = i$ exactly when L divides 6. The surprising fact that makes the problem easy is that the length of the cycle through a given element is uniformly distributed.

Work it out. Fix i and a length L with $1 \le L \le 10$. Permutations in which i lies in a cycle of length L: the cycle is $i \to a_1 \to \cdots \to a_{L-1} \to i$ with $a_1, \ldots, a_{L-1}$ distinct elements chosen in order, in $9 \cdot 8 \cdots (11-L) = \frac{9!}{(10-L)!}$ ways, and the other $10-L$ elements are permuted in $(10-L)!$ ways. That is 9! permutations for every L, so each L has probability $\frac{9!}{10!} = \frac{1}{10}$.

The lengths dividing 6 are $1, 2, 3, 6$, so $P(\sigma^6(i) = i) = \frac{4}{10}$. By linearity of expectation over the 10 elements, the expected number is $10 \cdot \frac{4}{10} = 4$.

Check. For σ^1 the same argument gives $10 \cdot \frac{1}{10} = 1$ expected fixed point, the familiar answer. A tempting wrong route adds the expected numbers of cycles of lengths $1, 2, 3, 6$, namely $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2$; but a cycle of length L contributes L fixed elements of σ^6, and $1 \cdot 1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{6} = 4$.

21. Answer: (A) $12\sqrt{5}$**. The idea.** The reflection of H in BC lies on the circumcircle of ABC. So reflecting the circumcircle of ABC in BC gives a circle through B, C and H: the circle through H, B, C is the mirror image of the circumcircle, and its centre O_A is the mirror image of the circumcentre O.

Work it out. First the reflection fact. The triangle is acute ($9^2 < 7^2 + 8^2$), so H is inside it. Let H' be the reflection of H in BC. Then $\angle BH'C = \angle BHC = 180° - \angle A$, since the angle between

the altitudes from B and C is the supplement of $\angle A$. So $\angle BH'C + \angle A = 180°$, with H' and A on opposite sides of BC: $ABH'C$ is cyclic and H' is on the circumcircle.

Hence the circumcircle of HBC is the reflection in line BC of the circumcircle of ABC (both pass through B, C and the mirror images $H \leftrightarrow H'$). Its centre is $O_A = 2M_A - O$, where M_A is the midpoint of BC (the foot of the perpendicular from O). Likewise $O_B = 2M_B - O$ and $O_C = 2M_C - O$.

So $O_AO_BO_C$ is the image of the medial triangle $M_AM_BM_C$ under the homothety centred at O with ratio 2, and its area is $4 \cdot \frac{[ABC]}{4} = [ABC]$.

By Heron with $s = 12$: $[ABC] = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{720} = 12\sqrt{5}$.

Check. Even more is true: $O_AO_BO_C$ is congruent to ABC, since $O_BO_C = 2M_BM_C = BC$ and so on. In coordinates with $B = (0,0)$, $C = (7,0)$, $A = (\frac{33}{7}, \frac{24\sqrt{5}}{7})$, one gets $O = (\frac{7}{2}, \frac{7\sqrt{5}}{5})$ and $O_A = (\frac{7}{2}, -\frac{7\sqrt{5}}{5})$, and indeed $O_AB^2 = \frac{49}{4} + \frac{49}{5} = \frac{441}{20} = R^2$, where $R = \frac{abc}{4K} = \frac{504}{48\sqrt{5}} = \frac{21\sqrt{5}}{10}$ is the circumradius; the same holds for O_AC by symmetry, and $O_AH = R$ follows from the reflection.

22. Answer: (D) 5731**. The idea.** The frog visits exactly the multiples of $g = \gcd(k, 100)$, so $V(k) = \frac{100}{g}$. Instead of summing over k, group the k by the value $m = V(k)$: exactly $\varphi(m)$ values of k give orbit size m, and the sum becomes $\sum_{m|100} m\,\varphi(m)$.

Work it out. After j jumps the frog is on pad jk mod 100. The values jk mod 100 are exactly the multiples of $g = \gcd(k, 100)$ (by Bezout), and there are $\frac{100}{g}$ of them, so $V(k) = \frac{100}{g}$.

For a divisor m of 100, $V(k) = m$ means $\gcd(k, 100) = \frac{100}{m}$, i.e. $k = \frac{100}{m}\,j$ with $1 \le j \le m$ and $\gcd(j, m) = 1$. There are $\varphi(m)$ such k. Therefore

$\sum_{k=1}^{100} V(k) = \sum_{m|100} m\,\varphi(m)$.

