MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

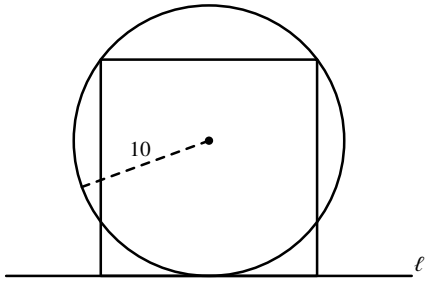
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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Two bacteria cultures are started at the same moment. Culture A begins with 1000 cells and triples every 4 hours; culture B begins with 8000 cells and doubles every 3 hours. Both grow smoothly, so after t hours culture A has $1000 \cdot 3^{t/4}$ cells and culture B has $8000 \cdot 2^{t/3}$ cells. Culture B starts far ahead, but A grows faster. After how many hours do the two cultures have exactly the same number of cells?

(A) $\frac{36}{\log_2(81/8)}$ **(B)** $\frac{12}{\log_2(27/16)}$ **(C)** $\frac{24}{\log_2(27/16)}$ **(D)** $\frac{36}{\log_2(27/16)}$ **(E)** $\frac{72}{\log_2(27/16)}$

2. A circle of radius 10 is tangent to a line ℓ. A square lies on the same side of ℓ as the circle: one side of the square lies along ℓ, and the two vertices of the square that are not on ℓ both lie on the circle. (The square is partly inside and partly outside the circle.) What is the area of the square?

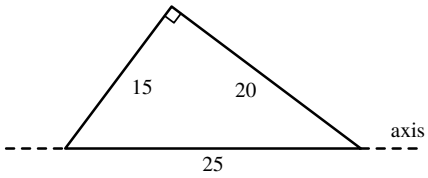


(A) 100 **(B)** 144 **(C)** 200 **(D)** 256 **(E)** 400

3. Two different numbers are chosen at random from $\{1, 2, 3, \ldots, 12\}$, with every pair of numbers equally likely. What is the probability that the larger of the two numbers is at least twice the smaller?

(A) $\frac{5}{11}$ **(B)** $\frac{6}{13}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{6}{11}$ **(E)** $\frac{2}{3}$

4. A right triangle has legs of length 15 and 20 and hypotenuse of length 25. The triangle is rotated one full turn about the line containing its hypotenuse, sweeping out a solid. What is the volume of that solid?



(A) 960π **(B)** 1200π **(C)** 1500π **(D)** 2000π **(E)** 3600π

5. Let $P(x) = \left(1 + x^3\right)\left(x^2 - \frac{2}{x}\right)^9$. When $P(x)$ is expanded and written as a sum of terms $c\,x^k$ with k an integer, one of the terms is a constant (the term with $k = 0$). What is that constant?

(A) -4608 **(B)** -4032 **(C)** -768 **(D)** 672 **(E)** 768

6. In triangle ABC, none of the angles is a right angle, and the tangents of all three angles, $\tan A$, $\tan B$ and $\tan C$, are integers. The longest side of the triangle has length 10. What is the area of triangle ABC?

(A) 25 **(B)** 30 **(C)** $\frac{100}{3}$ **(D)** 40 **(E)** 50

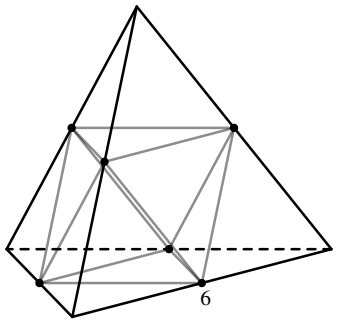
7. Three forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ act on one point in a plane and balance exactly, so $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$. Their magnitudes are $|\vec{F}_1| = 3$, $|\vec{F}_2| = 5$ and $|\vec{F}_3| = 7$ newtons. What is the magnitude of the vector $2\vec{F}_1 - \vec{F}_2$?

(A) $\sqrt{19}$ **(B)** $\sqrt{31}$ **(C)** 7 **(D)** $\sqrt{61}$ **(E)** $\sqrt{91}$

8. A rectangle whose sides are parallel to the coordinate axes has all four of its vertices on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Among all such rectangles, what is the largest possible perimeter?

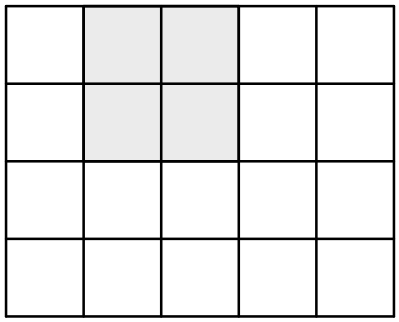
(A) 10 **(B)** $12\sqrt{2}$ **(C)** $\frac{96}{5}$ **(D)** $14\sqrt{2}$ **(E)** 20

9. A regular tetrahedron has edge length 6. The midpoints of its six edges are the six vertices of a regular octahedron. What is the volume of this octahedron?



(A) $\frac{9\sqrt{2}}{2}$ **(B)** $6\sqrt{2}$ **(C)** $9\sqrt{2}$ **(D)** $\frac{27\sqrt{2}}{2}$ **(E)** $18\sqrt{2}$

10. Each cell of a grid with 4 rows and 5 columns is coloured black or white. The colouring must put exactly 2 black cells in every 2×2 block of four cells (two adjacent rows and two adjacent columns); there are 12 such blocks. How many colourings are possible?

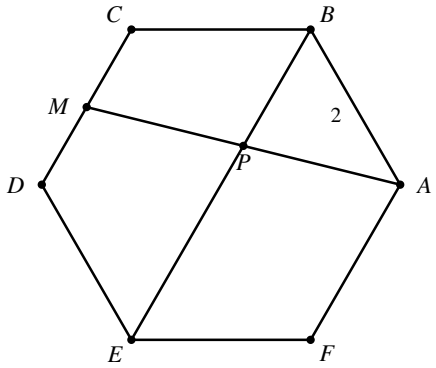


(A) 46 **(B)** 48 **(C)** 50 **(D)** 62 **(E)** 256

11. Find the sum of all real numbers x that satisfy $4 \cdot 27^x + 9 \cdot 8^x = 11 \cdot 18^x + 6 \cdot 12^x$.

(A) 0 **(B)** 1 **(C)** 2 **(D)** $\log_{3/2} 3$ **(E)** $\frac{11}{4}$

12. Regular hexagon $ABCDEF$ has side length 2, with its vertices labelled in order around it. Let M be the midpoint of side $\overline{CD}$. The segment $\overline{AM}$ crosses the long diagonal $\overline{BE}$ at a point P. What is the length BP?



(A) $\frac{1}{2}$ **(B)** $\frac{3}{4}$ **(C)** 1 **(D)** $\frac{4}{3}$ **(E)** $\frac{3}{2}$

13. Consider the equation $x^4 - 5x^3 + 8x^2 - 5x + 1 = 0$. What is the sum of all the different real numbers x that satisfy it? (A root that appears more than once is counted only once.)

