## MathCastle

# AMC 12

Practice Contest

25 questions • 75 minutes

**Name** _____

**School** _____

**Grade** _____

**INSTRUCTIONS**

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

**DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.**

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**1.** The equation $4^x - 12 \cdot 2^x + 18 = 0$ has exactly two real solutions $x_1$ and $x_2$. What is $x_1 + x_2$?

**(A)** $\log_2 6$ **(B)** $\log_2 12$ **(C)** $\log_2 18$ **(D)** 6 **(E)** 12

**2.** Three fair six-sided dice are rolled. What is the probability that the largest of the three numbers shown is exactly twice the smallest?

**(A)** $\frac{1}{12}$ **(B)** $\frac{9}{56}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{7}{36}$ **(E)** $\frac{2}{9}$

**3.** A triangle has two sides of lengths 5 and 8, and its area is 12. This information does not determine the length of the third side. What is the sum of the squares of all the possible lengths of the third side?

**(A)** 25 **(B)** 89 **(C)** 153 **(D)** 169 **(E)** 178

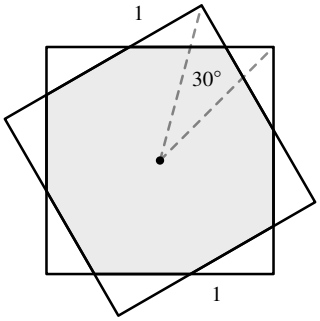
**4.** A real number $x > 0$ satisfies the equation $9^x + 9^{-x} = 23$. What is the value of the difference $27^x - 27^{-x}$?

**(A)** $18\sqrt{21}$ **(B)** $21\sqrt{21}$ **(C)** $22\sqrt{21}$ **(D)** $23\sqrt{21}$ **(E)** $24\sqrt{21}$

**5.** What is the sum of all positive divisors $d$ of 720 for which the quotient $\frac{720}{d}$ is an odd number?

**(A)** 78 **(B)** 624 **(C)** 1248 **(D)** 2340 **(E)** 2418

**6.** Two squares of side length 1 have the same centre. The second square is obtained from the first by rotating it 30° about that centre. What is the area of the region that lies inside both squares?




**(A)** $4 - 2\sqrt{3}$ **(B)** $\frac{3}{4}$ **(C)** $2\sqrt{2} - 2$ **(D)** $2 - \frac{2\sqrt{3}}{3}$ **(E)** $\frac{\sqrt{3}}{2}$

**7.** Complex numbers $z$ and $w$ both lie on the unit circle, so $|z| = |w| = 1$, and their sum is $z + w = 1 + \frac{1}{2}i$. What is the product $zw$?

**(A)** $\frac{3}{5} + \frac{4}{5}i$ **(B)** $\frac{3}{5} - \frac{4}{5}i$ **(C)** $\frac{4}{5} + \frac{3}{5}i$ **(D)** $-\frac{3}{5} + \frac{4}{5}i$ **(E)** $\frac{3}{4} + i$

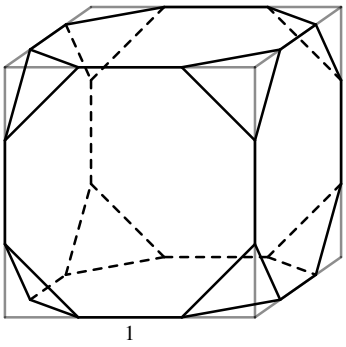
**8.** For how many integers $k$ does the equation $x^2 - kx + 2k = 0$ have two different real roots that both lie strictly between $-10$ and 10?

**(A)** 4 **(B)** 8 **(C)** 12 **(D)** 13 **(E)** 14

**9.** An integer $N$ is chosen uniformly at random from $\{1, 2, 3, \ldots, 1000\}$. What is the probability that the sum of the digits of $N$ and the sum of the digits of $N + 1$ are both odd or both even?

**(A)** $\frac{9}{100}$ **(B)** $\frac{91}{1000}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{111}{1000}$ **(E)** $\frac{1}{2}$

**10.** Each of the eight corners of a cube of edge length 1 is sliced off by a plane that cuts the three edges meeting at that corner, all at the same distance $t$ from the corner. The number $t$ is chosen so that each of the six faces of the cube becomes a regular octagon. What is the volume of the solid that remains?




**(A)** $\frac{5}{6}$ **(B)** $\frac{31-20\sqrt{2}}{3}$ **(C)** $\frac{14\sqrt{2}-17}{3}$ **(D)** $\frac{77}{81}$ **(E)** $\frac{7(\sqrt{2}-1)}{3}$

**11.** The numbers $1, 11, 111, 1111, \ldots$ are written using only the digit 1; the $n$th of them has $n$ digits. Among the first 2026 of these numbers (the last one has 2026 ones), how many are divisible by 123?

**(A)** 81 **(B)** 135 **(C)** 136 **(D)** 405 **(E)** 675

**12.** A sequence is defined by $a_0 = 2$, $a_1 = 2$, and $a_n = 2a_{n-1} - 4a_{n-2}$ for every $n \geq 2$. For how many integers $n$ with $1 \leq n \leq 100$ is $a_n$ positive?

**(A)** 17 **(B)** 33 **(C)** 34 **(D)** 49 **(E)** 50

**13.** Here $\gcd(a, b)$ denotes the greatest common divisor of $a$ and $b$. How many ordered pairs $(a, b)$ of positive integers satisfy $a + b + \gcd(a, b) = 60$?

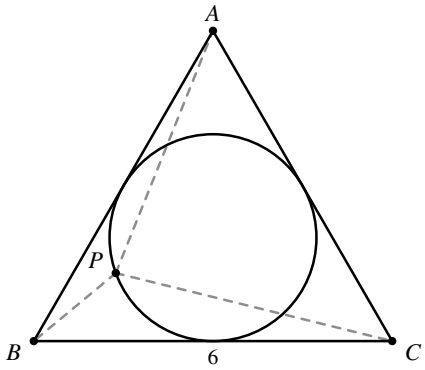
**(A)** 135 **(B)** 136 **(C)** 145 **(D)** 150 **(E)** 270

**14.** The complex number $z$ moves once around the circle $|z| = 2$, and for each position we plot the point $w = z + \dfrac{1}{z}$ in the complex plane. The points $w$ trace out an ellipse. What is the area of the region enclosed by this ellipse?

**(A)** $3\pi$ **(B)** $\frac{15\pi}{4}$ **(C)** $4\pi$ **(D)** $5\pi$ **(E)** $\frac{25\pi}{4}$

**15.** Equilateral triangle $ABC$ has side length 6. A point $P$ is chosen anywhere on the circle inscribed in the triangle. It turns out that $PA^2 + PB^2 + PC^2$ does not depend on where $P$ is on that circle. What is its value?

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**(A)** 36 **(B)** 42 **(C)** 45 **(D)** 54 **(E)** 72

**16.** A fair six-sided die is rolled repeatedly, stopping right after the first roll that shows a 6. What is the expected number of different faces that appear during the whole process, counting the final 6? (For example, the rolls 3, 1, 3, 6 show 3 different faces.)

