MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

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1. Let $i = \sqrt{-1}$. What is the value of the sum $1 \cdot i + 2 \cdot i^2 + 3 \cdot i^3 + 4 \cdot i^4 + \cdots + 2026 \cdot i^{2026}$, in which the kth term is $k \cdot i^k$?

(A) $-2026 + 2025i$ **(B)** $-1014 + 1013i$ **(C)** $-1013 + 1013i$ **(D)** $1012 - 1012i$ **(E)** $1013 - 1014i$

2. For each integer $b \geq 2$, the numeral 1001 read in base b stands for the number $b^3 + 1$. For example, 1001 in base 2 is 9. For how many integers b with $2 \leq b \leq 100$ is the number written 1001 in base b divisible by 7?

(A) 14 **(B)** 28 **(C)** 42 **(D)** 49 **(E)** 57

3. In triangle ABC, $\cos A = \frac{4}{5}$ and $\cos B = \frac{7}{25}$. The area of triangle ABC is 468. What is the perimeter of triangle ABC?

(A) 104 **(B)** 112 **(C)** 117 **(D)** 120 **(E)** 156

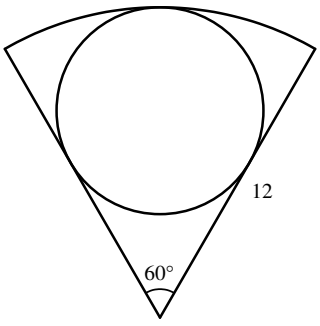
4. Ten lamps stand in a row, and each lamp is switched either on or off, giving 2^{10} possible settings. Two lamps next to each other that are both on form a bright pair. For example, if lamps 3, 4 and 5 are on and lamps 2 and 6 are off, that makes two bright pairs: lamps 3 and 4, and lamps 4 and 5. How many of the 2^{10} settings have exactly one bright pair?

(A) 88 **(B)** 144 **(C)** 235 **(D)** 256 **(E)** 306

5. For how many ordered pairs (m, n) of integers with $1 \leq m \leq 24$ and $1 \leq n \leq 24$ is the complex number $(1 + i)^m \, (\sqrt{3} + i)^n$ a positive real number?

(A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 72

6. A sector of a circle has radius 12 and central angle $60°$. A smaller circle lies inside the sector and is tangent to both straight edges of the sector and to its arc. What is the area of the smaller circle?



(A) 12π **(B)** 16π **(C)** 24π **(D)** 36π **(E)** 64π

7. A complex number z is exactly twice as far from 3 as it is from $3i$, that is, $|z - 3| = 2\,|z - 3i|$. What is the largest possible value of $|z|$?

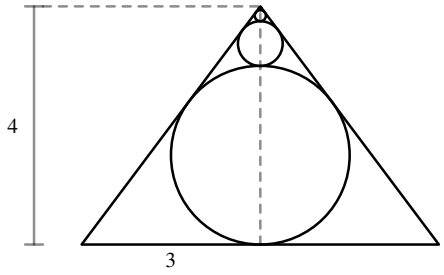
(A) $\sqrt{17}$ **(B)** $\sqrt{17} + \sqrt{2}$ **(C)** $4 + 2\sqrt{2}$ **(D)** $\sqrt{17} + 2\sqrt{2}$ **(E)** 8

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8. A four-digit positive integer is chosen at random from all four-digit positive integers whose digits are all even (the digit 0 counts as even, but the first digit cannot be 0). What is the probability that the chosen number is divisible by 8?

(A) $\frac{3}{25}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{6}{25}$ **(E)** $\frac{8}{25}$

9. A solid right circular cone has base radius 3 and height 4 and stands on its base. Ball 1 is the ball inside the cone that touches the base and touches the curved surface all the way around. For each $k \geq 1$, ball $k+1$ sits on top of ball k: it touches ball k and touches the curved surface of the cone all the way around. This produces an endless stack of balls, all with their centers on the axis of the cone. What fraction of the volume of the cone is filled by all the balls together?



(A) $\frac{8}{63}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{8}{21}$

10. A regular hexagon is drawn in the complex plane. Its centre is the point 2, and one of its vertices is the point $1+i$. What is the product of the six complex numbers at its vertices?

(A) $64-8i$ **(B)** 64 **(C)** $64+8i$ **(D)** $-64+8i$ **(E)** $-8i$

11. Real numbers a and b satisfy $\sin a + \sin b = 1$ and $\cos a + \cos b = \frac{1}{2}$. What is the value of $\sin a \cdot \sin b$? (It is the same for every such pair.)

(A) $\frac{9}{80}$ **(B)** $\frac{9}{40}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{39}{80}$ **(E)** $\frac{3}{5}$

12. How many ordered triples (a, b, c) of positive integers satisfy $a+b+c = 30$ and have greatest common divisor 1, that is, no integer greater than 1 divides all three of a, b and c?

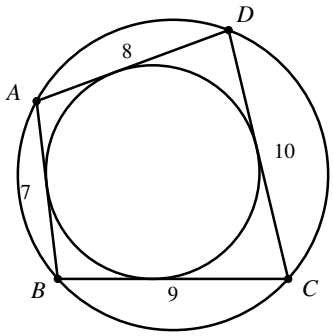
(A) 262 **(B)** 269 **(C)** 276 **(D)** 285 **(E)** 406

13. How many ordered pairs of integers (x, y), positive, negative, or zero, satisfy the equation $x^2+6xy+5y^2 = 2016$?

(A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 72

14. Quadrilateral $ABCD$ has side lengths $AB = 7$, $BC = 9$, $CD = 10$ and $DA = 8$. Its four vertices lie on a circle, and it also has an inscribed circle, a circle inside it that is tangent to all four sides. What is the radius of the inscribed circle?

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(A) $\frac{6\sqrt{35}}{17}$ **(B)** 4 **(C)** $\frac{12\sqrt{35}}{17}$ **(D)** $\frac{72}{17}$ **(E)** $\frac{9}{2}$

15. A box holds three coins that look exactly alike: a fair coin, a coin with heads on both sides, and a bent coin that lands heads with probability $\frac{3}{4}$. Mia takes one of the three coins at random, flips it twice, and gets heads both times. She then flips the same coin a third time. What is the probability that the third flip is also heads?

(A) $\frac{33}{64}$ **(B)** $\frac{16}{29}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{29}{36}$ **(E)** $\frac{99}{116}$

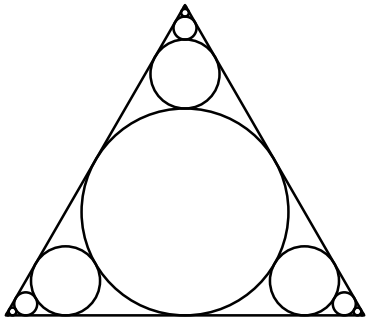
16. Let $z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. What is the value of $\left|1 + 2z + 2z^2 + 2z^4\right|^2$?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 11

17. Let $f(x) = x^2 - 2$. Consider all real numbers x that return to themselves after three applications of f but not after one, that is, $f(f(f(x))) = x$ but $f(x) \neq x$. What is the product of all such real numbers x?