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

14. An angle θ satisfies $\tan\theta = 2$. (The angle could lie in the first or the third quadrant; the answer is the same either way.) What is the value of

$$\frac{\sin^5\theta + \cos^5\theta}{\sin\theta + \cos\theta}\,?$$

(A) $\frac{1}{25}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{7}{25}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{11}{25}$

15. For some integers $n > 1$, the decimal expansion of $\frac{1}{n}$ is purely periodic: a block of digits starting right after the decimal point repeats forever, with nothing in front of it. For example, $\frac{1}{7} = 0.\overline{142857}$ is purely periodic, and its shortest repeating block, 142857, has 6 digits; but $\frac{1}{6} = 0.1\overline{6}$ is not purely periodic, and $\frac{1}{11} = 0.\overline{09}$ has a shortest repeating block of only 2 digits. For how many integers $n > 1$ is $\frac{1}{n}$ purely periodic with a shortest repeating block of exactly 6 digits?

(A) 48 **(B)** 53 **(C)** 56 **(D)** 58 **(E)** 64

16. The numbers $2^{120} - 1$ and $2^{84} - 1$ are both enormous (they have 37 and 26 digits), but they share some divisors; for example, both are odd multiples of 3. How many positive integers divide both $2^{120} - 1$ and $2^{84} - 1$?

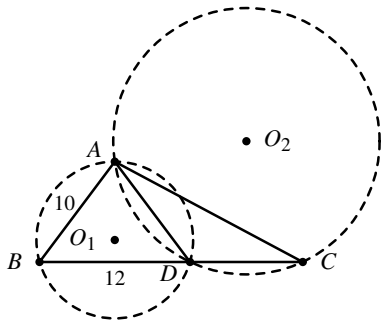
(A) 6 **(B)** 12 **(C)** 16 **(D)** 24 **(E)** 32

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17. Six different balls are to be placed into three boxes labelled 1, 2 and 3. Every ball goes into some box, no box may be left empty, and box 1 must end up holding more balls than box 2. How many such placements are there? (Two placements are different if some ball is in a different box.)

(A) 150 **(B)** 210 **(C)** 240 **(D)** 270 **(E)** 420

18. Triangle ABC has side lengths $AB = 10$, $BC = 21$ and $CA = 17$. Point D lies on side $\overline{BC}$ with $BD = 12$. Let O_1 be the centre of the circle through A, B and D, and let O_2 be the centre of the circle through A, C and D. What is the distance O_1O_2?



(A) $\frac{105}{8}$ **(B)** $\frac{135}{8}$ **(C)** $\frac{35}{2}$ **(D)** $\frac{147}{8}$ **(E)** $\frac{105}{4}$

19. A sequence is defined by $a_1 = 1$, $a_2 = 3$, and
$a_{n+2} = \dfrac{a_{n+1}^2 + 2}{a_n}$ for every $n \geq 1$.
So $a_3 = 11$, $a_4 = 41$, and it turns out that every term is a positive integer. What is the remainder when a_{2026} is divided by 11?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 10

20. Point $A = (5, 0)$ is fixed on the circle $x^2 + y^2 = 25$ centred at the origin O. Points B and C move along the same circle, always staying 8 units apart ($BC = 8$) and always different from A. Let H be the orthocenter of triangle ABC (the common point of its three altitudes). What is the greatest possible value of the distance OH?

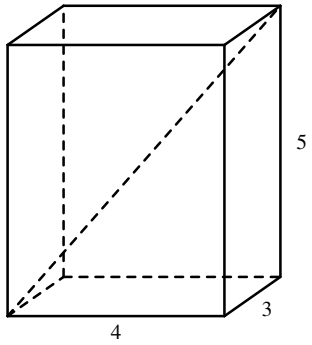
(A) 8 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 15

21. Two players take turns rolling one fair six-sided die, with the first player rolling first. A player loses the game as soon as they roll a number that has already come up earlier in the game (on a roll by either player). What is the probability that the first player loses?

(A) $\frac{155}{324}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{169}{324}$ **(D)** $\frac{5}{9}$ **(E)** $\frac{7}{12}$

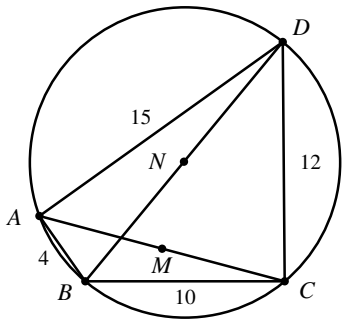
22. A solid rectangular box measures $3 \times 4 \times 5$. Parallel rays of light shine on it in the direction of one of its space diagonals (the segment joining two opposite corners of the box), and the box casts a shadow on a flat screen perpendicular to the rays. What is the area of the shadow?

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(A) 20 **(B)** $12\sqrt{3}$ **(C)** $9\sqrt{6}$ **(D)** $18\sqrt{2}$ **(E)** $\sqrt{769}$

23. Quadrilateral $ABCD$ is inscribed in a circle, with its vertices in that order around the circle, and its sides are $AB = 4$, $BC = 10$, $CD = 12$ and $DA = 15$. Let M be the midpoint of diagonal $\overline{AC}$ and N the midpoint of diagonal $\overline{BD}$. What is the length MN?



(A) $\frac{9}{2}$ **(B)** 5 **(C)** $\frac{11}{2}$ **(D)** $\frac{9\sqrt{2}}{2}$ **(E)** 9

24. A sequence is defined by $a_1 = 1$ and $a_{n+1} = a_n + \dfrac{1}{a_n}$ for $n \geq 1$, so $a_2 = 2$, $a_3 = \frac{5}{2}$, and the terms increase without bound. What is the smallest n for which $a_n > 63$?

(A) 1981 **(B)** 1982 **(C)** 1983 **(D)** 1984 **(E)** 1985

25. How many subsets of $\{1, 2, 3, \ldots, 24\}$, including the empty set, have the property that the sum of their elements is divisible by 13? (The empty set has sum 0.)

(A) 1290555 **(B)** 1290556 **(C)** 1290557 **(D)** 1290558 **(E)** 1290559

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Answer Key — AMC 12 Practice Contest 3

1. (D) $\frac{36}{\log_2(27/16)}$

2. (D) 256

3. (D) $\frac{6}{11}$

4. (B) 1200π

5. (E) 768

6. (C) $\frac{100}{3}$

7. (B) $\sqrt{31}$

8. (E) 20

9. (C) $9\sqrt{2}$

10. (A) 46

11. (C) 2

12. (E) $\frac{3}{2}$

13. (A) 4

14. (E) $\frac{11}{25}$

15. (B) 53

16. (D) 24

17. (B) 210

18. (A) $\frac{105}{8}$

19. (E) 10

20. (C) 11

21. (A) $\frac{155}{324}$

22. (D) $18\sqrt{2}$

23. (A) $\frac{9}{2}$

24. (C) 1983

25. (B) 1290556

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Solutions — AMC 12 Practice Contest 3

1. Answer: (D) $\frac{36}{\log_2(27/16)}$**. The idea.** Divide the two formulas so the question becomes "when does a single exponential equal 8?", then take $\log_2$ of both sides. The unknown sits in the exponents, so logarithms turn it into a linear equation.