**(A)** 3 **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 6

**17.** Positive real numbers $x$, $y$ and $z$ satisfy $x^{\log_2 y} = 8$, $y^{\log_2 z} = 16$, $z^{\log_2 x} = 4096$. What is the sum of all possible values of the product $xyz$?

**(A)** $\frac{65537}{256}$ **(B)** 257 **(C)** $\frac{65537}{128}$ **(D)** 1024 **(E)** 4096

**18.** A positive integer $n$ has exactly four positive divisors, and the sum of these four divisors is 2016. What is the smallest possible value of $n$?

**(A)** 1342 **(B)** 1509 **(C)** 1757 **(D)** 1837 **(E)** 1909

**19.** Square $ABCD$ has side length 1. A point $P$ moves around the circle inscribed in the square. What is the largest possible value of the product $PA \cdot PB \cdot PC \cdot PD$?

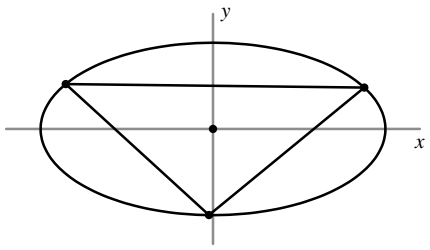
**(A)** $\frac{3}{16}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{9}{32}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{9}{16}$

**20.** In triangle $ABC$ the cotangents of the angles satisfy $\cot A + \cot C = 2\cot B$. What is the largest possible measure of angle $B$?

**(A)** 60° **(B)** 75° **(C)** 90° **(D)** 120° **(E)** 150°

**21.** A triangle has all three of its vertices on the ellipse $x^2 + 4y^2 = 36$, and the centroid of the triangle is the origin (the centre of the ellipse). What is the area of the triangle?

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**(A)** $\frac{27\sqrt{3}}{2}$ **(B)** $18\sqrt{3}$ **(C)** $\frac{81\sqrt{3}}{4}$ **(D)** $36$ **(E)** $27\sqrt{3}$

**22.** A fair six-sided die is rolled repeatedly, and a running total of the numbers rolled is kept. The rolling stops as soon as the running total is at least 7. What is the expected number of rolls?

**(A)** $\left(\frac{7}{6}\right)^6 - 1$ **(B)** $\left(\frac{7}{6}\right)^4$ **(C)** $2$ **(D)** $\left(\frac{7}{6}\right)^5$ **(E)** $\left(\frac{7}{6}\right)^6$

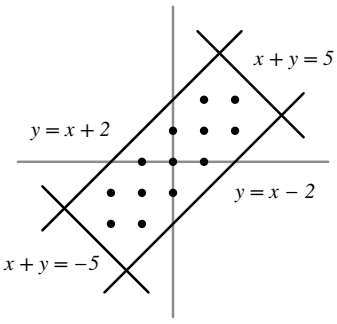
**23.** For every real number $x$, define $f(x) = \sum_{k=0}^{\infty} \frac{\sin^4\left(2^k x\right)}{4^k} = \sin^4 x + \frac{\sin^4 2x}{4} + \frac{\sin^4 4x}{16} + \cdots$. What is the value of $f\left(\frac{\pi}{7}\right) \cdot f\left(\frac{2\pi}{7}\right) \cdot f\left(\frac{4\pi}{7}\right)$?

**(A)** $\frac{49}{4096}$ **(B)** $\frac{1}{64}$ **(C)** $\frac{7}{128}$ **(D)** $\frac{7}{64}$ **(E)** $\frac{\sqrt{7}}{8}$

**24.** A regular tetrahedron has all six edges of length 12. A sphere is tangent to all six edges (each edge touches the sphere at exactly one point). Parts of this sphere lie inside the tetrahedron and parts lie outside it. What is the total surface area of the part of the sphere that lies outside the tetrahedron?

**(A)** $24\pi\left(2\sqrt{3} - 3\right)$ **(B)** $12\pi\left(3 - \sqrt{3}\right)$ **(C)** $48\pi\left(3 - \sqrt{6}\right)$ **(D)** $48\pi$ **(E)** $48\pi\left(3 - \sqrt{3}\right)$

**25.** A bug starts at the origin of the coordinate plane. Every second it moves one unit up, down, left or right, each with probability $\frac{1}{4}$, independently of its earlier moves. It stops as soon as it reaches a point on one of the four lines $x + y = 5$, $x + y = -5$, $y = x + 2$, $y = x - 2$. What is the probability that it stops on one of the lines $x + y = \pm 5$?




**(A)** $\frac{1}{58}$ **(B)** $\frac{1}{41}$ **(C)** $\frac{1}{32}$ **(D)** $\frac{1}{29}$ **(E)** $\frac{1}{5}$

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# Answer Key — AMC 12 Practice Contest 5

**1.** (C) $\log_2 18$

**2.** (C) $\frac{1}{6}$

**3.** (E) 178

**4.** (E) $24\sqrt{21}$

**5.** (C) 1248

**6.** (D) $2 - \frac{2\sqrt{3}}{3}$

**7.** (A) $\frac{3}{5} + \frac{4}{5}i$

**8.** (C) 12

**9.** (B) $\frac{91}{1000}$

**10.** (E) $\frac{7(\sqrt{2}-1)}{3}$

**11.** (B) 135

**12.** (D) 49

**13.** (A) 135

**14.** (B) $\frac{15\pi}{4}$

**15.** (C) 45

**16.** (B) $\frac{7}{2}$

**17.** (A) $\frac{65537}{256}$

**18.** (B) 1509

**19.** (D) $\frac{5}{16}$

**20.** (A) $60^\circ$

**21.** (A) $\frac{27\sqrt{3}}{2}$

**22.** (E) $\left(\frac{7}{6}\right)^6$

**23.** (D) $\frac{7}{64}$

**24.** (E) $48\pi\left(3 - \sqrt{3}\right)$

**25.** (D) $\frac{1}{29}$

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# Solutions — AMC 12 Practice Contest 5

**1. Answer: (C)** $\log_2 18$**. The idea.** Since $4^x = (2^x)^2$, the substitution $t = 2^x$ turns the equation into a quadratic in $t$. The two values of $t$ are ugly, but we never need them: adding exponents is the same as multiplying powers, and the product of the two $t$-values is given by Vieta's formulas.

**Work it out.** With $t = 2^x > 0$ the equation is $t^2 - 12t + 18 = 0$. Its discriminant is $144 - 72 = 72 > 0$, and the roots $t = 6 \pm 3\sqrt{2}$ are both positive (their sum 12 and product 18 are positive), so each gives exactly one real $x$. If $2^{x_1} = t_1$ and $2^{x_2} = t_2$, then $2^{x_1+x_2} = t_1 t_2 = 18$, so $x_1 + x_2 = \log_2 18$.

**Check.** Numerically $t_1 \approx 10.243$ and $t_2 \approx 1.757$, giving $x_1 \approx 3.356$ and $x_2 \approx 0.813$, whose sum is $4.170 \approx \log_2 18$. The choice $\log_2 12$ comes from using the sum of the roots instead of their product, and 12 and 6 confuse $t$ with $x$.

**2. Answer: (C)** $\frac{1}{6}$**. The idea.** Sort by the smallest value $m$. The largest is then $2m \le 6$, so $m \in \{1, 2, 3\}$, and for each $m$ we need the rolls whose values all lie between $m$ and $2m$ and which actually use both $m$ and $2m$. Counting rolls that use both ends of a range is a job for inclusion-exclusion.