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

18. The incircle of an equilateral triangle is drawn. Then, in each of the three corners, an endless chain of circles is drawn toward the vertex: the first circle of the chain is tangent to the two sides meeting at that vertex and to the incircle, and each later circle is tangent to the same two sides and to the circle just before it in the chain, each one smaller than the one before. What fraction of the area of the triangle is covered by the incircle together with all the circles of the three chains?



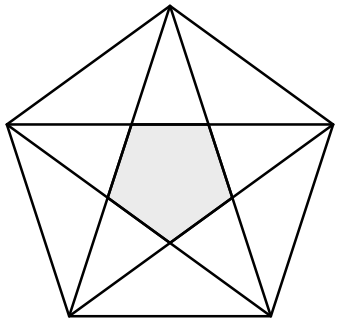
(A) $\frac{\sqrt{3}\pi}{9}$ **(B)** $\frac{5\sqrt{3}\pi}{36}$ **(C)** $\frac{4\sqrt{3}\pi}{27}$ **(D)** $\frac{11\sqrt{3}\pi}{72}$ **(E)** $\frac{\sqrt{3}\pi}{6}$

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19. Positive real numbers a, b and c satisfy $8a + 3b^2 + 8c^3 = 44$. What is the largest possible value of the product abc?

(A) 4 **(B)** 5 **(C)** $4\sqrt[3]{2}$ **(D)** $3\sqrt{3}$ **(E)** 6

20. All five diagonals of a regular pentagon are drawn. They form a five-pointed star, and the region at the middle of the star, bounded by pieces of the five diagonals, is a smaller regular pentagon. What is the ratio of the area of the small pentagon to the area of the original pentagon?



(A) $\frac{7}{2} - \frac{3\sqrt{5}}{2}$ **(B)** $-2 + \sqrt{5}$ **(C)** $\frac{1}{4}$ **(D)** $-\frac{1}{4} + \frac{\sqrt{5}}{4}$ **(E)** $\frac{3}{2} - \frac{\sqrt{5}}{2}$

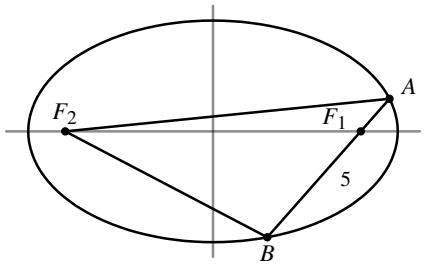
21. Six cards are dealt at random from a standard 52-card deck, which has 13 cards in each of the 4 suits; every set of six cards is equally likely. What is the probability that all four suits appear among the six cards?

(A) $\frac{24167}{195755}$ **(B)** $\frac{59319}{195755}$ **(C)** $\frac{1181}{3290}$ **(D)** $\frac{83486}{195755}$ **(E)** $\frac{11936}{27965}$

22. For a positive integer n, let $\varphi(n)$ be the number of integers k with $1 \le k \le n$ and $\gcd(k, n) = 1$. For example, $\varphi(1) = 1$ and $\varphi(12) = 4$, counting $1, 5, 7, 11$. For how many positive integers $n \le 1000$ is n divisible by $\varphi(n)$?

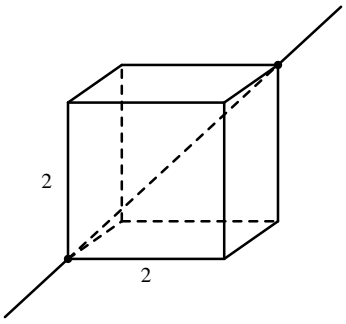
(A) 24 **(B)** 33 **(C)** 34 **(D)** 35 **(E)** 40

23. The ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ has foci $F_1 = (4, 0)$ and $F_2 = (-4, 0)$. A chord AB of the ellipse passes through F_1 and has length $AB = 5$. What is the area of triangle ABF_2?



(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 20

24. A solid cube has edge length 2. It is rotated by 60° about the line through two opposite vertices of the cube (the line containing a space diagonal). What is the volume of the region inside the original cube, the rotated cube, or both?



(A) 8 **(B)** 9 **(C)** $\frac{37}{4}$ **(D)** $\frac{28}{3}$ **(E)** 10

25. A fair six-sided die is rolled repeatedly, and the rolling stops as soon as each of the six faces has appeared at least once. When the rolling stops, some faces may have appeared exactly twice. What is the expected number of faces that have appeared exactly twice?

(A) 1 **(B)** $\frac{4669}{3600}$ **(C)** $\frac{29}{20}$ **(D)** $\frac{49}{20}$ **(E)** $\frac{13489}{3600}$

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Answer Key — AMC 12 Practice Contest 4

1. (B) $-1014 + 1013i$

2. (C) 42

3. (A) 104

4. (C) 235

5. (B) 24

6. (B) 16π

7. (D) $\sqrt{17} + 2\sqrt{2}$

8. (E) $\frac{8}{25}$

9. (E) $\frac{8}{21}$

10. (A) $64 - 8i$

11. (A) $\frac{9}{80}$

12. (C) 276

13. (B) 24

14. (C) $\frac{12\sqrt{35}}{17}$

15. (E) $\frac{99}{116}$

16. (D) 7

17. (A) -1

18. (D) $\frac{11\sqrt{3}\pi}{72}$

19. (E) 6

20. (A) $\frac{7}{2} - \frac{3\sqrt{5}}{2}$

21. (D) $\frac{83486}{195755}$

22. (C) 34

23. (D) 15

24. (E) 10

25. (B) $\frac{4669}{3600}$

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Solutions — AMC 12 Practice Contest 4

1. Answer: (B) $-1014 + 1013i$**. The idea.** The powers of i repeat with period 4: $i, -1, -i, 1$. Without the weights, each block of four terms would cancel to 0. With the weights they do not cancel, but every block of four consecutive terms leaves the *same* remainder, because moving a block up by 4 adds $4(i - 1 - i + 1) = 0$ to it.

Work it out. The block starting at $k = 4m+1$ is $(4m+1)i-(4m+2)-(4m+3)i+(4m+4) = 2-2i$, whatever m is. Since $2024 = 4 \cdot 506$, the terms $k = 1, \ldots, 2024$ form 506 blocks and contribute $506(2-2i) = 1012-1012i$. The two leftover terms are $2025\, i^{2025} = 2025i$ (as $2025 \equiv 1 \pmod 4$) and $2026\, i^{2026} = -2026$ (as $2026 \equiv 2$). The total is $1012-1012i+2025i-2026 = -1014+1013i$.

Check. Try the same method on a short sum: $1 \cdot i+2i^2+3i^3+4i^4+5i^5+6i^6 = i-2-3i+4+5i-6 = -4+3i$, and the method predicts one block $2-2i$ plus leftovers $5i-6$, which is $-4+3i$. The choice $1012 - 1012i$ forgets the two leftover terms, $-2026 + 2025i$ keeps only them, and $1013 - 1014i$ swaps the signs of the block sum.