Work it out. Setting $1000 \cdot 3^{t/4} = 8000 \cdot 2^{t/3}$ and dividing by $1000 \cdot 2^{t/3}$ gives $\frac{3^{t/4}}{2^{t/3}} = 8$. Take $\log_2$: $\frac{t}{4}\log_2 3 - \frac{t}{3} = 3$. Multiply by 12: $t\,(3\log_2 3 - 4) = 36$. Now $3\log_2 3 - 4 = \log_2 27 - \log_2 16 = \log_2 \frac{27}{16}$, so $t = \frac{36}{\log_2(27/16)}$.

Check. Per hour, A grows by the factor $3^{1/4} \approx 1.316$ and B by $2^{1/3} \approx 1.260$, so the ratio A/B grows by $\frac{3^{1/4}}{2^{1/3}}$ per hour, and $\log_2$ of that is $\frac{1}{12}\log_2 \frac{27}{16} \approx 0.0629$. A must gain a factor 8, i.e. 3 in $\log_2$, which takes $\frac{3}{0.0629} \approx 47.7$ hours, matching $\frac{36}{\log_2(27/16)} \approx \frac{36}{0.755} \approx 47.7$. (Swapping the two periods would give the wrong choice $\frac{36}{\log_2(81/8)}$.)

2. Answer: (D) 256**. The idea.** The top side of the square is a chord of the circle, parallel to ℓ, so it is bisected by the vertical line through the centre. That gives one right triangle with the radius as hypotenuse, and a single equation for the side s.

Work it out. Put ℓ on the x-axis and the centre at $(0, 10)$. The top vertices of a square of side s are at height s, and by symmetry they are $(\pm\frac{s}{2}, s)$. Being on the circle means $\left(\frac{s}{2}\right)^2 + (s-10)^2 = 100$, i.e. $\frac{s^2}{4} + s^2 - 20s = 0$, so $\frac{5}{4}s^2 = 20s$ and $s = 16$. The area is $16^2 = 256$.

Check. The corner $(8, 16)$ is at distance $\sqrt{8^2 + 6^2} = 10$ from the centre $(0, 10)$, as required. In general the side is $\frac{8}{5}$ of the radius: bigger than the square inscribed in the circle (area 200, side $10\sqrt{2} \approx 14.1$), because the bottom of this square sticks out of the circle at its corners.

3. Answer: (D) $\frac{6}{11}$**. The idea.** Organise the count by the smaller number a. Once a is fixed, the good partners are the numbers from $2a$ up to 12, and there are $12 - 2a + 1 = 13 - 2a$ of them (when $2a \le 12$).

Work it out. For $a = 1, 2, 3, 4, 5, 6$ the counts $13 - 2a$ are $11, 9, 7, 5, 3, 1$; for $a \ge 7$ there are none, since $2a > 12$. That is $11 + 9 + 7 + 5 + 3 + 1 = 36$ good pairs out of $\binom{12}{2} = 66$, so the probability is $\frac{36}{66} = \frac{6}{11}$.

Check. The sum of the first six odd numbers is $6^2 = 36$, as found. The complement is the 30 pairs with $a < b < 2a$: for $a = 2, \ldots, 12$ that is $\min(a-1, 12-a)$ partners, $1+2+3+4+5+5+4+3+2+1+0 = 30$, and $36 + 30 = 66$. Requiring the larger to be *more* than twice the smaller drops the 6 exact doubles and gives $\frac{30}{66} = \frac{5}{11}$; letting the two numbers be equal (so 78 pairs) gives $\frac{36}{78} = \frac{6}{13}$.

4. Answer: (B) 1200π**. The idea.** Only the right-angle vertex travels in a big circle. The altitude from it to the hypotenuse splits the triangle into two right triangles, and each one sweeps out a cone. So the solid is two cones glued base to base, sharing the radius h (the altitude to the hypotenuse), with heights that add up to the hypotenuse, so the two heights never need to be found separately.

Work it out. Twice the area is $15 \cdot 20 = 300 = 25h$, so $h = 12$. If the foot of the altitude cuts the hypotenuse into pieces a and b with $a + b = 25$, the volume is $\frac{1}{3}\pi h^2 a + \frac{1}{3}\pi h^2 b = \frac{1}{3}\pi \cdot 144 \cdot 25 = 1200\pi$.

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Check. The pieces are $a = \frac{15^2}{25} = 9$ and $b = \frac{20^2}{25} = 16$ (and $9^2 + 12^2 = 15^2$, $16^2 + 12^2 = 20^2$), giving cones of volume 432π and 768π, total 1200π. Pappus agrees: the centroid is $\frac{12}{3} = 4$ from the axis, so the volume is $150 \cdot 2\pi \cdot 4 = 1200\pi$. Spinning about the leg of length 20 or 15 instead gives 1500π or 2000π; forgetting the $\frac{1}{3}$ gives 3600π; using radius 12 with height 20 gives 960π.

5. Answer: (E) 768**. The idea.** A constant term of the product can come from two places: the constant term of $\left(x^2 - \frac{2}{x}\right)^9$ times 1, and the x^{-3} term of $\left(x^2 - \frac{2}{x}\right)^9$ times x^3. Find both with the general binomial term.

Work it out. The general term of $\left(x^2 - \frac{2}{x}\right)^9$ is $\binom{9}{k}(x^2)^{9-k}\left(-\frac{2}{x}\right)^k = \binom{9}{k}(-2)^k x^{18-3k}$. The exponent is 0 at $k = 6$: $\binom{9}{6} \cdot 64 = 84 \cdot 64 = 5376$. The exponent is -3 at $k = 7$: $\binom{9}{7} \cdot (-128) = 36 \cdot (-128) = -4608$. Multiplying by $1 + x^3$, the constant term is $5376 + (-4608) = 768$.

Check. Every exponent $18 - 3k$ is a multiple of 3, so the only terms of $\left(x^2 - \frac{2}{x}\right)^9$ that can pair with 1 or x^3 to give x^0 are x^0 and x^{-3}; nothing else contributes. Getting both signs backwards gives -768, and keeping only the x^{-3} term gives -4608.

6. Answer: (C) $\frac{100}{3}$**. The idea.** Since $A + B + C = 180°$, $\tan C = -\tan(A + B)$, which rearranges to the identity $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Integer solutions of $x + y + z = xyz$ that can be tangents of a triangle are very limited.