**Work it out.** Fix $m$. The rolls with every value in $\{m, m+1, \ldots, 2m\}$ number $(m+1)^3$. Remove those that miss $m$ (values in a set of size $m$: $m^3$ rolls) and those that miss $2m$ (also $m^3$), then add back those that miss both ($(m-1)^3$). So the count is $(m+1)^3 - 2m^3 + (m-1)^3$.

$m = 1$: $8 - 2 + 0 = 6$. $m = 2$: $27 - 16 + 1 = 12$. $m = 3$: $64 - 54 + 8 = 18$.

In total $6 + 12 + 18 = 36$ of the 216 equally likely rolls work, so the probability is $\frac{36}{216} = \frac{1}{6}$.

**Check.** The formula simplifies to $(m+1)^3 + (m-1)^3 - 2m^3 = 6m$, matching 6, 12, 18. Directly for $m = 1$: the rolls use only 1s and 2s and both appear, $2^3 - 2 = 6$ of them. The choice $\frac{1}{12} = \frac{18}{216}$ counts only rolls with three different numbers, and $\frac{9}{56}$ counts the 9 unordered outcomes out of 56 as if they were equally likely, which they are not.

**3. Answer: (E)** 178**. The idea.** The area fixes the sine of the angle between the sides 5 and 8, and a sine value belongs to two angles, one acute and one obtuse, whose cosines are negatives of each other. In the law of cosines the cosine terms then cancel when the two squares are added.

**Work it out.** Let $\theta$ be the angle between the sides 5 and 8. The area is $\frac{1}{2} \cdot 5 \cdot 8 \sin\theta = 20 \sin\theta = 12$, so $\sin\theta = \frac{3}{5}$ and $\cos\theta = \pm\frac{4}{5}$. The third side $c$ satisfies $c^2 = 25 + 64 - 80\cos\theta = 89 \mp 64$, so $c^2 = 25$ or $c^2 = 153$. Both give genuine triangles (the angle $\theta$ is between $0^\circ$ and $180^\circ$), so the lengths are 5 and $\sqrt{153}$, and the sum of their squares is $25 + 153 = 178$.

**Check.** The sum is $2(25 + 64) = 178$ whatever the area is, since the $\cos\theta$ terms cancel. Directly: the 5-5-8 isosceles triangle has height $\sqrt{25 - 16} = 3$ over the base 8, area 12; for the other triangle put the side 8 on the $x$-axis from $(0, 0)$ to $(8, 0)$ and the third vertex at $(-4, 3)$, which is 5 from the origin: its distance to $(8, 0)$ is $\sqrt{144 + 9} = \sqrt{153}$.

**4. Answer: (E)** $24\sqrt{21}$**. The idea.** Everything is built from $u = 3^x$: the given quantity is $u^2 + \frac{1}{u^2}$ and the wanted one is $u^3 - \frac{1}{u^3}$. Both connect to the simpler number $d = u - \frac{1}{u}$: squaring $d$ brings in $u^2 + \frac{1}{u^2}$, and a difference of cubes factors through $d$. The condition $x > 0$ fixes the sign of $d$.

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**Work it out.** First, $d^2 = u^2 - 2 + \frac{1}{u^2} = 23 - 2 = 21$. Since $x > 0$, $u = 3^x > 1$, so $u > \frac{1}{u}$ and $d > 0$: $d = \sqrt{21}$. Then, by the difference of cubes, $u^3 - \frac{1}{u^3} = \left(u - \frac{1}{u}\right)\left(u^2 + 1 + \frac{1}{u^2}\right) = \sqrt{21} \cdot (23 + 1) = 24\sqrt{21}$.

**Check.** Another route: $d^3 = u^3 - 3u + \frac{3}{u} - \frac{1}{u^3}$, so $u^3 - \frac{1}{u^3} = d^3 + 3d = 21\sqrt{21} + 3\sqrt{21} = 24\sqrt{21}$. Numerically $u = \frac{5+\sqrt{21}}{2} \approx 4.7913$ (it satisfies $u^2 + u^{-2} = 23$), and $u^3 - u^{-3} \approx 109.9909 - 0.0091 = 109.98 \approx 24\sqrt{21}$. The choice $21\sqrt{21}$ is $d^3$ alone, $18\sqrt{21}$ is $d^3 - 3d$ (a sign error in the cube), $22\sqrt{21}$ uses $u^2 - 1 + u^{-2}$ (the sum-of-cubes factor), and $23\sqrt{21}$ forgets the +1 in the factor $u^2 + 1 + u^{-2}$.

**5. Answer: (C)** 1248**. The idea.** Write $720 = 2^4 \cdot 45$. The quotient $\frac{720}{d}$ is odd exactly when $d$ takes *all* four factors of 2, so the divisors in question are 16 times the divisors of the odd part 45.

**Work it out.** If $d = 2^j m$ with $m$ odd, then $\frac{720}{d} = 2^{4-j} \cdot \frac{45}{m}$, which is odd exactly when $j = 4$. So the divisors are $d = 16m$ with $m \mid 45$, that is $m \in \{1, 3, 5, 9, 15, 45\}$, whose sum is 78. The required sum is $16 \cdot 78 = 1248$.

**Check.** Listing them: $16+48+80+144+240+720 = 1248$, and the quotients are $45, 15, 9, 5, 3, 1$, all odd. The choice 78 adds the odd divisors (the case where $d$ is odd instead of the quotient), $2418 = \sigma(720)$ adds every divisor, 2340 adds the even divisors, and 624 uses $2^3$ instead of $2^4$.

**6. Answer: (D)** $2 - \frac{2\sqrt{3}}{3}$**. The idea.** The common region is the first square with four small corners removed. Each corner of the first square pokes out past one side of the second square, and the piece that pokes out is a right triangle (the corner of a square is a right angle) whose hypotenuse lies on a side of the second square and whose acute angles are 30° and 60°. The whole picture is symmetric under swapping the two squares, and that symmetry gives one equation for the triangle's size.

**Work it out.** Let each removed triangle have hypotenuse $c$, so its legs are $c\cos 30° = \frac{\sqrt{3}}{2}c$ and $c \sin 30° = \frac{1}{2}c$. Walk along one side of the second square. It is made of the hypotenuse of one triangle cut from the first square, and at its two ends, one leg each of two triangles cut from the second square; by the swap symmetry those are congruent to the first square's triangles, one contributing its long leg and one its short leg. So $1 = c + \frac{\sqrt{3}}{2}c + \frac{1}{2}c = \frac{3+\sqrt{3}}{2}c$, and $c = \frac{2}{3+\sqrt{3}} = \frac{3-\sqrt{3}}{3}$. Each triangle has area $\frac{1}{2} \cdot \frac{\sqrt{3}}{2}c \cdot \frac{1}{2}c = \frac{\sqrt{3}}{8}c^2$, and $c^2 = \frac{12-6\sqrt{3}}{9} = \frac{4-2\sqrt{3}}{3}$. The four triangles together have area $\frac{\sqrt{3}}{2}c^2 = \frac{4\sqrt{3}-6}{6} = \frac{2\sqrt{3}-3}{3}$. The overlap is $1 - \frac{2\sqrt{3}-3}{3} = 2 - \frac{2\sqrt{3}}{3} \approx 0.845$.