2. Answer: (C) 42**. The idea.** We need $b^3 \equiv -1 \pmod 7$. Cubes modulo 7 take very few values, so only certain residues of b work. Equivalently, $b^3 + 1 = (b + 1)(b^2 - b + 1)$, and 7 can divide either factor. Stopping at $7 \mid b + 1$ misses two thirds of the answers.

Work it out. Cube the residues modulo 7: $1^3 = 1$, $2^3 = 8 \equiv 1$, $3^3 = 27 \equiv 6$, $4^3 = 64 \equiv 1$, $5^3 = 125 \equiv 6$, $6^3 = 216 \equiv 6$, and $0^3 = 0$. So $b^3 \equiv -1 \equiv 6$ exactly when $b \equiv 3, 5$ or $6 \pmod 7$. Among $2, 3, \ldots, 100$: the numbers $\equiv 3$ are $3, 10, \ldots, 94$ (14 of them), the numbers $\equiv 5$ are $5, 12, \ldots, 96$ (14), and the numbers $\equiv 6$ are $6, 13, \ldots, 97$ (14). The total is 42.

Check. For $b = 3$: $1001_3 = 28 = 4 \cdot 7$. For $b = 5$: $1001_5 = 126 = 18 \cdot 7$. For $b = 6$: $1001_6 = 217 = 31 \cdot 7$. For $b = 4$: $1001_4 = 65$, which is not a multiple of 7. In factored form, $b = 3$ and $b = 5$ work through $b^2 - b + 1$ (7 and 21), and $b = 6$ works through $b + 1$.

3. Answer: (A) 104**. The idea.** The two cosines fix the shape of the triangle. By the law of sines, the sides are in the ratio $\sin A : \sin B : \sin C$, and $\sin C = \sin(A + B)$. The area then fixes the size.

Work it out. Both angles are acute, so $\sin A = \frac{3}{5}$ and $\sin B = \frac{24}{25}$. Then $\sin C = \sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \cdot \frac{7}{25} + \frac{4}{5} \cdot \frac{24}{25} = \frac{21+96}{125} = \frac{117}{125}$. Hence $BC : CA : AB = \frac{3}{5} : \frac{24}{25} : \frac{117}{125} = 75 : 120 : 117 = 25 : 40 : 39$. Write $BC = 25k$, $CA = 40k$ and $AB = 39k$. The area is $\frac{1}{2} \cdot BC \cdot CA \cdot \sin C = \frac{1}{2} \cdot 25k \cdot 40k \cdot \frac{117}{125} = 468k^2$. Setting this equal to 468 gives $k = 1$, so the perimeter is $25 + 40 + 39 = 104$.

Check. Drop the altitude from C to AB. Its length is $CA \sin A = 40 \cdot \frac{3}{5} = 24$. It cuts AB into $CA \cos A = 32$ and $CB \cos B = 25 \cdot \frac{7}{25} = 7$, and indeed $32 + 7 = 39 = AB$. The area is $\frac{1}{2} \cdot 39 \cdot 24 = 468$. The sides $24, 32, 40$ and $7, 24, 25$ are right triangles, as they should be.

4. Answer: (C) 235**. The idea.** Look at the maximal blocks of consecutive lit lamps. Exactly one bright pair means exactly one block has length 2 and every other block has length 1 (a block of length 3 or more already gives two pairs). So count by the number of blocks r. Place the blocks into the gaps between the dark lamps, and then choose which block is the double one.

Work it out. With r blocks there are $r + 1$ lit lamps and $10 - (r + 1) = 9 - r$ dark ones. The dark lamps create $10 - r$ gaps, counting the two ends. Different blocks must be separated by at least

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one dark lamp, so they go into r different gaps, which can be chosen in $\binom{10-r}{r}$ ways. Then choose which of the r blocks is the pair, in r ways. The total is $\sum_r r\binom{10-r}{r} = 1\cdot\binom{9}{1}+2\binom{8}{2}+3\binom{7}{3}+4\binom{6}{4}+5\binom{5}{5} = 9+56+105+60+5 = 235$. For $r = 6$ we would need $\binom{4}{6} = 0$.

Check. The same formula for 4 lamps gives $1\cdot\binom{3}{1}+2\binom{2}{2} = 5$. By listing: 1100, 0110, 0011, 1101, 1011. That is five settings, as predicted. A common slip is to forget the factor r (which block is the pair), which gives $9+28+35+15+1 = 88$.

5. Answer: (B) 24**. The idea.** Work in polar form. Multiplying complex numbers adds their arguments. The moduli are positive whatever m and n are, so only the total angle matters, and it must be a whole number of full turns.

Work it out. $1+i = \sqrt{2}\,(\cos 45^\circ + i\sin 45^\circ)$ and $\sqrt{3}+i = 2(\cos 30^\circ + i\sin 30^\circ)$. The product has argument $45m+30n$ degrees. It is a positive real exactly when $45m+30n$ is a multiple of 360, that is, $3m+2n \equiv 0 \pmod{24}$. Then $3m$ must be even, so m is even. Write $m = 2t$ with $1 \le t \le 12$. The condition becomes $6t+2n \equiv 0 \pmod{24}$, or $n \equiv -3t \pmod{12}$. For each t this has exactly two solutions n in $1,\dots,24$. So there are $12\cdot 2 = 24$ pairs.

Check. For $m = 2$ (angle 90°), we need $30n \equiv 270 \pmod{360}$, so $n = 9$ or 21. Indeed $(1+i)^2(\sqrt{3}+i)^9 = 2i\cdot 2^9 i^3 = 2^{10}$, which is positive. Allowing negative reals as well would ask only for a multiple of 180° and would double the count to 48, which is the trap.

6. Answer: (B) 16π**. The idea.** By symmetry the center of the small circle lies on the bisector of the 60° angle. Its distance to either edge is the radius r, and those edges make a 30° angle with the bisector, which pins down how far the center is from the vertex. Tangency to the arc then says that this distance plus r is the big radius.

Work it out. Let O be the vertex of the sector and P the center of the small circle, with $OP = d$. Drop the perpendicular from P to one edge. It has length r, and the right triangle it forms has angle 30° at O, so $r = d\sin 30^\circ = \frac{d}{2}$, i.e. $d = 2r$. The small circle touches the arc at the point of the arc on the ray OP, beyond P, so $d+r = 12$. Hence $3r = 12$, $r = 4$, and the area is 16π.

Check. The center is at distance 8 from O along the bisector, and the tangent points on the edges are at distance $8\cos 30^\circ = 4\sqrt{3} \approx 6.9 < 12$ from O, so they really lie on the edges of the sector. The farthest point of the small circle from O is at distance $8+4 = 12$, on the arc. Taking the radius to be half the sector's radius gives the trap 36π, and using $r = d$ instead of $r = \frac{d}{2}$ gives $r = 6$ as well.

7. Answer: (D) $\sqrt{17}+2\sqrt{2}$**. The idea.** The points whose distances to two fixed points are in a fixed ratio other than 1 form a circle (a circle of Apollonius). Once we have its center and radius, the farthest point from 0 lies on the line through 0 and the center, on the far side of the center.