Work it out. At most one angle is obtuse, so at most one tangent is negative, and none is 0. If one were negative, say $x = -a$ with $a \geq 1$ and $y, z \geq 1$, then $y+z-a = -ayz$, i.e. $y+z = a(1-yz) \leq 0$, impossible. So all three are positive integers; order them $1 \leq x \leq y \leq z$. Then $xyz = x+y+z \leq 3z$ gives $xy \leq 3$, so (x, y) is $(1, 1)$, $(1, 2)$ or $(1, 3)$. $(1, 1)$ gives $2 + z = z$, impossible; $(1, 2)$ gives $3 + z = 2z$, $z = 3$; $(1, 3)$ gives $4 + z = 3z$, $z = 2 < y$, a repeat. So the tangents are $1, 2, 3$. The largest angle (tangent 3) is opposite the longest side, so place that side on a line with length 10 and drop the altitude h from the third vertex. The feet split the side into $\frac{h}{\tan 45°} + \frac{h}{2} = h + \frac{h}{2} = 10$, so $h = \frac{20}{3}$ and the area is $\frac{1}{2} \cdot 10 \cdot \frac{20}{3} = \frac{100}{3}$.

Check. With the base from $(0, 0)$ to $(10, 0)$, the lines $y = x$ and $y = -2(x - 10)$ meet at $\left(\frac{20}{3}, \frac{20}{3}\right)$. The two other sides have lengths $\frac{20\sqrt{2}}{3} \approx 9.43$ and $\frac{10\sqrt{5}}{3} \approx 7.45$, both shorter than 10, as they must be. The third angle has $\tan C = -\tan(A + B) = -\frac{1+2}{1-2} = 3$.

7. Answer: (B) $\sqrt{31}$**. The idea.** Every length we want is a combination of $|\vec{F}_1|^2$, $|\vec{F}_2|^2$ and the one unknown dot product $\vec{F}_1 \cdot \vec{F}_2$. The balance condition pins that dot product down: $\vec{F}_1 + \vec{F}_2 = -\vec{F}_3$ has length 7.

Work it out. Squaring, $|\vec{F}_1 + \vec{F}_2|^2 = 9 + 25 + 2\,\vec{F}_1 \cdot \vec{F}_2 = 49$, so $\vec{F}_1 \cdot \vec{F}_2 = \frac{15}{2}$. Then $|2\vec{F}_1 - \vec{F}_2|^2 = 4 \cdot 9 + 25 - 4\,\vec{F}_1 \cdot \vec{F}_2 = 36 + 25 - 30 = 31$, so the magnitude is $\sqrt{31}$.

Check. The dot product says $\cos\theta = \frac{15/2}{3 \cdot 5} = \frac{1}{2}$, a 60° angle between $\vec{F}_1$ and $\vec{F}_2$. Take $\vec{F}_1 = (3, 0)$ and $\vec{F}_2 = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$: then $\vec{F}_1 + \vec{F}_2 = \left(\frac{11}{2}, \frac{5\sqrt{3}}{2}\right)$ has length $\sqrt{\frac{121+75}{4}} = 7$, and $2\vec{F}_1 - \vec{F}_2 = \left(\frac{7}{2}, -\frac{5\sqrt{3}}{2}\right)$ has length $\sqrt{\frac{49+75}{4}} = \sqrt{31}$. A sign slip (+30 instead of −30) gives $\sqrt{91}$.

8. Answer: (E) 20**. The idea.** By symmetry the rectangle is determined by its first-quadrant

corner (x, y), and its perimeter is $4(x + y)$. Parametrize the ellipse as $(4\cos\theta, 3\sin\theta)$: then $x + y = 4\cos\theta + 3\sin\theta$, a single sinusoid whose largest value is $\sqrt{4^2 + 3^2}$.

Work it out. An axis-parallel rectangle inscribed in the ellipse is symmetric about both axes (its vertical sides are chords symmetric about the x-axis, and the horizontal ones about the y-axis), so its corners are $(\pm x, \pm y)$ and its perimeter is $4x + 4y$. With $x = 4\cos\theta$, $y = 3\sin\theta$ for $0 < \theta < \frac{\pi}{2}$, write $4\cos\theta + 3\sin\theta = 5\cos(\theta - \varphi)$ where $\cos\varphi = \frac{4}{5}$, $\sin\varphi = \frac{3}{5}$. Its maximum is 5, reached at $\theta = \varphi$, i.e. at the corner $\left(\frac{16}{5}, \frac{9}{5}\right)$. The largest perimeter is $4 \cdot 5 = 20$.

Check. The corner $\left(\frac{16}{5}, \frac{9}{5}\right)$ is on the ellipse: $\frac{256}{400} + \frac{81}{225} = 0.64 + 0.36 = 1$, and $4\left(\frac{16}{5} + \frac{9}{5}\right) = 20$. The rectangle of largest *area* (corner $(2\sqrt{2}, \frac{3\sqrt{2}}{2})$) has the smaller perimeter $14\sqrt{2} \approx 19.8$; the inscribed square (corner $\left(\frac{12}{5}, \frac{12}{5}\right)$) has perimeter $\frac{96}{5} = 19.2$; and taking the perimeter as $2(x + y)$ gives only 10.

9. Answer: (C) $9\sqrt{2}$**. The idea.** The octahedron is what is left of the tetrahedron after slicing off its four corners. Each corner piece is cut by the plane through the midpoints of the three edges at that vertex, so it is a regular tetrahedron with half the edge, and hence $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ of the volume.

Work it out. A regular tetrahedron of edge s has volume $\frac{s^3}{6\sqrt{2}}$, so edge 6 gives $\frac{216}{6\sqrt{2}} = \frac{36}{\sqrt{2}} = 18\sqrt{2}$. Removing four corners of $\frac{1}{8}$ each leaves $1 - \frac{4}{8} = \frac{1}{2}$ of it: $\frac{1}{2} \cdot 18\sqrt{2} = 9\sqrt{2}$.

Check. The octahedron's edges join midpoints of edges sharing a face, so they have length 3. A regular octahedron of edge e is two square pyramids of base e^2 and height $\frac{e}{\sqrt{2}}$, volume $\frac{\sqrt{2}}{3}e^3 = \frac{\sqrt{2}}{3} \cdot 27 = 9\sqrt{2}$, which agrees. Removing only two of the four corners would leave $\frac{3}{4} \cdot 18\sqrt{2} = \frac{27\sqrt{2}}{2}$; taking one of the two square pyramids alone gives $\frac{9\sqrt{2}}{2}$; and $18\sqrt{2}$ is the whole tetrahedron.

10. Answer: (A) 46**. The idea.** Look at how one row controls the row below it. Where two neighbouring cells of the upper row have the same colour, the two cells below them must both have the other colour; where they differ, the two cells below must differ too. So each new row is either a copy or the complement of the row above, and a copy is allowed only when the upper row alternates colours all the way along.