Term by term: $m = 1, 2, 4, 5, 10, 20, 25, 50, 100$ give $1, 2, 8, 20, 40, 160, 500, 1000, 4000$, which sum to 5731.

Check. The function $f(n) = \sum_{m|n} m\,\varphi(m)$ is multiplicative, and $f(4) = 1 + 2 + 8 = 11$, $f(25) = 1 + 20 + 500 = 521$, so $f(100) = 11 \cdot 521 = 5731$. The $\varphi(m)$ counts add up correctly too: $1+1+2+4+4+8+20+20+40 = 100$ values of k. Note $k = 100$ gives $V = 1$, and the 40 values of k coprime to 100 give $V = 100$.

23. Answer: (E) 71**. The idea.** If $r^n = s^n$ with $r \ne s$, then $z = \frac{r}{s}$ is a root of unity other than 1. Its "trace" $z + \frac{1}{z} = \frac{r^2+s^2}{rs} = \frac{b^2-2c}{c}$ is rational, and a rational number of the form $z + \frac{1}{z}$ with z a root of unity must be one of $-2, -1, 0, 1, 2$. That leaves only three shapes of quadratic, which are easy to count.

Work it out. Note $rs = c \ne 0$, so $z = \frac{r}{s}$ is defined, and $z^n = 1$, so $|z| = 1$ and $t = z + \frac{1}{z} = z + \bar{z}$ is real with $|t| \le 2$. By Vieta, $t = \frac{(r+s)^2-2rs}{rs} = \frac{b^2-2c}{c}$, a rational number.

Why t must be an integer: the numbers $t_k = z^k + z^{-k}$ satisfy $t_0 = 2$, $t_1 = t$ and $t_{k+1} = t\,t_k - t_{k-1}$, so $t_k = P_k(t)$ for a monic polynomial P_k with integer coefficients ($P_2 = t^2 - 2$, $P_3 = t^3 - 3t$, …). From $z^n = 1$ we get $P_n(t) = 2$, so the rational number t is a root of the monic integer polynomial $P_n(x) - 2$. By the rational root theorem, t is an integer, so $t \in \{-2, -1, 0, 1, 2\}$.

$t = 2$ means $z = 1$, i.e. $r = s$: excluded. $t = -2$ means $z = -1$, i.e. $r + s = 0$, so $b = 0$: impossible. The other values give $b^2 - 2c = tc$, that is $b^2 = (t + 2)c$:

$t = -1$: $c = b^2$ (then z is a cube root of unity). Need $b^2 \le 1000$: $b = 1, \ldots, 31$, which is 31 pairs.

$t = 0$: $c = \frac{b^2}{2}$ (then $z = \pm i$). Need b even and $b^2 \le 2000$: $b = 2, 4, \ldots, 44$, which is 22 pairs.

$t = 1$: $c = \frac{b^2}{3}$ (then z is a primitive sixth root of unity). Need $3 \mid b$ and $b^2 \le 3000$: $b = 3, 6, \ldots, 54$, which is 18 pairs.

Conversely each of these pairs works: $b^2 < 4c$ in every case, so the roots are distinct complex conjugates, and z satisfies $z^2 - tz + 1 = 0$, making z^3, z^4 or z^6 equal to 1. All values of b are within $1 \le b \le 100$. The total is $31 + 22 + 18 = 71$.

Check. $x^2 + 2x + 2$ has roots $-1 \pm i$, and $(-1 + i)^4 = (-2i)^2 = -4 = (-1 - i)^4$. $x^2 + 3x + 3$ has roots $\frac{-3 \pm i\sqrt{3}}{2} = \sqrt{3}\, e^{\pm 150^\circ i}$, whose sixth powers are both $27e^{\pm 900^\circ i} = -27$. By contrast $x^2 + x + 2$ has $t = \frac{1-4}{2} = -\frac{3}{2}$, not an integer, and indeed its roots have ratio of infinite order. Missing one of the three families gives 40, 49 or 53; only the obvious family $c = b^2$ gives 31.

24. Answer: (A) -4054**. The idea.** Each term simplifies: $\tan x \tan 2x = \sec 2x - 1$. The problem then becomes a sum of secants of the angles $\frac{2\pi j}{2027}$, which is a sum over the 2027th roots of unity and can be computed exactly with the logarithmic derivative of $z^{2027} - 1$.