Work it out. Write $z = x+yi$ and square: $(x-3)^2+y^2 = 4\left(x^2+(y-3)^2\right)$. Expanding gives $x^2-6x+9+y^2 = 4x^2+4y^2-24y+36$, so $3x^2+3y^2+6x-24y+27 = 0$, i.e. $x^2+y^2+2x-8y+9 = 0$. Completing squares gives $(x+1)^2+(y-4)^2 = 8$. So z runs over the circle with center $-1+4i$ and radius $2\sqrt{2}$. The center is at distance $\sqrt{1+16} = \sqrt{17}$ from the origin, so the largest possible $|z|$ is $\sqrt{17}+2\sqrt{2} \approx 6.95$.

Check. The circle should cross the segment from 3 to $3i$ at the point that divides it in the ratio $2:1$, which is $1+2i$. Indeed $(1+1)^2+(2-4)^2 = 8$. The origin lies outside the circle ($\sqrt{17} > 2\sqrt{2}$), so the smallest value is $\sqrt{17}-2\sqrt{2}$, and both extremes lie on the line through 0 and the center.

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8. Answer: (E) $\frac{8}{25}$**. The idea.** Divisibility by 8 depends only on the last three digits, since 1000 is a multiple of 8. When the digits are even, even more collapses: the hundreds digit contributes a multiple of 200, which is a multiple of 8. So only the tens and units digits matter.

Work it out. There are $4 \cdot 5 \cdot 5 \cdot 5 = 500$ such numbers, and the last two digits t and u are each equally likely to be any of $0, 2, 4, 6, 8$, independently. Write $t = 2j$. Modulo 8, the number is $\equiv 10t + u = 20j + u \equiv 4j + u$. If $t \in \{0, 4, 8\}$, then j is even, $4j \equiv 0$, and we need $u \equiv 0 \pmod 8$: $u = 0$ or 8. If $t \in \{2, 6\}$, then j is odd, $4j \equiv 4$, and we need $u \equiv 4 \pmod 8$: $u = 4$. That makes $3 \cdot 2 + 2 \cdot 1 = 8$ good endings out of 25, so the probability is $\frac{8}{25}$.

Check. The good two-digit endings are $00, 08, 24, 40, 48, 64, 80, 88$, and each is a multiple of 8 (for example $24 = 3 \cdot 8$ and $64 = 8 \cdot 8$). With even digits in front, $2224 = 8 \cdot 278$, $2248 = 8 \cdot 281$ and $2208 = 8 \cdot 276$ are all multiples of 8, as predicted. Forgetting the tens digits 2 and 6 (which pair with units digit 4) leaves only 6 endings and gives $\frac{6}{25}$; allowing only the units digit 0 gives $\frac{3}{25}$; and the naive guess $\frac{1}{8}$ ignores that the digits are restricted.

9. Answer: (E) $\frac{8}{21}$**. The idea.** Above the top of ball 1 there is a small cone, cut off by the horizontal plane that touches the top of ball 1. This small cone is similar to the whole cone, and ball 2 sits in it exactly as ball 1 sits in the whole cone. So every ball is the previous one scaled by the same factor, and the volumes form a geometric series.

Work it out. Cut the cone through its axis. The cross-section is an isosceles triangle with base 6, height 4 and slant sides 5. Ball 1 appears as the inscribed circle of this triangle, whose radius is $\frac{\text{area}}{\text{semiperimeter}} = \frac{12}{8} = \frac{3}{2}$. The top of ball 1 is at height $2 \cdot \frac{3}{2} = 3$, so the small cone above it has height $4 - 3 = 1$, which is $\frac{1}{4}$ of the whole. Ball 2 touches the base of the small cone (the tangent plane at the top of ball 1) and its curved surface, so ball 2 is ball 1 shrunk by the factor $\frac{1}{4}$. Repeating the argument, each ball is $\frac{1}{4}$ the size of the one below it, so each volume is $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ of the one below. Ball 1 has volume $\frac{4}{3}\pi\left(\frac{3}{2}\right)^3 = \frac{9\pi}{2}$, so all the balls together have volume $\frac{9\pi}{2} \cdot \frac{1}{1-\frac{1}{64}} = \frac{9\pi}{2} \cdot \frac{64}{63} = \frac{32\pi}{7}$. The cone has volume $\frac{1}{3}\pi \cdot 3^2 \cdot 4 = 12\pi$, so the fraction is $\frac{32\pi/7}{12\pi} = \frac{8}{21}$.

Check. Measure distance from the apex. The slant side makes an angle θ with the axis where $\sin\theta = \frac{3}{5}$, so a ball centered on the axis at distance d from the apex and touching the curved surface has radius $\frac{3}{5}d$. Ball 1: center at distance $4 - \frac{3}{2} = \frac{5}{2}$, radius $\frac{3}{5} \cdot \frac{5}{2} = \frac{3}{2}$. Its top is at distance 1 from the apex. Ball 2 has $d + \frac{3}{5}d = 1$, so $d = \frac{5}{8}$ and radius $\frac{3}{8} = \frac{1}{4} \cdot \frac{3}{2}$, as claimed. The balls fill $\frac{9\pi/2}{12\pi} = \frac{3}{8}$ of the cone with ball 1 alone, and slightly more with the rest, so $\frac{8}{21} \approx 0.381$ is sensible. Forgetting the factor $\frac{1}{3}$ in the cone's volume gives $\frac{8}{63}$, and computing the cone with its radius and height swapped gives $\frac{2}{7}$.

10. Answer: (A) $64-8i$**. The idea.** Measured from the centre, the six vertices are $u, u\omega, u\omega^2, \ldots, u\omega^5$, where u is one vertex minus the centre and ω is a rotation by $60°$. These are exactly the six solutions of $w^6 = u^6$. So the vertices z are exactly the roots of the polynomial $(z-2)^6 - u^6$, and the product of the roots of a monic polynomial of degree 6 is its constant term. There is no need to find a single vertex other than the one given.

Work it out. Here $u = (1+i) - 2 = -1+i$. Then $u^2 = 1 - 2i + i^2 = -2i$ and $u^6 = (-2i)^3 = -8i^3 = 8i$. So the vertices are the six roots of $(z-2)^6 - 8i = 0$. Its constant term is $(-2)^6 - 8i = 64 - 8i$, and for a monic polynomial of even degree the product of the roots equals the constant term (the sign factor is $(-1)^6 = 1$). The product of the vertices is $64 - 8i$.

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Check. Pair each vertex $2 + w$ with the opposite vertex $2 - w$: their product is $4 - w^2$. For the given vertex, $w = -1 + i$, and indeed $(1 + i)(3 - i) = 4 + 2i = 4 - (-2i)$. The three values of w^2 are the three cube roots of $u^6 = 8i$, i.e. the roots of $t^3 - 8i$, so the product of the three pairs is $(4^3 - 8i) = 64 - 8i$ again. The choice $-8i$ is the product of the six vectors from the centre, forgetting to shift by 2, and 64 treats the hexagon as if it had a vertex on the real line through the centre.