Work it out. Take a block with upper cells u_1, u_2 and lower cells d_1, d_2. If $u_1 = u_2$ (both black or both white), the block already has 2 or 0 black cells on top, so $d_1 = d_2$ must be the opposite colour. If $u_1 \neq u_2$, the top has one black cell, so the bottom needs exactly one: $d_1 \neq d_2$. In both cases $d_1 = d_2$ exactly when $u_1 = u_2$, so going along the row, the lower row changes colour exactly where the upper row does: it is a copy or the complement of the upper row. If the upper row has two equal neighbours anywhere, the copy fails at that block and only the complement works. If the upper row alternates, both the copy and the complement work, and both alternate again. Count by the top row. Top row alternating (2 choices): each of the other 3 rows is a copy or complement, $2 \cdot 2^3 = 16$ colourings. Top row not alternating ($2^5 - 2 = 30$ choices): every later row is forced to be the complement of the one above, 30 colourings. Total $16 + 30 = 46$.

Check. Swapping the roles of rows and columns gives the same count as $2^5 + 2^4 - 2 = 46$ (columns alternating or not), a good consistency check, and a computer search of all 2^{20} colourings finds exactly 46. Adding the 2 checkerboards instead of subtracting them gives $2^5 + 2^4 + 2 = 50$, and using 2^5 for the columns too gives 62; letting every row be copied or complemented freely gives $2^5 \cdot 2^3 = 256$.

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11. Answer: (C) 2. The idea. Every term is a product of three factors each equal to 2^x or 3^x: $27^x = (3^x)^3$, $18^x = (3^x)^2 2^x$, $12^x = 3^x(2^x)^2$, $8^x = (2^x)^3$. The equation is homogeneous of degree 3, so dividing by 8^x turns it into a cubic in the single variable $t = \left(\frac{3}{2}\right)^x > 0$.

Work it out. Divide by 8^x: $4t^3 + 9 = 11t^2 + 6t$, i.e. $4t^3 - 11t^2 - 6t + 9 = 0$. Testing $t = -1$: $-4 - 11 + 6 + 9 = 0$, so $4t^3 - 11t^2 - 6t + 9 = (t+1)(4t^2 - 15t + 9) = (t+1)(t-3)(4t-3)$. Since $t = \left(\frac{3}{2}\right)^x$ is positive, $t = -1$ is rejected and $t = 3$ or $t = \frac{3}{4}$. The corresponding x are $\log_{3/2} 3$ and $\log_{3/2} \frac{3}{4}$, whose sum is $\log_{3/2}\left(3 \cdot \frac{3}{4}\right) = \log_{3/2} \frac{9}{4} = 2$.

Check. Each positive t gives exactly one x, since $\left(\frac{3}{2}\right)^x$ is increasing, so there are exactly two real solutions. Their sum is 2 exactly because $\left(\frac{3}{2}\right)^{x_1+x_2} = t_1 t_2 = \frac{9}{4} = \left(\frac{3}{2}\right)^2$. Numerically $x_1 \approx 2.710$ and $x_2 \approx -0.710$. Adding all three t-roots gives the wrong choice $\frac{11}{4}$, and keeping only $t = 3$ gives $\log_{3/2} 3$.

12. Answer: (E) $\frac{3}{2}$. The idea. Put the hexagon's centre O at the origin. The long diagonal $\overline{BE}$ passes through O and has length 4, so P is found by intersecting two lines, and BP is read off as a fraction of BE.

Work it out. Take $A = (2, 0)$, $B = (1, \sqrt{3})$, $C = (-1, \sqrt{3})$, $D = (-2, 0)$, $E = (-1, -\sqrt{3})$. Then $M = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$. Points of $\overline{AM}$ are $\left(2 - \frac{7}{2}s, \frac{\sqrt{3}}{2}s\right)$ for $0 \le s \le 1$, and line BE is $y = \sqrt{3}\,x$. Setting $\frac{\sqrt{3}}{2}s = \sqrt{3}\left(2 - \frac{7}{2}s\right)$ gives $\frac{s}{2} = 2 - \frac{7s}{2}$, so $s = \frac{1}{2}$ and $P = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$. Then $BP = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3\sqrt{3}}{4}\right)^2} = \sqrt{\frac{9+27}{16}} = \frac{3}{2}$.

Check. Notice $P = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) = \frac{1}{4}B$ as a vector, so P lies on $\overline{OB}$ with $OP = \frac{1}{4} \cdot OB = \frac{1}{2}$, and $BP = OB - OP = 2 - \frac{1}{2} = \frac{3}{2}$. Also $s = \frac{1}{2}$ says P is the midpoint of $\overline{AM}$.

13. Answer: (A) 4. The idea. The coefficients $1, -5, 8, -5, 1$ read the same in both directions. If x is a root then so is $\frac{1}{x}$, which suggests dividing by x^2 and working with $y = x + \frac{1}{x}$.

Work it out. $x = 0$ is not a root, so divide by x^2: $\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 8 = 0$. With $y = x + \frac{1}{x}$ we have $x^2 + \frac{1}{x^2} = y^2 - 2$, so $y^2 - 5y + 6 = 0$, giving $y = 2$ or $y = 3$. From $y = 2$: $x^2 - 2x + 1 = 0$, so $x = 1$, a double root. From $y = 3$: $x^2 - 3x + 1 = 0$, so $x = \frac{3 \pm \sqrt{5}}{2}$, both real. The different real roots are 1, $\frac{3+\sqrt{5}}{2}$, $\frac{3-\sqrt{5}}{2}$, with sum $1 + 3 = 4$.

Check. By Vieta the four roots counted with multiplicity add to $5 = 1 + 1 + 3$, which matches; the trap answer 5 counts the double root twice. Also $(x-1)^2(x^2 - 3x + 1) = x^4 - 5x^3 + 8x^2 - 5x + 1$.

14. Answer: (E) $\frac{11}{25}$. The idea. Do not find $\sin\theta$ and $\cos\theta$ separately. Divide the sum of fifth powers by $\sin\theta + \cos\theta$ algebraically; what is left depends only on $p = \sin\theta\cos\theta$, and p can be read straight from $\tan\theta$.

Work it out. Write $s = \sin\theta$, $c = \cos\theta$. Since $\tan\theta \neq -1$, $s + c \neq 0$, and $s^5 + c^5 = (s+c)(s^4 - s^3 c + s^2 c^2 - s c^3 + c^4)$. The second factor is $(s^4 + c^4) - sc(s^2 + c^2) + s^2 c^2 = (1 - 2p^2) - p + p^2 = 1 - p - p^2$, using $s^2 + c^2 = 1$ and $s^4 + c^4 = (s^2 + c^2)^2 - 2s^2 c^2$. Next, $p = \dfrac{sc}{s^2 + c^2} = \dfrac{\tan\theta}{1 + \tan^2\theta} = \dfrac{2}{5}$ (divide top and bottom by c^2), whichever quadrant θ is in. So the value is $1 - \frac{2}{5} - \frac{4}{25} = \frac{11}{25}$.