**Check.** Measure the triangle's height instead. A corner of the first square is $\frac{\sqrt{2}}{2}$ from the centre, in a direction $45° - 30° = 15°$ away from the perpendicular to the nearest side of the second square, which is $\frac{1}{2}$ from the centre. So the corner sticks out by $h = \frac{\sqrt{2}}{2}\cos 15° - \frac{1}{2} = \frac{\sqrt{3}+1}{4} - \frac{1}{2} = \frac{\sqrt{3}-1}{4}$. A 30-60-90 triangle with altitude $h$ to the hypotenuse has legs $2h$ and $\frac{2h}{\sqrt{3}}$, area $\frac{2h^2}{\sqrt{3}} = \frac{2\sqrt{3}-3}{12}$, and four of them give $\frac{2\sqrt{3}-3}{3}$ again. The choice $2\sqrt{2} - 2$ is the answer for a 45° turn.

**7. Answer: (A)** $\frac{3}{5} + \frac{4}{5}i$**. The idea.** On the unit circle the conjugate of a number is its reciprocal: $\overline{z} = \frac{1}{z}$. So conjugating the sum gives $\frac{1}{z} + \frac{1}{w} = \frac{z+w}{zw}$, an equation that involves the product directly.

**Work it out.** Let $s = z + w = 1 + \frac{1}{2}i$. Then $\overline{s} = \overline{z} + \overline{w} = \frac{1}{z} + \frac{1}{w} = \frac{s}{zw}$, so $zw = \frac{s}{\overline{s}} = \frac{s^2}{|s|^2}$. Here $s^2 = 1 + i - \frac{1}{4} = \frac{3}{4} + i$ and $|s|^2 = 1 + \frac{1}{4} = \frac{5}{4}$, so $zw = \frac{4}{5}\left(\frac{3}{4} + i\right) = \frac{3}{5} + \frac{4}{5}i$.

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**Check.** Such $z$ and $w$ exist because $|s| = \frac{\sqrt{5}}{2} < 2$. Solving $t^2 - st + \left(\frac{3}{5} + \frac{4}{5}i\right) = 0$ gives $z \approx 0.871 - 0.492i$ and $w \approx 0.129 + 0.992i$; both have modulus 1 (for instance $0.871^2 + 0.492^2 \approx 1.000$), and their sum is $1 + 0.5i$. Also $|zw| = 1$, as it must be, and the argument of $zw$ is twice the argument of $s$: the points $z$ and $w$ sit symmetrically about the direction of their midpoint $\frac{s}{2}$, so their arguments average to that of $s$. The choice $\frac{3}{4} + i$ forgets to divide by $|s|^2$, and $\frac{3}{5} - \frac{4}{5}i$ is $\frac{\overline{s}}{s}$, the reciprocal of the answer.

**8. Answer: (C)** 12**. The idea.** Do not solve for the roots. The graph of $f(x) = x^2 - kx + 2k$ is an upward parabola, and it crosses the $x$-axis twice inside the window $-10 < x < 10$ exactly when it dips below the axis (positive discriminant), its vertex lies inside the window, and it is above the axis at both ends of the window.

**Work it out.** The discriminant is $k^2 - 8k = k(k - 8)$, which is positive exactly when $k < 0$ or $k > 8$. At the ends, $f(-10) = 100 + 12k > 0$ needs $k > -\frac{25}{3}$, so $k \geq -8$, and $f(10) = 100 - 8k > 0$ needs $k < \frac{25}{2}$, so $k \leq 12$. The vertex is at $x = \frac{k}{2}$, which is inside the window for every $k$ between $-20$ and 20, so it adds nothing new. The allowed integers are $k = -8, -7, \ldots, -1$ (eight values) and $k = 9, 10, 11, 12$ (four values), for a total of 12.

**Check.** For $k = -8$ the roots are $-4 \pm 4\sqrt{2} \approx -9.66$ and 1.66, both inside; for $k = -9$ one root is $\frac{-9-\sqrt{153}}{2} \approx -10.7$, outside. For $k = 12$ the roots $6 \pm 2\sqrt{3} \approx 2.54, 9.46$ are inside, while $k = 13$ gives a root near 10.53. The values $k = 0$ and $k = 8$ give a double root, which is why 13 and 14 are wrong, and 4 forgets the negative values of $k$.

**9. Answer: (B)** $\frac{91}{1000}$**. The idea.** Adding 1 usually raises the digit sum by exactly 1, which flips its parity. The exception is a carry: if $N$ ends in exactly $k$ nines, those $k$ nines become zeros and the next digit goes up by 1, so the digit sum changes by $1 - 9k$. The parities agree exactly when $1 - 9k$ is even, that is, when $k$ is odd.

**Work it out.** For $N \leq 1000$, the number $k$ of trailing nines is at most 3, so we need $k = 1$ or $k = 3$.

$k = 1$: the last digit is 9 and the tens digit is not 9. Among $1, \ldots, 1000$ there are 100 numbers ending in 9, of which 10 end in 99, leaving 90.

$k = 3$: only $N = 999$.

So $90 + 1 = 91$ of the 1000 choices work, and the probability is $\frac{91}{1000}$.

**Check.** Examples: $19 \to 20$ has digit sums 10 and 2 (both even), $99 \to 100$ has 18 and 1 (different, $k = 2$), and $999 \to 1000$ has 27 and 1 (both odd, $k = 3$). The choice $\frac{9}{100}$ forgets 999, $\frac{1}{10}$ counts every number ending in 9, $\frac{111}{1000}$ counts every number with at least one trailing nine as $100 + 10 + 1$, and $\frac{1}{2}$ assumes the parities are independent.

**10. Answer: (E)** $\frac{7(\sqrt{2}-1)}{3}$**. The idea.** Each face of the cube loses its four corners and becomes an octagon whose sides alternate between what is left of the cube's edges and the cuts. Making those two kinds of side equal fixes $t$; the angles of such an octagon are automatically 135°, so equal sides make it regular. Each piece removed is a small right-corner tetrahedron.

**Work it out.** On a face, a leftover edge has length $1 - 2t$, and a cut side joins two points at distance $t$ from a corner along perpendicular edges, so it has length $t\sqrt{2}$. Setting $1 - 2t = t\sqrt{2}$ gives $t = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2} \approx 0.293$; since $t < \frac{1}{2}$ the eight cuts do not overlap. Each removed

piece is a tetrahedron with three mutually perpendicular edges of length $t$, so its volume is $\frac{t^3}{6}$. Now $t = 1 - \frac{\sqrt{2}}{2}$, so $t^3 = 1 - \frac{3\sqrt{2}}{2} + \frac{3}{2} - \frac{\sqrt{2}}{4} = \frac{5}{2} - \frac{7\sqrt{2}}{4}$. The remaining volume is $1 - 8 \cdot \frac{t^3}{6} = 1 - \frac{4}{3}\left(\frac{5}{2} - \frac{7\sqrt{2}}{4}\right) = 1 - \frac{10}{3} + \frac{7\sqrt{2}}{3} = \frac{7(\sqrt{2}-1)}{3} \approx 0.9665$.