11. Answer: (A) $\frac{9}{80}$**. The idea.** Product-to-sum gives $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$. Each of these two cosines can be pulled out of the data by its own trick. Squaring and adding gives $\cos(a-b)$. Dividing the sum-to-product forms gives $\tan\frac{a+b}{2}$, and from that $\cos(a+b)$.

Work it out. Square both equations and add: $(\sin^2 a + \cos^2 a) + (\sin^2 b + \cos^2 b) + 2(\sin a \sin b + \cos a \cos b) = 1 + \frac{1}{4}$. So $2 + 2\cos(a-b) = \frac{5}{4}$ and $\cos(a-b) = -\frac{3}{8}$. Next, $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$ and $\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$. Dividing gives $\tan\frac{a+b}{2} = 2$. (The common factor is not 0, since the sums are not both 0.) Hence $\cos(a+b) = \frac{1-\tan^2\frac{a+b}{2}}{1+\tan^2\frac{a+b}{2}} = \frac{1-4}{1+4} = -\frac{3}{5}$. Therefore $\sin a \sin b = \frac{1}{2}\left(-\frac{3}{8} + \frac{3}{5}\right) = \frac{1}{2}\cdot\frac{9}{40} = \frac{9}{80}$.

Check. The same two cosines give $\cos a \cos b = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right) = \frac{1}{2}\left(-\frac{3}{8} - \frac{3}{5}\right) = -\frac{39}{80}$. Then $\cos a \cos b - \sin a \sin b = -\frac{39}{80} - \frac{9}{80} = -\frac{48}{80} = -\frac{3}{5} = \cos(a+b)$, which is consistent. Such a, b exist, since $\cos^2\frac{a-b}{2} = \frac{1+\cos(a-b)}{2} = \frac{5}{16}$ lies between 0 and 1.

12. Answer: (C) 276**. The idea.** Any common divisor of a, b, c also divides their sum 30. So a triple is bad exactly when 2, 3 or 5 divides all three numbers. Count all triples, then remove the bad ones by inclusion–exclusion over these three primes.

Work it out. By stars and bars, the number of ordered triples of positive integers with sum s is $\binom{s-1}{2}$. If d divides all three numbers, write $a = da'$, $b = db'$, $c = dc'$, so that $a' + b' + c' = \frac{30}{d}$, giving $\binom{30/d-1}{2}$ triples. Now: all triples, $\binom{29}{2} = 406$; all divisible by 2, $\binom{14}{2} = 91$; by 3, $\binom{9}{2} = 36$; by 5, $\binom{5}{2} = 10$; by 6, $\binom{4}{2} = 6$; by 10, $\binom{2}{2} = 1$; by 15, $\binom{1}{2} = 0$; by 30, 0. Inclusion–exclusion gives $406 - (91 + 36 + 10) + (6 + 1 + 0) - 0 = 276$.

Check. Count the bad triples a second way, sorting them into disjoint groups. First, every entry even: 91. Second, every entry a multiple of 3 but not all even: $36 - 6 = 30$. Third, every entry a multiple of 5 but neither all even nor all multiples of 3: $10 - 1 - 0 = 9$. That is $91 + 30 + 9 = 130$ bad triples, and $406 - 130 = 276$. Subtracting the counts for every divisor $2, 3, 5, 6, 10, 15, 30$ instead of adding back the overlaps would give the wrong answer 262.

13. Answer: (B) 24**. The idea.** The left side factors as $(x + y)(x + 5y)$. Call the factors $a = x + y$ and $b = x + 5y$. Not every factorization $ab = 2016$ comes from integers x, y: since $b - a = 4y$, the two factors must be congruent mod 4. That condition, pushed through the powers of 2 in $2016 = 2^5 \cdot 63$, is the whole problem.

Work it out. Conversely, if $ab = 2016$ and $a \equiv b \pmod 4$, then $y = \frac{b-a}{4}$ and $x = a - y = \frac{5a-b}{4}$ are integers (as $5a - b \equiv a - b \equiv 0$), so pairs (x, y) match ordered factorizations (a, b) with $a \equiv b \pmod 4$ one-to-one. The product is even, so one factor is even, and then $a \equiv b \pmod 2$ forces both to be even: $a = 2a'$, $b = 2b'$ with $a'b' = 504$ and $a' \equiv b' \pmod 2$. Again 504 is even, so both a', b' are even: $a = 4p$, $b = 4q$ with $pq = 126$. Now $a \equiv b \equiv 0 \pmod 4$ automatically, so

every ordered factorization $pq = 126$ works. $126 = 2 \cdot 3^2 \cdot 7$ has $2 \cdot 3 \cdot 2 = 12$ positive divisors, and p, q may both be positive or both negative: $2 \cdot 12 = 24$ pairs.

Check. With $p = 1$, $q = 126$: $y = q - p = 125$, $x = 5p - q = -121$, and $(x + y)(x + 5y) = 4 \cdot 504 = 2016$. A factorization that fails: $a = 2$, $b = 1008$ gives $b - a = 1006$, not a multiple of 4. Counting all 36 divisors of 2016 with both signs gives 72; forgetting negatives gives 12; stopping after "both even" gives $2 \cdot 24 = 48$.

14. Answer: (C) $\frac{12\sqrt{35}}{17}$**. The idea.** Each circle gives one fact. The inscribed circle means the area is $K = rs$, where s is the semiperimeter (cut the quadrilateral into four triangles with apex at the incenter, each of height r). The circumscribed circle means Brahmagupta's formula $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ applies. The two together make the area formula especially neat.

Work it out. The semiperimeter is $s = \frac{7+9+10+8}{2} = 17$. A quadrilateral with an inscribed circle has equal sums of opposite sides (equal tangent lengths from each vertex), and indeed $7 + 10 = 9 + 8 = 17$. So $s - a = c$, $s - b = d$ and so on, and Brahmagupta's formula collapses to $K = \sqrt{abcd} = \sqrt{7 \cdot 9 \cdot 10 \cdot 8} = \sqrt{5040} = 12\sqrt{35}$. Then $r = \frac{K}{s} = \frac{12\sqrt{35}}{17} \approx 4.18$.

Check. Compute the area directly. Opposite angles B and D are supplementary, so $\cos D = -\cos B$, and the diagonal AC computed two ways gives $49+81-126\cos B = 100+64+160\cos B$, so $\cos B = -\frac{17}{143}$ and $\sin B = \frac{\sqrt{20160}}{143} = \frac{24\sqrt{35}}{143}$. The area is $\frac{1}{2}(7\cdot 9+10\cdot 8)\sin B = \frac{143}{2}\cdot\frac{24\sqrt{35}}{143} = 12\sqrt{35}$. Dividing by the full perimeter 34 instead of s gives the trap $\frac{6\sqrt{35}}{17}$.

15. Answer: (E) $\frac{99}{116}$**. The idea.** The two heads are evidence about which coin Mia holds: they make the two-headed coin more likely and the fair coin less likely. The cleanest way to use this is the definition of conditional probability, $P(\text{3rd H} \mid \text{HH}) = \frac{P(\text{HHH})}{P(\text{HH})}$, with each probability averaged over the three equally likely coins.