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Check. In the first quadrant $s = \frac{2}{\sqrt{5}}$, $c = \frac{1}{\sqrt{5}}$, so $s^5 + c^5 = \frac{33}{25\sqrt{5}}$ and $s + c = \frac{3}{\sqrt{5}}$; the quotient is $\frac{33}{75} = \frac{11}{25}$. In the third quadrant both s and c change sign, and so do the numerator and the denominator.

15. Answer: (B) 53**. The idea.** A purely periodic decimal with a repeating block of length k is a fraction with denominator $10^k - 1$: $0.\overline{d_1 \dots d_k} = \frac{d_1 \dots d_k}{10^k - 1}$. So $\frac{1}{n}$ repeats in blocks of 6 (not necessarily shortest) exactly when $n \mid 10^6 - 1 = 999999$. The shortest block then divides 6, so it is exactly 6 unless $\frac{1}{n}$ already repeats in blocks of 2 or 3 (a block of 1 is also a block of 2), that is, unless $n \mid 99$ or $n \mid 999$.

Work it out. Indeed, if $\frac{1}{n} = \frac{m}{10^k - 1}$ then $n \mid 10^k - 1$; conversely if $n \mid 10^k - 1$ then $\frac{1}{n} = \frac{(10^k - 1)/n}{10^k - 1}$ repeats in blocks of k. Factor $999999 = 999 \cdot 1001 = 3^3 \cdot 37 \cdot 7 \cdot 11 \cdot 13$, which has $4 \cdot 2^4 = 64$ divisors. Remove the divisors of $999 = 3^3 \cdot 37$ (8 of them) and of $99 = 3^2 \cdot 11$ (6 of them), adding back the common divisors, which are the divisors of $\gcd(999, 99) = 9$ (3 of them): $64 - 8 - 6 + 3 = 53$. The removed set contains $n = 1$, so all 53 remaining values are greater than 1.

Check. $n = 7$: $7 \mid 999999$ but $7 \nmid 99$ and $7 \nmid 999$, and indeed $\frac{1}{7}$ has period 6. $n = 27$: $27 \mid 999$, and $\frac{1}{27} = 0.\overline{037}$ has period 3, correctly excluded. A computer check over all $n < 10^6$ coprime to 10, testing when the order of 10 modulo n is 6, also gives 53.

16. Answer: (D) 24**. The idea.** The common divisors of two numbers are exactly the divisors of their greatest common divisor, and numbers of the form $2^m - 1$ have a beautiful gcd rule: $\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1$. The rule comes from running the Euclidean algorithm on the exponents.

Work it out. If $m > n$, then $2^m - 1 = 2^{m-n}(2^n - 1) + (2^{m-n} - 1)$, so $\gcd(2^m - 1, 2^n - 1) = \gcd(2^n - 1, 2^{m-n} - 1)$: subtracting exponents does not change the gcd, exactly as in the Euclidean algorithm for $\gcd(m, n)$. Here $120 \to 84 \to (84, 36) \to (48, 36) \to (36, 12) \to \dots \to (12, 12)$, so the gcd is $2^{12} - 1 = 4095$. Factor: $4095 = 4096 - 1 = (2^6 - 1)(2^6 + 1) = 63 \cdot 65 = 3^2 \cdot 5 \cdot 7 \cdot 13$. Its number of divisors is $(2 + 1)(1 + 1)(1 + 1)(1 + 1) = 24$.

Check. Both numbers really are multiples of 4095: $2^{12} \equiv 1 \pmod{4095}$, and $120 = 12 \cdot 10$, $84 = 12 \cdot 7$, so $2^{120} \equiv 2^{84} \equiv 1$. The trap answers: 12 is $\gcd(120, 84)$ itself, 6 counts its divisors, and 16 forgets that 3 appears squared in 4095.

17. Answer: (B) 210**. The idea.** Count all placements with no empty box, remove the ones where boxes 1 and 2 hold equally many balls, and use symmetry: swapping the contents of boxes 1 and 2 matches each placement with more in box 1 to one with more in box 2.

Work it out. All placements: $3^6 = 729$. By inclusion-exclusion, those leaving some box empty number $3 \cdot 2^6 - 3 \cdot 1^6 = 189$, so $729 - 189 = 540$ placements leave no box empty. Ties between boxes 1 and 2: the sizes are $(1, 1, 4)$ or $(2, 2, 2)$. For $(1, 1, 4)$ choose the ball for box 1 and the ball for box 2: $6 \cdot 5 = 30$ ways. For $(2, 2, 2)$: $\binom{6}{2}\binom{4}{2} = 15 \cdot 6 = 90$ ways. So 120 placements are ties, and the rest split evenly: $\frac{540 - 120}{2} = 210$.

Check. Directly by sizes (n_1, n_2, n_3) with $n_1 > n_2 \geq 1$, $n_3 \geq 1$: $(2, 1, 3)$: 60; $(3, 1, 2)$: 60; $(4, 1, 1)$: 30; $(3, 2, 1)$: 60. Total 210. Halving 540 without removing ties gives the trap answer 270; forgetting to halve gives 420.

18. Answer: (A) $\frac{105}{8}$**. The idea.** Do not look for the two centres separately; look at the segment

$\overline{O_1O_2}$ from two directions. Each centre lies on the perpendicular bisector of its chord on line BC: O_1 is directly above the midpoint of $\overline{BD}$ and O_2 directly above the midpoint of $\overline{DC}$. Those midpoints are always $\frac{BD}{2} + \frac{DC}{2} = \frac{BC}{2}$ apart, so the shadow of $\overline{O_1O_2}$ on line BC has length $\frac{21}{2}$ wherever D is. And $\overline{O_1O_2}$ is perpendicular to the common chord $\overline{AD}$ of the two circles, which fixes its tilt.

Work it out. Let H be the foot of the altitude from A. From $AB^2 - BH^2 = AC^2 - CH^2$: $100 - BH^2 = 289 - (21 - BH)^2$, so $42\,BH = 252$, $BH = 6$ and $AH = 8$. Then $HD = 12 - 6 = 6$ and $AD = \sqrt{8^2 + 6^2} = 10$. Since $\overline{O_1O_2} \perp \overline{AD}$, the angle between $\overline{O_1O_2}$ and line BC equals the angle between $\overline{AD}$ and the altitude $\overline{AH}$, whose cosine is $\frac{AH}{AD} = \frac{8}{10} = \frac{4}{5}$. So the shadow $\frac{21}{2}$ equals $O_1O_2 \cdot \frac{4}{5}$, and $O_1O_2 = \frac{21}{2} \cdot \frac{5}{4} = \frac{105}{8}$.