**Check.** Numerically $t^3 \approx 0.02513$, the eight corners total $\frac{4}{3}t^3 \approx 0.0335$, and $1 - 0.0335 = 0.9665$. The choice $\frac{5}{6}$ cuts at the midpoints (giving triangles and squares, not octagons), $\frac{77}{81}$ cuts at $t = \frac{1}{3}$, $\frac{31-20\sqrt{2}}{3}$ uses $t = \sqrt{2} - 1$ from solving $1 - t = t\sqrt{2}$, and $\frac{14\sqrt{2}-17}{3}$ takes each corner's volume to be $\frac{t^3}{3}$.

**11. Answer: (B)** 135**. The idea.** Factor $123 = 3 \cdot 41$ and handle each prime separately. Divisibility by 3 is a digit-sum test. For 41, the number with $n$ ones is $R_n = \frac{10^n - 1}{9}$, so it is a multiple of 41 exactly when $10^n \equiv 1 \pmod{41}$, which depends on the order of 10 modulo 41.

**Work it out.** The digit sum of $R_n$ is $n$, so $3 \mid R_n$ exactly when $3 \mid n$. Next, powers of 10 modulo 41: $10^2 = 100 \equiv 18$, $10^3 \equiv 180 \equiv 16$, $10^4 \equiv 160 \equiv 37$, $10^5 \equiv 370 \equiv 1$. So 10 has order 5 modulo 41, and since $\gcd(9, 41) = 1$, $41 \mid R_n$ exactly when $5 \mid n$. (Indeed $11111 = 41 \cdot 271$.) Both conditions hold exactly when $15 \mid n$. Since $2025 = 15 \cdot 135$, there are 135 such $n$ from 1 to 2026.

**Check.** The first one is $R_{15} = 111111111111111 = 123 \cdot 903342366757$, and $R_5 = 11111$ is divisible by 41 but not by 3, while $R_3 = 111 = 3 \cdot 37$ is divisible by 3 but not by 41. The choice $405 = \lfloor 2026/5 \rfloor$ ignores the factor 3, $675 = \lfloor 2026/3 \rfloor$ ignores 41, and 81 comes from requiring $25 \mid n$.

**12. Answer: (D)** 49**. The idea.** A linear recurrence is solved by powers of the roots of its characteristic equation. Here those roots are complex conjugates, $1 \pm i\sqrt{3} = 2e^{\pm i\pi/3}$, so $a_n$ is a sum of conjugate powers, a real number $2^{n+1}\cos\frac{n\pi}{3}$ whose sign repeats with period 6.

**Work it out.** The characteristic equation $r^2 - 2r + 4 = 0$ has roots $r = 1 \pm i\sqrt{3}$. The sequence $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n$ satisfies the recurrence and has the right starting values 2 and $(1+i\sqrt{3}) + (1-i\sqrt{3}) = 2$, so it is $a_n$. Since $1 \pm i\sqrt{3} = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$, De Moivre gives $a_n = 2^{n+1}\cos\frac{n\pi}{3}$. As $n$ runs through the residues $0, 1, 2, 3, 4, 5$ modulo 6, $\cos\frac{n\pi}{3}$ is $1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}$; it is never 0. So $a_n > 0$ exactly when $n \equiv 0, 1$ or $5 \pmod 6$. From 1 to 100: $n \equiv 1$ gives $1, 7, \ldots, 97$ (17 values), $n \equiv 5$ gives $5, 11, \ldots, 95$ (16), and $n \equiv 0$ gives $6, 12, \ldots, 96$ (16). The total is 49.

**Check.** Direct computation: $a_2 = 4 - 8 = -4$, $a_3 = -8 - 8 = -16$, $a_4 = -32 + 16 = -16$, $a_5 = -32 + 64 = 32$, $a_6 = 64 + 64 = 128$, $a_7 = 256 - 128 = 128$, matching $2^{n+1}\cos\frac{n\pi}{3}$ and the sign pattern $+, -, -, -, +, +, +$ for $n = 1, \ldots, 7$. The choice 33 forgets the multiples of 6, and 50 assumes half the terms are positive.

**13. Answer: (A)** 135**. The idea.** Pull out the greatest common divisor: write $g = \gcd(a, b)$, $a = gx$ and $b = gy$ with $\gcd(x, y) = 1$. The equation becomes $g(x + y + 1) = 60$, so $g$ is a divisor of 60 and $x + y$ is fixed by $g$. What remains is to count coprime ordered pairs with a given sum $m$, and since $\gcd(x, m - x) = \gcd(x, m)$, those are counted by Euler's $\varphi(m)$.

**Work it out.** For each divisor $g$ of 60, put $m = x + y = \frac{60}{g} - 1$. We need $m \geq 2$, so $g \leq 20$ ($g = 30$ gives $m = 1$ and $g = 60$ gives $m = 0$, both impossible). For a fixed $m \geq 2$, the pair is determined by $x \in \{1, \ldots, m - 1\}$, and $\gcd(x, y) = \gcd(x, m - x) = \gcd(x, m)$, so the good choices of $x$ are those coprime to $m$: there are $\varphi(m)$ of them (for $m \geq 2$ the value $x = m$ is never coprime

to $m$, so nothing is lost by stopping at $m-1$). The divisors give $g = 1, 2, 3, 4, 5, 6, 10, 12, 15, 20$ and $m = 59, 29, 19, 14, 11, 9, 5, 4, 3, 2$, with $\varphi(m) = 58, 28, 18, 6, 10, 6, 4, 2, 2, 1$. The total is $58+28+18+6+10+6+4+2+2+1 = 135$.

**Check.** Spot checks: $g = 20$ gives $a = b = 20$, and $20+20+20 = 60$; $g = 4$, $x = 3$ gives $(a, b) = (12, 44)$ with gcd = 4 and $12+44+4 = 60$; but $g = 4$, $x = 2$ would give $(8, 48)$, whose gcd is 8, not 4, which is why the coprimality matters. Ignoring it (counting all $m-1$ splits) gives 145 instead. A direct computer count of all pairs with $a+b < 60$ also gives 135.

**14. Answer: (B) $\frac{15\pi}{4}$. The idea.** On the circle, $1/z$ is a scaled copy of the conjugate of $z$, so $z + 1/z$ adds a big and a small circle turning in opposite directions. Their real parts add and their imaginary parts partly cancel, which squashes the circle into an ellipse whose semi-axes are easy to read.

**Work it out.** Write $z = 2(\cos\theta + i\sin\theta)$. Then $\frac{1}{z} = \frac{1}{2}(\cos\theta - i\sin\theta)$, so $w = \left(2+\frac{1}{2}\right)\cos\theta + i\left(2-\frac{1}{2}\right)\sin\theta = \frac{5}{2}\cos\theta + i\cdot\frac{3}{2}\sin\theta$. With $w = X + iY$ this is $\frac{X^2}{(5/2)^2} + \frac{Y^2}{(3/2)^2} = 1$, an ellipse with semi-axes $\frac{5}{2}$ and $\frac{3}{2}$, traced once as $\theta$ goes around. Its area is $\pi\cdot\frac{5}{2}\cdot\frac{3}{2} = \frac{15\pi}{4}$.