Work it out. $P(\text{HH}) = \frac{1}{3}\left(\frac{1}{4} + 1 + \frac{9}{16}\right) = \frac{1}{3}\cdot\frac{29}{16}$ and $P(\text{HHH}) = \frac{1}{3}\left(\frac{1}{8} + 1 + \frac{27}{64}\right) = \frac{1}{3}\cdot\frac{99}{64}$. So the answer is $\frac{99/64}{29/16} = \frac{99}{116} \approx 0.853$.

Check. Bayes' rule in two steps. After HH the chances of the three coins are proportional to $\frac{1}{4} : 1 : \frac{9}{16} = 4 : 16 : 9$, i.e. $\frac{4}{29}, \frac{16}{29}, \frac{9}{29}$. The next flip is heads with probability $\frac{4}{29}\cdot\frac{1}{2} + \frac{16}{29}\cdot 1 + \frac{9}{29}\cdot\frac{3}{4} = \frac{8+64+27}{116} = \frac{99}{116}$. Ignoring the evidence gives the average $\frac{3}{4}$; updating on only one head gives $\frac{29}{36}$.

16. Answer: (D) 7**. The idea.** Do not compute cosines. The six nonreal seventh roots of unity split into $w = z + z^2 + z^4$ and its conjugate $\overline{w} = z^6 + z^5 + z^3$ (since $\overline{z^k} = z^{7-k}$). Their sum and product are easy, so w is a root of a known quadratic.

Work it out. Since $1+z+\cdots+z^6 = 0$, we get $w+\overline{w} = -1$. Multiplying out, $w\overline{w} = (z+z^2+z^4)(z^3+z^5+z^6)$ has nine terms with exponents $4, 6, 7, 5, 7, 8, 7, 9, 10$, i.e. $z^4+z^6+1+z^5+1+z+1+z^2+z^3 = 3 + (-1) = 2$. So w and $\overline{w}$ are the roots of $t^2 + t + 2 = 0$, namely $\frac{-1\pm i\sqrt{7}}{2}$. Hence $1 + 2w = \pm i\sqrt{7}$, and $|1 + 2w|^2 = 7$.

Check. Without solving: $|1 + 2w|^2 = (1 + 2w)(1 + 2\overline{w}) = 1 + 2(w + \overline{w}) + 4w\overline{w} = 1 - 2 + 8 = 7$. Guessing $w = -\frac{1}{2}$ (real) gives 0; guessing $w\overline{w} = 1$ gives 3; forgetting the cross term $2(w + \overline{w})$ with $|w|^2 = 3$ leads to answers like 11.

17. Answer: (A) -1**. The idea.** The polynomial $f(f(f(x))) - x$ has degree 8, so the product of all its roots is easy (Vieta). The real difficulty is that the question is about *real* roots: we must

know that all eight roots are real. The substitution $x = 2\cos\theta$ shows this, because $f(2\cos\theta) = 4\cos^2\theta - 2 = 2\cos 2\theta$.

Work it out. If $|x| > 2$ then $f(x) > 2$ and the iterates only grow, so they never return; every real solution has $|x| \le 2$ and can be written $x = 2\cos\theta$ with $0 \le \theta \le \pi$. Then $f(f(f(x))) = 2\cos 8\theta$, and we need $\cos 8\theta = \cos\theta$, so $8\theta = \pm\theta + 2\pi k$: $\theta = \frac{2\pi k}{7}$ or $\theta = \frac{2\pi k}{9}$. In $[0, \pi]$ these are $0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$ and $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}$ (the value 0 appears in both lists), eight different angles giving eight different real x. A degree-8 polynomial has at most 8 roots, so these are all the roots and all are real.

The polynomial is monic of even degree, so the product of its roots is its constant term, $f(f(f(0))) - 0 = f(f(-2)) = f(2) = 2$. The fixed points, from $x^2 - 2 = x$, are $x = 2$ and $x = -1$ (the angles 0 and $\frac{2\pi}{3} = \frac{6\pi}{9}$), with product -2. Removing them leaves $\frac{2}{-2} = -1$.

Check. The six remaining numbers form two 3-cycles, the roots of $x^3 + x^2 - 2x - 1$ (the $2\cos\frac{2\pi k}{7}$) and of $x^3 - 3x + 1$ (the $2\cos\frac{2\pi k}{9}$ other than -1). Indeed $((x^2-2)^2-2)^2 - 2 - x = (x-2)(x+1)(x^3 + x^2 - 2x - 1)(x^3 - 3x + 1)$, and the two cubics have root products 1 and -1: total -1. Forgetting to remove the fixed points gives the trap 2.

18. Answer: (D) $\frac{11\sqrt{3}\pi}{72}$**. The idea.** Every circle tangent to both sides of a 60° corner has its center on the angle bisector, and its distance to the vertex is exactly twice its radius (the bisector makes 30° with each side, and $\sin 30° = \frac{1}{2}$). This fixes the ratio between neighbouring circles in a chain, and the incircle itself is the start of all three chains.

Work it out. Take the incircle radius $r = 1$. Two neighbouring circles of a chain, radii $\rho > \rho'$, have centers at distances 2ρ and $2\rho'$ from the vertex along the same line, and they touch, so $2\rho - 2\rho' = \rho + \rho'$, giving $\rho' = \frac{\rho}{3}$. The incircle touches both sides at every corner, so each chain has radii $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$, and its areas add to $\pi\left(\frac{1}{9} + \frac{1}{81} + \cdots\right) = \pi \cdot \frac{1/9}{1-1/9} = \frac{\pi}{8}$. All the circles together cover $\pi + 3 \cdot \frac{\pi}{8} = \frac{11\pi}{8}$. The triangle with inradius 1 has side $2\sqrt{3}$ and area $\frac{\sqrt{3}}{4} \cdot 12 = 3\sqrt{3}$. The fraction is $\frac{11\pi}{8 \cdot 3\sqrt{3}} = \frac{11\sqrt{3}\,\pi}{72} \approx 0.831$.

Check. The incircle alone covers $\frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\,\pi}{9} \approx 0.605$, and the three chains add $\frac{3\pi/8}{3\sqrt{3}} \approx 0.227$, below 1 as it must be. The incenter is $2r = 2$ from each vertex, and the first corner circle ($\rho = \frac{1}{3}$) has its center $2\rho = \frac{2}{3}$ from the vertex, so the two centers are $2 - \frac{2}{3} = \frac{4}{3} = 1 + \frac{1}{3}$ apart, exactly the sum of the radii. Keeping only the first circle in each corner gives the trap $\frac{4\sqrt{3}\,\pi}{27}$.

19. Answer: (E) 6**. The idea.** AM-GM turns a fixed sum into a bound on a product, so we want to write 44 as a sum of equal-weight pieces whose product is a power of abc. The three terms carry a, b^2 and c^3. Taking six pieces that each contain a, three that each contain b^2 and two that each contain c^3 makes every letter appear to the sixth power. So cut $8a$ into six equal parts, $3b^2$ into three and $8c^3$ into two: eleven pieces in all.