Check. Coordinates $B = (0, 0)$, $C = (21, 0)$, $A = (6, 8)$, $D = (12, 0)$. O_1 has $x = 6$ and is equally far from B and A: $36 + y^2 = (y - 8)^2$ gives $y = \frac{7}{4}$. O_2 has $x = \frac{33}{2}$ and is equally far from D and C as from A: $\frac{81}{4} + y^2 = \frac{441}{4} + (y - 8)^2$ gives $y = \frac{77}{8}$. Then $O_1O_2^2 = \left(\frac{21}{2}\right)^2 + \left(\frac{63}{8}\right)^2 = \frac{7056+3969}{64} = \frac{11025}{64}$, so $O_1O_2 = \frac{105}{8}$. The traps: $\frac{105}{4}$ uses the whole of BC as the shadow, $\frac{147}{8} = \frac{21}{2} + \frac{63}{8}$ adds the two legs instead of using Pythagoras, $\frac{35}{2}$ uses $\frac{3}{5}$ in place of $\frac{4}{5}$, and $\frac{135}{8}$ is the sum of the two radii $\frac{25}{4} + \frac{85}{8}$.

19. Answer: (E) 10**. The idea.** The rule says $a_{n+2}a_n - a_{n+1}^2 = 2$ for every n. Writing the same identity one step earlier and subtracting uncovers a linear recurrence, which is easy to run modulo 11.

Work it out. From $a_{n+2}a_n - a_{n+1}^2 = 2 = a_{n+1}a_{n-1} - a_n^2$ we get $a_n(a_{n+2} + a_n) = a_{n+1}(a_{n+1} + a_{n-1})$, i.e. $\dfrac{a_{n+2} + a_n}{a_{n+1}} = \dfrac{a_{n+1} + a_{n-1}}{a_n}$. This ratio is therefore constant, equal to $\frac{a_3+a_1}{a_2} = \frac{12}{3} = 4$. So $a_{n+2} = 4a_{n+1} - a_n$ (which also explains why every term is an integer). Modulo 11, starting from a_1, a_2: $1, 3, 0, 8, 10, 10, 8, 0, 3, 1, 1, 3, \ldots$ Since $(a_{11}, a_{12}) \equiv (a_1, a_2)$, the sequence repeats with period 10. As $2026 \equiv 6 \pmod{10}$, $a_{2026} \equiv a_6 \equiv 10 \pmod{11}$.

Check. Directly: $a_5 = \frac{41^2+2}{11} = 153$ and $a_6 = \frac{153^2+2}{41} = 571 = 11 \cdot 51 + 10$, which agrees with both the linear rule ($4 \cdot 153 - 41 = 571$) and the residue 10. A computer running the original rule exactly up to a_{2026} gives the same remainder.

20. Answer: (C) 11**. The idea.** When the circumcentre is the origin, the orthocentre is simply $H = A + B + C$ (as vectors). Then the only thing that moves is $B + C$, which is twice the midpoint of the chord BC.

Work it out. Let $H = A+B+C$. Then $(H-A)\cdot(B-C) = (B+C)\cdot(B-C) = |B|^2 - |C|^2 = 25 - 25 = 0$, so line AH is perpendicular to BC: it is the altitude from A. By the same computation H is on the altitudes from B and C, so this H is the orthocentre. Let M be the midpoint of $\overline{BC}$. A chord of length 8 in a circle of radius 5 is at distance $\sqrt{25 - 16} = 3$ from the centre, so $|M| = 3$ and $H = A + 2M$ lies on the circle of radius 6 about A. Therefore $OH \le OA + 6 = 11$, with equality when $M = (3, 0)$, i.e. $B = (3, 4)$, $C = (3, -4)$, a genuine triangle.

Check. For $A = (5, 0)$, $B = (3, 4)$, $C = (3, -4)$: the altitude from A is the x-axis, and the altitude from B is perpendicular to $\overrightarrow{CA} = (2, 4)$, i.e. $2(x - 3) + 4(y - 4) = 0$; at $y = 0$ this gives $x = 11$, so $H = (11, 0)$. Also Euler's formula $OH^2 = 9R^2 - (a^2 + b^2 + c^2) = 225 - (64 + 20 + 20) = 121$.

21. Answer: (A) $\frac{155}{324}$**. The idea.** Forget the players and look only at the sequence of rolls: the game

ends at the first roll that repeats an earlier number, and the first player loses exactly when that roll has an odd position. The first repeat must come by roll 7, since only six numbers exist, so there are just a few cases.

Work it out. For roll k to be the first repeat, rolls $1, \dots, k-1$ must all be different, which has probability $\frac{6}{6} \cdot \frac{5}{6} \cdots \frac{6-k+2}{6}$, and then roll k must match one of those $k-1$ numbers, probability $\frac{k-1}{6}$. The first player rolls $k = 1, 3, 5, 7$ (roll 1 can never repeat).

$k = 3$: $\frac{5}{6} \cdot \frac{2}{6} = \frac{10}{36} = \frac{2160}{7776}$.

$k = 5$: $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{4}{6} = \frac{240}{1296} = \frac{1440}{7776}$.

$k = 7$: $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \cdot \frac{6}{6} = \frac{120}{7776}$.

The total is $\frac{3720}{7776} = \frac{155}{324}$.

Check. The even cases are $k = 2$: $\frac{1}{6} = \frac{1296}{7776}$; $k = 4$: $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{2160}{7776}$; $k = 6$: $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{5}{6} = \frac{600}{7776}$; these add to $\frac{4056}{7776} = \frac{169}{324}$, and $155 + 169 = 324$, so the game always ends. The first player is less likely to lose, even though they roll first, because the second player faces the first chance to repeat. Dropping the factor $\frac{1}{6}$ from the $k = 7$ case gives the trap $\frac{4320}{7776} = \frac{5}{9}$; mixing up the players gives $\frac{169}{324}$.

22. Answer: (D) $18\sqrt{2}$**. The idea.** Seen along the light direction, the shadow is covered exactly once by the three faces that face the light (the three faces at the lit corner), and the other three faces cover it again from behind. Projecting a flat face onto a perpendicular screen multiplies its area by $|\cos\varphi|$, where φ is the angle between the face's normal and the light.

Work it out. Put the box along the axes with edges $3, 4, 5$ along x, y, z. The light direction is the space diagonal $(3, 4, 5)$, of length $\sqrt{50} = 5\sqrt{2}$. The 4×5 face has normal along x, so its shadow has area $20 \cdot \frac{3}{5\sqrt{2}}$; the 3×5 face gives $15 \cdot \frac{4}{5\sqrt{2}}$; the 3×4 face gives $12 \cdot \frac{5}{5\sqrt{2}}$. Each is $\frac{60}{5\sqrt{2}}$, and the shadows of the three lit faces fit together without overlap into the whole shadow (a hexagon), so the area is $\frac{180}{5\sqrt{2}} = \frac{36}{\sqrt{2}} = 18\sqrt{2}$.

Check. For a unit cube the same method gives $3 \cdot \frac{1}{\sqrt{3}} = \sqrt{3}$, the well-known regular-hexagon shadow of side $\sqrt{2/3}$. Projecting all eight corners of the 3×4×5 box onto the screen and computing the area of their convex hull exactly also gives $18\sqrt{2} \approx 25.46$. The tempting $\sqrt{12^2 + 15^2 + 20^2} = \sqrt{769}$ is the largest shadow over all directions, not this one.