**Check.** The four points $z = 2, 2i, -2, -2i$ give $w = \frac{5}{2}, \frac{3}{2}i, -\frac{5}{2}, -\frac{3}{2}i$: exactly the ends of the two axes. As a sanity check, the ellipse sits between the circles of radius $\frac{3}{2}$ and $\frac{5}{2}$, and $\frac{9\pi}{4} < \frac{15\pi}{4} < \frac{25\pi}{4}$.

**15. Answer: (C) 45. The idea.** Put the origin at the centre $O$ of the triangle (its centroid, which is also the incentre and circumcentre) and use vectors. Because the three vertex vectors add to zero, the cross terms vanish and the sum depends only on $|OP|$, which is fixed on the incircle.

**Work it out.** Let $\vec{p}$ be the vector to $P$ and $\vec{a}, \vec{b}, \vec{c}$ the vectors to the vertices, so $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}| = R$. Then $PA^2+PB^2+PC^2 = \sum|\vec{p}-\vec{a}|^2 = 3|\vec{p}|^2 - 2\,\vec{p}\cdot(\vec{a}+\vec{b}+\vec{c}) + 3R^2 = 3|\vec{p}|^2+3R^2$. For side 6, the circumradius is $R = \frac{6}{\sqrt{3}} = 2\sqrt{3}$ and the inradius is $r = \frac{R}{2} = \sqrt{3}$. On the incircle $|\vec{p}| = r$, so the sum is $3\cdot 3 + 3\cdot 12 = 45$.

**Check.** Take $P$ to be the point where the incircle touches $BC$, the midpoint of $BC$. Then $PB = PC = 3$ and $PA$ is the altitude $3\sqrt{3}$, so the sum is $9+9+27 = 45$.

**16. Answer: (B) $\frac{7}{2}$. The idea.** Count by linearity of expectation: the answer is the sum over the faces of the probability that each face appears. The 6 always appears. For a face $k \neq 6$, only the relative order of the first $k$ and the first 6 matters.

**Work it out.** Fix $k \in \{1, 2, 3, 4, 5\}$. Ignore every roll that is neither $k$ nor 6; the first roll that is one of the two is equally likely to be $k$ or 6. Face $k$ appears exactly when it comes first, so it appears with probability $\frac{1}{2}$. By linearity, the expected number of different faces is $1 + 5\cdot\frac{1}{2} = \frac{7}{2}$.

**Check.** Track how many new faces have shown. With $j$ non-six faces seen, the next roll that is not a repeat is equally likely to be the 6 or any of the $5-j$ unseen faces. Let $E_j$ be the expected final count; then $E_5 = 6$ and $E_j = \frac{1}{6-j}(j+1) + \frac{5-j}{6-j}E_{j+1}$, giving $E_4 = \frac{11}{2}$, $E_3 = 5$, $E_2 = \frac{9}{2}$, $E_1 = 4$ and $E_0 = \frac{1}{6} + \frac{5}{6}\cdot 4 = \frac{7}{2}$.

**17. Answer: (A) $\frac{65537}{256}$. The idea.** Take $\log_2$ of each equation. Since $\log_2\left(x^{\log_2 y}\right) = \log_2 x\cdot\log_2 y$, the three equations become statements about products of the three unknowns $u = \log_2 x$, $v = \log_2 y$, $w = \log_2 z$. Multiplying all three pins down $uvw$ up to a sign, and that sign is the whole point.

**Work it out.** The equations become $uv = 3$, $vw = 4$ and $wu = 12$. Multiplying, $(uvw)^2 = 144$, so $uvw = 12$ or $uvw = -12$. Dividing by each equation: if $uvw = 12$ then $u = \frac{12}{vw} = 3$, $v = \frac{12}{wu} = 1$, $w = \frac{12}{uv} = 4$; if $uvw = -12$ then $(u, v, w) = (-3, -1, -4)$. Both are valid, because $x = 2^u$, $y = 2^v$, $z = 2^w$ are positive for any real exponents. Since $\log_2(xyz) = u + v + w = \pm 8$, the product $xyz$ is $2^8 = 256$ or $2^{-8} = \frac{1}{256}$, and the sum of the possible values is $256 + \frac{1}{256} = \frac{65537}{256}$.

**Check.** For $(x, y, z) = \left(\frac{1}{8}, \frac{1}{2}, \frac{1}{16}\right)$: $\left(\frac{1}{8}\right)^{\log_2(1/2)} = \left(\frac{1}{8}\right)^{-1} = 8$, $\left(\frac{1}{2}\right)^{\log_2(1/16)} = \left(\frac{1}{2}\right)^{-4} = 16$ and $\left(\frac{1}{16}\right)^{\log_2(1/8)} = \left(\frac{1}{16}\right)^{-3} = 4096$. So the negative solution is genuine and must be counted. The choice $4096 = 2^{12}$ comes from taking $xyz = 2^{uvw}$ instead of $2^{u+v+w}$.

**18. Answer: (B)** 1509**. The idea.** A number with exactly four divisors is either $p^3$ or $pq$ for distinct primes $p, q$. In the second case the divisor sum factors as $(p + 1)(q + 1)$, so we are looking for factorisations of 2016 into two factors that are each one more than a prime.

**Work it out.** If $n = p^3$, the sum $1 + p + p^2 + p^3 = (1 + p)(1 + p^2)$ is 1464 for $p = 11$ and 2380 for $p = 13$, so it never equals 2016. If $n = pq$, then $(p + 1)(q + 1) = 2016 = 2^5 \cdot 3^2 \cdot 7$. Take $p < q$. If $p = 2$ then $q + 1 = 672$, $q = 671 = 11 \cdot 61$, not prime. Otherwise both $p$ and $q$ are odd, so write $p + 1 = 2u$ and $q + 1 = 2v$ with $uv = 504$ and $2 \le u < v$ (so $u \le 22$). The options are $u = 2, 3, 4, 6, 7, 8, 9, 12, 14, 18, 21$, giving $(p, q) = (3, 503), (5, 335), (7, 251), (11, 167), (13, 143), (15, 125), (17, 111$ Discarding pairs with a composite entry ($335 = 5{\cdot}67$, $143 = 11{\cdot}13$, $15$, $111 = 3{\cdot}37$, $27$, $35$) leaves $(3, 503), (7, 251), (11, 167), (23, 83), (41, 47)$, with products $1509, 1757, 1837, 1909, 1927$. Because $pq = (p + 1)(q + 1) - (p + q) - 1 = 2015 - (p + q)$, the product is smallest when $p + q$ is largest, which is the pair $(3, 503)$: $n = 1509$.

**Check.** $1509 = 3{\cdot}503$ has divisors $1, 3, 503, 1509$, summing to 2016. The tempting $2{\cdot}671 = 1342$ fails because $671 = 11 \cdot 61$, so 1342 has eight divisors.

**19. Answer: (D)** $\frac{5}{16}$**. The idea.** Place the square in the complex plane with its centre at 0. Its vertices are then the four roots of a single equation $z^4 = c$, so the product of the distances from $P$ to them is $|P^4 - c|$, which is easy to maximise when $|P|$ is fixed.