Work it out. Apply AM-GM to the eleven positive numbers $\frac{4a}{3}$ (six times), b^2 (three times) and $4c^3$ (twice). They add to $8a + 3b^2 + 8c^3 = 44$, so their mean is 4, and $\left(\frac{4a}{3}\right)^6 (b^2)^3 (4c^3)^2 \le 4^{11}$, that is $\frac{4^8}{3^6}(abc)^6 \le 4^{11}$. So $(abc)^6 \le 4^3 \cdot 3^6 = 64 \cdot 729 = 46656 = 6^6$, and $abc \le 6$. Equality needs all eleven pieces equal to 4: $\frac{4a}{3} = 4$, $b^2 = 4$, $4c^3 = 4$, so $a = 3$, $b = 2$, $c = 1$. Indeed $8 \cdot 3 + 3 \cdot 4 + 8 \cdot 1 = 44$, and $abc = 6$ is reached.

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Check. Other points on the surface do worse. With $a = b = c = t$ we need $8t + 3t^2 + 8t^3 = 44$, so $t \approx 1.475$ and $abc \approx 3.2$. With $a = b = 2$ we get $8c^3 = 16$, $c = \sqrt[3]{2}$ and $abc = 4\sqrt[3]{2} \approx 5.04$. A calculus check agrees: with the multiplier λ, $bc = 8\lambda$, $ac = 6\lambda b$, $ab = 24\lambda c^2$ give $8a = 6b^2 = 24c^3$, which with the constraint again forces $(3, 2, 1)$.

20. Answer: (A) $\frac{7}{2} - \frac{3\sqrt{5}}{2}$**. The idea.** Everything in a regular pentagon is measured by the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$: with side 1, each diagonal is φ. Each diagonal is cut by two others into three pieces, and the middle piece is a side of the small pentagon. Angle chasing with $36°$ and $72°$ finds the pieces.

Work it out. Let the pentagon be $ABCDE$ with side 1 and diagonal d. Ptolemy's theorem in the cyclic quadrilateral $ABCD$ gives $d \cdot d = d \cdot 1 + 1 \cdot 1$, so $d^2 = d + 1$ and $d = \varphi$. Each interior angle is $108°$, and triangle ABC is isosceles, so $\angle BAC = \angle BCA = 36°$; likewise $\angle CBD = 36°$. Let AC meet BD at P. In triangle ABP, $\angle BAP = 36°$ and $\angle ABP = 108° - 36° = 72°$, so $\angle APB = 72°$ and $AP = AB = 1$. Then $PC = \varphi - 1$. By symmetry the other crossing point Q on AC (with BE) has $AQ = \varphi - 1$, so the middle piece is $PQ = 1 - (\varphi - 1) = 2 - \varphi = \frac{1}{\varphi^2}$ (because $\varphi^2 = \varphi + 1$). The small pentagon is similar to the large one with ratio $\frac{1}{\varphi^2}$, so the area ratio is $\frac{1}{\varphi^4} = \left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{14-6\sqrt{5}}{4} = \frac{7-3\sqrt{5}}{2} \approx 0.146$.

Check. Trigonometry directly: the inner pentagon is the outer one rotated $36°$ and scaled, with circumradius ratio $\frac{\cos 72°}{\cos 36°} = \frac{(\sqrt{5}-1)/4}{(\sqrt{5}+1)/4} = \frac{3-\sqrt{5}}{2}$, the same $\frac{1}{\varphi^2}$. Forgetting to square the length ratio gives the trap $\frac{3-\sqrt{5}}{2}$.

21. Answer: (D) $\frac{83486}{195755}$**. The idea.** The tempting count ("one card of each suit, then any two more") counts the same hand several times, since it does not know which card was the "first" of its suit. Instead sort the hands by how many cards each suit gets. Six cards in four suits, each suit at least once, means the suit sizes are $3, 1, 1, 1$ or $2, 2, 1, 1$ in some order, and each shape is counted exactly once.

Work it out. Shape $3, 1, 1, 1$: choose the suit with three cards (4 ways), its cards ($\binom{13}{3} = 286$), and one card from each other suit (13^3): $4 \cdot 286 \cdot 2197 = 2513368$. Shape $2, 2, 1, 1$: choose the two suits with two cards ($\binom{4}{2} = 6$), their cards ($\binom{13}{2}^2 = 78^2$), and one card from each of the other two (13^2): $6 \cdot 6084 \cdot 169 = 6169176$. The total is 8682544 out of $\binom{52}{6} = 20358520$ hands, and $\frac{8682544}{20358520} = \frac{83486}{195755} \approx 0.426$.

Check. Inclusion–exclusion over missing suits: $\binom{52}{6} - 4\binom{39}{6} + 6\binom{26}{6} - 4\binom{13}{6} = 20358520 - 13031524 + 1380390 - 6864 = 8682544$. Dropping the last term gives the near miss $\frac{11936}{27965}$, and the overcount $13^4\binom{48}{2}$ is even larger than the number of hands.

22. Answer: (C) 34**. The idea.** The ratio $\frac{n}{\varphi(n)}$ depends only on which primes divide n: $\frac{n}{\varphi(n)} = \prod_{p \mid n} \frac{p}{p-1}$. The numerator is a product of distinct primes, so it has at most one factor of 2. Every odd prime p puts the even number $p - 1$ in the denominator. Counting factors of 2 leaves almost no room.

Work it out. Suppose $\frac{n}{\varphi(n)}$ is an integer. If n had two odd prime factors, the denominator would have at least two factors of 2 and the numerator at most one: impossible. If n had an odd prime factor p but no factor 2, the numerator would be odd and the denominator even: impossible. So

the prime set is $\{\}$, $\{2\}$ or $\{2, p\}$ with p odd. The empty set gives $n = 1$ (ratio 1), and $\{2\}$ gives ratio 2. For $\{2, p\}$ we need $\frac{2p}{p-1}$ to be an integer; since $\gcd(p-1, p) = 1$, $p-1$ must divide 2, so $p = 3$ (ratio 3). So the n that work are 1, the powers 2^a with $a \geq 1$, and the numbers $2^a 3^b$ with $a, b \geq 1$. Up to 1000: $2^1, \ldots, 2^9$ give 9. For $2^a 3^b \leq 1000$ the number of b for $a = 1, 2, \ldots, 8$ is $5, 5, 4, 3, 3, 2, 1, 1$, a total of 24. With $n = 1$ the count is $1 + 9 + 24 = 34$.

Check. Examples: $\varphi(48) = 16$ divides 48, $\varphi(96) = 32$ divides 96, but $\varphi(18) = 6$ divides 18 while $\varphi(9) = 6$ does not divide 9 and $\varphi(10) = 4$ does not divide 10. Forgetting $n = 1$ gives 33; wrongly allowing powers of 3 alone gives 40.