23. Answer: (A) $\frac{9}{2}$**. The idea.** For any quadrilateral, the sides, the diagonals and the segment MN are tied together by $AB^2+BC^2+CD^2+DA^2 = AC^2+BD^2+4MN^2$ (Euler's quadrilateral relation). So we only need the two diagonals, and for a cyclic quadrilateral Ptolemy's theorem gives their product while a sine-area argument gives their ratio.

Work it out. Relation: with vectors, $2(M - N) = A - B + C - D$. Expanding $|A - B + C - D|^2$ and comparing with $|A - B|^2 + |B - C|^2 + |C - D|^2 + |D - A|^2 - |A - C|^2 - |B - D|^2$ term by term shows they are equal (each $|X|^2$ appears with coefficient 1, and the dot products match).

Diagonals. Ptolemy: $AC \cdot BD = AB \cdot CD + BC \cdot DA = 48 + 150 = 198$. Opposite angles are supplementary, so $\sin D = \sin B$ and $\sin C = \sin A$. Computing the area twice, $[ABCD] = \frac{1}{2}\sin B\,(AB \cdot BC + CD \cdot DA) = \frac{1}{2}\sin A\,(AB \cdot AD + CB \cdot CD)$, and by the law of sines $AC : BD = \sin B : \sin A$. Hence $\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{AB \cdot BC + CD \cdot DA} = \frac{60+120}{40+180} = \frac{9}{11}$. So $AC^2 = 198 \cdot \frac{9}{11} = 162$ and $BD^2 = 198 \cdot \frac{11}{9} = 242$.

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Finally $4MN^2 = (16 + 100 + 144 + 225) - 162 - 242 = 81$, so $MN = \frac{9}{2}$.

Check. Law of cosines with $\cos D = -\cos B$: $AC^2 = 16+100-80\cos B = 225+144+360\cos B$ gives $\cos B = -\frac{253}{440}$ and $AC^2 = 116+46 = 162$, as found. Building the quadrilateral in coordinates and measuring MN directly also gives 4.5.

24. Answer: (C) 1983**. The idea.** Squaring the rule makes it almost additive: $a_{n+1}^2 = a_n^2 + 2 + \frac{1}{a_n^2}$. So a_n^2 is $2n-1$ plus a slowly growing error term $S_n = \sum_{k=1}^{n-1} \frac{1}{a_k^2}$, which behaves like $\frac{1}{2}\ln n$. The naive guess ignores S_n and says $2n-1 > 3969$, i.e. $n = 1985$. The real answer needs a two-sided estimate of S_n.

Work it out. Let $b_n = a_n^2$, so $b_n = 2n - 1 + S_n$ and $b_n \geq 2n - 1$. We need $b_n > 63^2 = 3969$.

Upper bound. $S_n \leq \sum_{k=1}^{n-1} \frac{1}{2k-1} \leq 1 + \int_1^{n-1} \frac{dx}{2x-1} = 1 + \frac{1}{2}\ln(2n-3)$. For $n = 1982$: $S_{1982} \leq 1 + \frac{1}{2}\ln 3961 < 1 + 4.15 < 6$, so $b_{1982} = 3963 + S_{1982} < 3969$ and $a_{1982} < 63$ (and so are all earlier terms, since the sequence increases).

Lower bound. The same estimate gives $S_k \leq 1 + \frac{1}{2}\ln 3963 < 6$ for every $k \leq 1983$, so $b_k = 2k - 1 + S_k < 2k + 5$ there, and $\frac{1}{a_k^2} > \frac{1}{2k+5}$. Using the exact first terms $b_1 = 1$, $b_2 = 4$, $b_3 = \frac{25}{4}$,

$$S_{1983} \geq 1 + \tfrac{1}{4} + \tfrac{4}{25} + \sum_{k=4}^{1982} \tfrac{1}{2k+5} \geq 1.41 + \tfrac{1}{2}\int_4^{1983} \tfrac{dx}{x+2.5} = 1.41 + \tfrac{1}{2}\ln\tfrac{1985.5}{6.5}.$$

Since $\frac{1985.5}{6.5} > 305 > e^{5.7}$, this is more than $1.41 + 2.85 > 4$. So $b_{1983} = 3965 + S_{1983} > 3969$ and $a_{1983} > 63$. The answer is 1983.

Check. Numerically $S_{1983} \approx 4.52$, so $a_{1983}^2 \approx 3969.5$ and $a_{1983} \approx 63.004$, while $a_{1982} \approx 62.99$. Running the recurrence to 50 significant digits confirms that 1983 is the first index past 63.

25. Answer: (B) 1290556**. The idea.** Use the generating function $f(x) = (1+x)(1+x^2)\cdots(1+x^{24})$, in which each subset contributes $x^{\text{its sum}}$, and the roots-of-unity filter: with $\omega = e^{2\pi i/13}$, the number of subsets with sum divisible by 13 is $\frac{1}{13}\sum_{j=0}^{12} f(\omega^j)$, because $\sum_j \omega^{jn}$ is 13 when $13 \mid n$ and 0 otherwise.

Work it out. $f(1) = 2^{24}$. Now fix $j \neq 0$ and let $z = \omega^j$, a primitive 13th root of unity. Since $x^{13} - 1 = \prod_{k=0}^{12}(x - z^k)$, putting $x = -1$ gives $-2 = -\prod_{k=0}^{12}(1 + z^k)$, so $\prod_{k=1}^{12}(1 + z^k) = 1$. Modulo 13 the numbers $1, \ldots, 24$ are: $1, \ldots, 12$ (product 1), then $13 \equiv 0$ (factor $1 + 1 = 2$), then $14, \ldots, 24 \equiv 1, \ldots, 11$ (product $\frac{1}{1+z^{12}} = \frac{1}{1+z^{-1}} = \frac{z}{1+z}$). So $f(z) = \frac{2z}{1+z} = 2 - \frac{2}{1+z}$. Pair z with z^{-1}: $\frac{1}{1+z} + \frac{1}{1+z^{-1}} = \frac{1}{1+z} + \frac{z}{z+1} = 1$. The 12 nontrivial roots form 6 such pairs, so $\sum_{j=1}^{12} \frac{1}{1+\omega^j} = 6$ and $\sum_{j=1}^{12} f(\omega^j) = 24 - 12 = 12$. The count is $\frac{2^{24}+12}{13} = \frac{16777228}{13} = 1290556$.

Check. The same method on $\{1, \ldots, 12\}$ alone gives $\frac{2^{12}+12}{13} = 316$, which a quick computer count confirms; for $\{1, \ldots, 24\}$ a dynamic-programming count of subsets by their sum modulo 13 also gives 1290556. Forgetting the empty set gives 1290555, and the plain guess $\frac{2^{24}}{13} \approx 1290555.1$ does not even land on an integer.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.