**Work it out.** Put the vertices at $\pm\frac{1}{2} \pm \frac{1}{2}i$. Each has modulus $\frac{\sqrt{2}}{2}$ and argument an odd multiple of $45°$, so each satisfies $z^4 = \left(\frac{\sqrt{2}}{2}\right)^4 (\cos 180° + i \sin 180°) = -\frac{1}{4}$. Hence $z^4 + \frac{1}{4} = (z - A)(z - B)(z - C)(z - D)$, and for any point $p$, $PA \cdot PB \cdot PC \cdot PD = \left|p^4 + \frac{1}{4}\right|$. On the inscribed circle $|p| = \frac{1}{2}$, so $p^4$ runs over the circle of radius $\frac{1}{16}$. By the triangle inequality $\left|p^4 + \frac{1}{4}\right| \le \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$, with equality when $p^4 = \frac{1}{16}$, for example $p = \frac{1}{2}$.

**Check.** At $p = \frac{1}{2}$ (the midpoint of a side) the two nearest vertices are at distance $\frac{1}{2}$ and the two far ones at distance $\sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$, so the product is $\frac{1}{4} \cdot \frac{5}{4} = \frac{5}{16}$. The minimum, $\frac{1}{4} - \frac{1}{16} = \frac{3}{16}$, happens when $P$ is closest to a vertex.

**20. Answer: (A)** 60°**. The idea.** Cotangents of a triangle's angles are rational in the sides: $\cot A = \frac{\cos A}{\sin A}$, with $\cos A$ from the law of cosines and $\sin A = \frac{2K}{bc}$ from the area $K$. This turns the angle condition into a relation among the squares of the sides, and then $\cos B$ can be bounded with AM-GM.

**Work it out.** With $a, b, c$ opposite $A, B, C$ and area $K$, $\cot A = \frac{(b^2 + c^2 - a^2)/(2bc)}{2K/(bc)} = \frac{b^2 + c^2 - a^2}{4K}$, and similarly for $B$ and $C$. The condition becomes $(b^2 + c^2 - a^2) + (a^2 + b^2 - c^2) = 2(a^2 + c^2 - b^2)$,

that is $2b^2 = 2(a^2 + c^2) - 2b^2$, so $a^2 + c^2 = 2b^2$. Then $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{a^2+c^2}{4ac} \geq \frac{2ac}{4ac} = \frac{1}{2}$, so $B \leq 60°$, with equality exactly when $a = c$, and then $b = a$: the equilateral triangle, for which $\cot A + \cot C = 2\cot B$ certainly holds.

**Check.** Take $a = 5$, $c = 7$; then $b^2 = \frac{25+49}{2} = 37$, and $5, \sqrt{37}, 7$ is a valid triangle. Here $\cos B = \frac{25+49-37}{70} = \frac{37}{70} > \frac{1}{2}$, so $B < 60°$, as the bound says; only the equilateral triangle reaches $60°$.

**21. Answer: (A) $\frac{27\sqrt{3}}{2}$. The idea.** Stretch the picture so the ellipse becomes a circle. The map $(x, y) \mapsto (x, 2y)$ sends lines to lines and midpoints to midpoints, so it sends centroids to centroids, and it multiplies every area by 2. On a circle, a triangle whose centroid is the centre must be equilateral, so the triangle is forced up to rotation and its area can be read off.

**Work it out.** Put $X = x$, $Y = 2y$. The ellipse $x^2 + 4y^2 = 36$ becomes the circle $X^2 + Y^2 = 36$ of radius 6 centred at $O$, and our triangle becomes a triangle $A'B'C'$ inscribed in this circle with centroid $O$, whose area is twice the original area. Now $O$ is both the circumcentre and the centroid of $A'B'C'$. Let $M$ be the midpoint of $B'C'$. The centroid lies on the median $A'M$, so $O$ is on line $A'M$; also $O \neq M$ (the centroid is strictly between $A'$ and $M$), and $OM \perp B'C'$ because $O$ is the circumcentre. So the median from $A'$ is perpendicular to $B'C'$, which means $A'B' = A'C'$. The same holds at every vertex, so $A'B'C'$ is equilateral. An equilateral triangle inscribed in a circle of radius 6 has side $6\sqrt{3}$ and area $\frac{\sqrt{3}}{4} \cdot 108 = 27\sqrt{3}$. Undoing the stretch halves the area: the answer is $\frac{27\sqrt{3}}{2}$, whichever such triangle we take.

**Check.** Take the vertex $(6, 0)$. The other two vertices must add to $(-6, 0)$; the pair $\left(-3, \pm\frac{3\sqrt{3}}{2}\right)$ works, since $9 + 4 \cdot \frac{27}{4} = 36$. The triangle has base $3\sqrt{3}$ on the line $x = -3$ and height 9, so its area is $\frac{1}{2} \cdot 3\sqrt{3} \cdot 9 = \frac{27\sqrt{3}}{2}$. Forgetting to undo the stretch gives $27\sqrt{3}$; a numerical search over many starting vertices finds the same area every time.

**22. Answer: (E) $\left(\frac{7}{6}\right)^6$. The idea.** Let $E_t$ be the expected number of rolls still needed when the running total is $t$. Each $E_t$ is 1 plus the average of the next six values, and subtracting two consecutive equations makes almost everything cancel, because every total of 7 or more contributes 0.

**Work it out.** $E_t = 0$ for $t \geq 7$, and for $0 \leq t \leq 6$, $E_t = 1 + \frac{1}{6}(E_{t+1} + E_{t+2} + \cdots + E_{t+6})$. For $0 \leq t \leq 5$, subtract the equation for $t + 1$: $E_t - E_{t+1} = \frac{1}{6}(E_{t+1} - E_{t+7}) = \frac{1}{6}E_{t+1}$, since $t + 7 \geq 7$. So $E_t = \frac{7}{6}E_{t+1}$. Also $E_6 = 1$ (any roll finishes). Therefore $E_0 = \left(\frac{7}{6}\right)^6 = \frac{117649}{46656} \approx 2.52$.

**Check.** Directly, $E = \sum_{n\geq 0} P(\text{the first } n \text{ rolls total at most } 6)$. The number of ways $n$ rolls total at most 6 is $\binom{6}{n}$ (stars and bars: $x_1 + \cdots + x_n + s = 6$ with $x_i \geq 1$, $s \geq 0$), so $E = \sum_{n=0}^{6} \binom{6}{n}\frac{1}{6^n} = \left(1 + \frac{1}{6}\right)^6$, the same value. Starting that sum at $n = 1$ instead of $n = 0$ drops the always-counted first roll and gives $\left(\frac{7}{6}\right)^6 - 1$.

**23. Answer: (D) $\frac{7}{64}$. The idea.** First find $f$ in closed form. The double-angle formula gives $\sin^2 2y = 4\sin^2 y \cos^2 y = 4\sin^2 y - 4\sin^4 y$, so $\sin^4 y = \sin^2 y - \frac{1}{4}\sin^2 2y$. With the weights $\frac{1}{4^k}$ this makes the series telescope, and $f(x) = \sin^2 x$. The problem then becomes a classical product of sines.