23. Answer: (D) 15**. The idea.** Measure points of the ellipse from the focus F_1 in polar form. A chord through F_1 in direction θ is made of two opposite focal radii, so its length is a simple function of $\cos\theta$; setting it equal to 5 pins down the direction. Then the area splits along the x-axis into two triangles with the common base F_1F_2.

Work it out. Here $a = 5$, $b = 3$, $c = 4$, $e = \frac{4}{5}$, and the directrix beside F_1 is $x = \frac{a}{e} = \frac{25}{4}$. For $P = F_1 + r(\cos\theta, \sin\theta)$ on the ellipse, $PF_1 = e \cdot$ (distance to the directrix) gives $r = \frac{4}{5}\left(\frac{9}{4} - r\cos\theta\right)$, so $r = \dfrac{9}{5 + 4\cos\theta}$. The opposite point uses $\theta + \pi$, so

$$AB = \frac{9}{5 + 4\cos\theta} + \frac{9}{5 - 4\cos\theta} = \frac{90}{25 - 16\cos^2\theta} = 5,$$

giving $\cos^2\theta = \frac{7}{16}$ and $\sin\theta = \frac{3}{4}$ (taking $0 < \theta < \pi$). A and B lie on opposite sides of the x-axis, and F_1F_2 lies on that axis, so triangle ABF_2 is the union of triangles AF_1F_2 and BF_1F_2:

$$[ABF_2] = \tfrac{1}{2} \cdot F_1F_2 \cdot (|y_A| + |y_B|) = \tfrac{1}{2} \cdot 8 \cdot (AB\sin\theta) = 4 \cdot 5 \cdot \tfrac{3}{4} = 15.$$

Check. The focal radii are $\frac{9}{5\pm\sqrt{7}} = \frac{5\mp\sqrt{7}}{2}$, and since $PF_1 + PF_2 = 10$ the triangle has sides 5, $\frac{15+\sqrt{7}}{2}$, $\frac{15-\sqrt{7}}{2}$, perimeter 20. Heron: $s = 10$, and $10 \cdot 5 \cdot \frac{5-\sqrt{7}}{2} \cdot \frac{5+\sqrt{7}}{2} = 50 \cdot \frac{18}{4} = 225$, area 15. Treating the chord as vertical (forgetting $\sin\theta$) gives 20; using half of F_1F_2 as the base gives 10.

24. Answer: (E) 10**. The idea.** A 60° turn is not a symmetry of the cube (only 120° turns about this line are), but it is half of one. Look down the diagonal from $O = (0, 0, 0)$ to $(2, 2, 2)$: the three vertices next to O (at level $x + y + z = 2$) project to a triangle, and the three vertices next to the far corner (at level 4) project to the same triangle turned by 60°. So rotating by 60° and then reflecting in the plane $x + y + z = 3$ (through the center, perpendicular to the diagonal) swaps these two triangles and the two end vertices: it maps the cube onto itself. Hence the rotated cube is simply the mirror image of the cube in that plane. Then slice both solids perpendicular to the diagonal.

Work it out. Slice at level $x + y + z = s$, $0 \leq s \leq 6$. The cube's slice is the set of points of that plane with all coordinates ≥ 0 (an upward triangle of size proportional to s) and all coordinates ≤ 2 (a downward triangle of size proportional to $6 - s$). Seen along the diagonal, call these $U(s)$ and $D(6-s)$; triangles of the same kind are nested, $U(a) \subseteq U(b)$ when $a \leq b$. The mirror cube's slice at level s is the cube's slice at level $6 - s$, namely $U(6 - s) \cap D(s)$. For $s \leq 3$ the common part is $U(s) \cap D(6 - s) \cap U(6 - s) \cap D(s) = U(s) \cap D(s)$: an equilateral triangle and its 180° turn about its center. They overlap in a regular hexagon with $\frac{2}{3}$ of the triangle's area (the triangle minus three corner triangles of $\frac{1}{9}$ each). $U(s)$ has side $s\sqrt{2}$ and area $\frac{\sqrt{3}}{2}s^2$, so the overlap has area

$\frac{\sqrt{3}}{3}s^2$. Neighbouring levels are $\frac{ds}{\sqrt{3}}$ apart along the diagonal, and the picture is symmetric about $s = 3$, so the common volume is $2\int_0^3 \frac{\sqrt{3}}{3}s^2 \cdot \frac{ds}{\sqrt{3}} = \frac{2}{3} \cdot \frac{27}{3} = 6$. The region covered by at least one cube has volume $8 + 8 - 6 = 10$.

Check. The same slicing applied to a single cube gives $\int_0^6 (\text{slice area})\frac{ds}{\sqrt{3}} = 8$, as it should. Also the common part is $\frac{6}{8} = \frac{3}{4}$ of the cube, which a random-point simulation of the actual 60° rotation confirms (about 0.75). Treating the 60° turn as a symmetry of the cube would give 8, and taking the hexagon as $\frac{3}{4}$ of the triangle instead of $\frac{2}{3}$ would give $\frac{37}{4}$.

25. Answer: (B) $\frac{4669}{3600}$. The idea. By linearity of expectation, the answer is 6 times the probability that one particular face, say f, ends with exactly two appearances. Two observations make that probability computable. First, by symmetry f is equally likely to be the 1st, 2nd, . . ., 6th different face to show up. Second, after f first appears, the only rolls that matter for f's final count are rolls of f and rolls of faces not yet seen; everything else can be ignored.

Work it out. Suppose f is the j-th new face, so when it first appears $m = 6 - j$ faces are still unseen. Consider the stage when k faces are unseen. Among the rolls that matter, the next one is f with probability $\frac{1}{k+1}$ and a new face with probability $\frac{k}{k+1}$. So in that stage f appears zero more times with probability $\frac{k}{k+1}$, and exactly once with probability $\frac{1}{k+1} \cdot \frac{k}{k+1}$. We need exactly one extra appearance over the stages $k = m, m-1, \ldots, 1$: one chosen stage s contributes one f and every other stage none. That probability is $\sum_{s=1}^{m} \frac{1}{s+1} \prod_{k=1}^{m} \frac{k}{k+1} = \frac{1}{m+1}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m+1}\right) = \frac{H_{m+1}-1}{m+1}$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Averaging over j (weight $\frac{1}{6}$ each) and multiplying by 6 faces, with $n = m+1 = 7-j$: $E = \sum_{n=1}^{6} \frac{H_n-1}{n} = 0 + \frac{1}{4} + \frac{5}{18} + \frac{13}{48} + \frac{77}{300} + \frac{29}{120} = \frac{900+1000+975+924+870}{3600} = \frac{4669}{3600} \approx 1.30$.

Check. The same method for faces seen exactly once gives $\sum_{n=1}^{6} \frac{1}{n} = H_6 = \frac{49}{20}$; the $n = 1$ term is the last new face, which always appears exactly once, as it must. For a two-sided coin the formula gives $\frac{1}{4}$, and directly: the first face repeats until the other face shows, and it appears exactly twice with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. An exact Markov chain on (faces unseen, seen once, seen twice) confirms $\frac{4669}{3600}$.

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