Work it out. From $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$, we get $\tan x \tan 2x = \frac{2\tan^2 x}{1-\tan^2 x} = \frac{2}{1-\tan^2 x} - 2 = \frac{2\cos^2 x}{\cos 2x} - 2 = \frac{1+\cos 2x}{\cos 2x} - 2 = \sec 2x - 1$. (All terms are defined: $n = 2027$ is odd, so neither $k\theta$ nor $2k\theta$ is an odd multiple of $\frac{\pi}{2}$.)

So the sum is $\sum_{k=1}^{2026} \sec \frac{2k\pi}{n} - 2026$. As k runs over $1, \ldots, n-1$, $2k \bmod n$ runs over $1, \ldots, n-1$ too (n is odd), and $\cos \frac{2\pi j}{n}$ depends only on $j \bmod n$. So we need $T - 1$ with $T = \sum_{j=0}^{n-1} \sec \frac{2\pi j}{n}$.

Let $z_j = e^{2\pi i j/n}$. Then $\sec \frac{2\pi j}{n} = \frac{2}{z_j + z_j^{-1}} = \frac{2z_j}{z_j^2+1} = \frac{1}{z_j - i} + \frac{1}{z_j + i}$. For $P(z) = z^n - 1 = \prod (z - z_j)$ and any w with $w^n \ne 1$, $\sum_j \frac{1}{w - z_j} = \frac{P'(w)}{P(w)} = \frac{nw^{n-1}}{w^n - 1}$.

Since $n = 2027 \equiv 3 \pmod 4$: $i^n = -i$, $i^{n-1} = -1$, $(-i)^n = i$, $(-i)^{n-1} = -1$. So

$\sum_j \frac{1}{z_j - i} = -\frac{n\, i^{n-1}}{i^n - 1} = \frac{n}{-i - 1}$ and $\sum_j \frac{1}{z_j + i} = -\frac{n(-i)^{n-1}}{(-i)^n - 1} = \frac{n}{i - 1}$.

Adding, $T = n\left(\frac{-1+i}{2} + \frac{-1-i}{2}\right) = -n = -2027$.

The requested sum is $(T - 1) - 2026 = -2028 - 2026 = -4054$.

Check. Try $n = 3$ ($\theta = 60^\circ$, also $\equiv 3 \pmod 4$): the terms are $\tan 60^\circ \tan 120^\circ = -3$ and $\tan 120^\circ \tan 240^\circ = -3$, total $-6 = -2n$, matching the pattern $-2n$ that the argument gives for every $n \equiv 3 \pmod 4$.

25. Answer: (A) $\frac{1067}{512}$**. The idea.** Write $E[M] = \sum_{k \ge 1} P(M \ge k)$. The event $M \ge k$ is handled by the reflection principle: a path that reaches k but ends below k can be reflected after its first visit to k, turning it into a path that ends above k. So everything reduces to the distribution of the final position S.

Work it out. Fix $k \ge 1$. Paths with $M \ge k$ either end at $S \ge k$ (and then automatically $M \ge k$) or end at $S < k$. For the second kind, reflect the part after the first visit to k in the line $y = k$: this is a bijection onto the paths ending at $S > k$ (each such path must pass k, and reflecting back recovers the original). All paths are equally likely, so

$P(M \ge k) = P(S \ge k) + P(S \ge k + 1)$.

Summing over $k \ge 1$: $E[M] = \sum_{k \ge 1} P(S \ge k) + \sum_{k \ge 1} P(S \ge k + 1) = E[\max(S, 0)] + E[\max(S - 1, 0)]$.

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With j up-moves, $S = 2j - 10$ and $P(j) = \binom{10}{j}/1024$. The positive values are $S = 2, 4, 6, 8, 10$ with counts $210, 120, 45, 10, 1$:

$E[\max(S,0)] = \frac{2\cdot 210+4\cdot 120+6\cdot 45+8\cdot 10+10\cdot 1}{1024} = \frac{1260}{1024}$,

$E[\max(S-1,0)] = \frac{1\cdot 210+3\cdot 120+5\cdot 45+7\cdot 10+9\cdot 1}{1024} = \frac{874}{1024}$.

So $E[M] = \frac{2134}{1024} = \frac{1067}{512} \approx 2.08$.

Check. For 2 moves the paths $++, +-, -+, --$ have maxima $2, 1, 0, 0$, so $E[M] = \frac{3}{4}$; the formula gives $E[\max(S,0)] + E[\max(S-1,0)] = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. The tempting shortcut $P(M \geq k) = 2P(S \geq k)$ ignores that S can equal k and gives $2 \cdot \frac{1260}{1024} = \frac{315}{128}$, too large.

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