**Work it out.** Let $a_k = \frac{\sin^2(2^k x)}{4^k}$. Applying the identity with $y = 2^k x$ and dividing by $4^k$ gives

$\frac{\sin^4(2^k x)}{4^k} = a_k - a_{k+1}$. The partial sums are $a_0 - a_{N+1}$, and $0 \le a_{N+1} \le 4^{-N-1} \to 0$, so $f(x) = a_0 = \sin^2 x$ for every $x$.

So we need $\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2$. Since $\sin\frac{4\pi}{7} = \sin\frac{3\pi}{7}$ and $\sin\frac{k\pi}{7} = \sin\frac{(7-k)\pi}{7}$, this square equals $\prod_{k=1}^{6}\sin\frac{k\pi}{7}$. To evaluate it, let $\omega = e^{2\pi i/7}$; then $|1-\omega^k| = 2\sin\frac{k\pi}{7}$, and $\prod_{k=1}^{6}(1-\omega^k) = 7$ (it is the value at $x = 1$ of $\frac{x^7-1}{x-1} = 1 + x + \cdots + x^6$). So $2^6\prod_{k=1}^{6}\sin\frac{k\pi}{7} = 7$, and the answer is $\frac{7}{64}$.

**Check.** Numerically $\sin^2\frac{\pi}{7} \approx 0.1883$, $\sin^2\frac{2\pi}{7} \approx 0.6113$, $\sin^2\frac{4\pi}{7} \approx 0.9505$, with product $\approx 0.1094 = \frac{7}{64}$. Note that $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{\sqrt{7}}{8}$ itself is not the answer: each $f$ value is a square.

**24. Answer: (E)** $48\pi\left(3-\sqrt{3}\right)$**. The idea.** By symmetry the sphere is centred at the centre $O$ of the tetrahedron. Its radius $\rho$ is the distance from $O$ to an edge, which is larger than the inradius $r$ (the distance from $O$ to a face). So the sphere pokes through each face in a spherical cap, and the part outside is four such caps. A cap of height $h$ on a sphere of radius $\rho$ has area $2\pi\rho h$ (Archimedes).

**Work it out.** Place the tetrahedron at alternate corners of a cube of side $s$; its edges are face diagonals, so $s\sqrt{2} = 12$ and $s = 6\sqrt{2}$. The centre of the cube is $O$, and the midpoint of each edge is the centre of a cube face, at distance $\frac{s}{2} = 3\sqrt{2}$ from $O$. So $\rho = 3\sqrt{2}$ (the sphere touches each edge at its midpoint). The inradius of a regular tetrahedron is a quarter of its height $12\sqrt{\frac{2}{3}} = 4\sqrt{6}$, so $r = \sqrt{6}$. Each face plane is at distance $\sqrt{6} < 3\sqrt{2}$ from $O$, and cuts the sphere in a circle of radius $\sqrt{18-6} = 2\sqrt{3}$, exactly the incircle of the face (inradius $\frac{12}{2\sqrt{3}} = 2\sqrt{3}$). So the circle lies inside the face, the four caps beyond the four faces are the whole outside part, and they meet only at the six tangency points. Each cap has height $h = 3\sqrt{2} - \sqrt{6}$ and area $2\pi\cdot 3\sqrt{2}\,(3\sqrt{2} - \sqrt{6}) = 2\pi(18 - 6\sqrt{3}) = 12\pi(3-\sqrt{3})$. Four caps give $48\pi(3-\sqrt{3})$.

**Check.** The whole sphere has area $4\pi\cdot 18 = 72\pi \approx 226.2$, and $48\pi(3-\sqrt{3}) \approx 191.2$, so most of the sphere is outside: plausible, since the sphere reaches out to the edges and the tetrahedron is thin near them. The inside part is $72\pi - 48\pi(3-\sqrt{3}) = 24\pi(2\sqrt{3}-3) \approx 35.0$, positive as it must be. (Replacing each cap by the flat incircle disc it covers would give $4\cdot\pi(2\sqrt{3})^2 = 48\pi$, too small: a cap is larger than its base disc.)

**25. Answer: (D)** $\frac{1}{29}$**. The idea.** Use the diagonal coordinates $u = x + y$ and $v = x - y$. A move right, left, up or down changes $(u, v)$ by $(1,1)$, $(-1,-1)$, $(1,-1)$ or $(-1,1)$: every one of the four sign patterns with probability $\frac{1}{4}$. So $u$ and $v$ are two *independent* fair $\pm1$ walks, and the question is a race: does $u$ reach $\pm5$ before $v$ reaches $\pm2$? The $v$-walk is simple enough that its survival probability is a clean power of 2, and weighting the $u$-walk by that power turns the race into one small linear system.

**Work it out.** After $n$ moves both $u$ and $v$ have the parity of $n$, so $|u| = 5$ can only happen at odd times and $|v| = 2$ only at even times: there are no ties. Every two moves, $v$ goes from 0 to $\pm1$ and then to $\pm2$ or back to 0, each with probability $\frac{1}{2}$. So $v$ has not yet stopped the bug after $2k$ moves with probability $2^{-k}$, and the same holds after $2k+1$ moves. If $T$ is the first time $|u| = 5$ (an odd number), the bug stops on $x + y = \pm5$ exactly when $v$ survives past $T$, which by independence has probability $2^{-(T-1)/2}$. Hence the answer is $\sqrt{2}\,E\left[s^T\right]$ with $s = \frac{1}{\sqrt{2}}$.

Let $w_j$ be $E\left[s^T\right]$ for the $u$-walk started at $j$, so $w_{\pm5} = 1$ and $w_j = \frac{s}{2}(w_{j-1} + w_{j+1})$, that is, $w_{j-1} + w_{j+1} = 2\sqrt{2}\,w_j$ for $|j| \le 4$. By symmetry $w_{-1} = w_1$, and stepping outward from $w_0$: $w_1 = \sqrt{2}\,w_0$, $\quad w_2 = 2\sqrt{2}\,w_1 - w_0 = 3w_0$, $\quad w_3 = 2\sqrt{2}\,w_2 - w_1 = 5\sqrt{2}\,w_0$, $w_4 = 2\sqrt{2}\,w_3 - w_2 =$

$17w_0$, $w_5 = 2\sqrt{2}\,w_4 - w_3 = 29\sqrt{2}\,w_0$. Since $w_5 = 1$, $w_0 = \frac{1}{29\sqrt{2}}$, and the probability is $\sqrt{2}\,w_0 = \frac{1}{29}$.

**Check.** The same recursion with the lines $x + y = \pm 1$ gives $\sqrt{2}w_0 = 1$ (the first move always lands on one of them, correct), and with $x + y = \pm 3$ gives $\frac{1}{5}$. The smallest term alone is already of the right size: $T = 5$ has probability $\frac{2}{32}$ and $v$ survives 5 moves with probability $\frac{1}{4}$, giving $\frac{1}{64}$ of the total $\frac{1}{29} \approx 0.0345$. Solving the 13-state absorbing chain on the lattice points between the four lines directly, with exact fractions, gives $\frac{1}{29}$ as well. Slips: using $2^{-T/2}$ instead of $2^{-(T-1)/2}$ for the survival of $v$ loses a factor $\sqrt{2}$ and gives $\frac{1}{29\sqrt{2}} \approx \frac{1}{41}$, and stopping the recursion at $x + y = \pm 3$ gives $\frac{1}{5}$.

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