MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

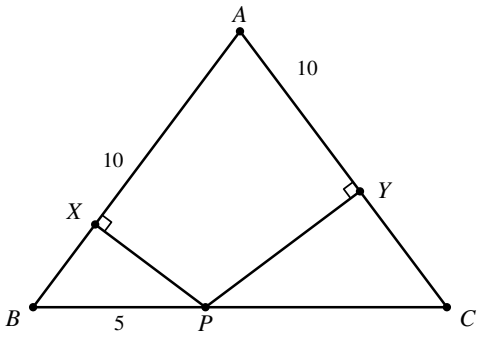
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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Positive real numbers a and b, neither equal to 1, satisfy $a < b$, $\log_a b + \log_b a = \frac{10}{3}$, and $ab = 81$. What is $a + b$?

(A) 12 **(B)** 28 **(C)** 30 **(D)** 36 **(E)** 84

2. Triangle ABC has $AB = AC = 10$ and $BC = 12$. Point P lies on side $\overline{BC}$ with $BP = 5$. Let X be the foot of the perpendicular from P to line AB, and let Y be the foot of the perpendicular from P to line AC. What is $PX + PY$?

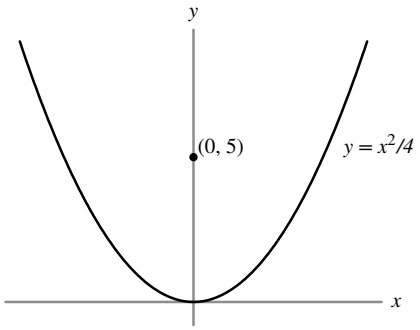


(A) $\frac{24}{5}$ **(B)** 8 **(C)** 9 **(D)** $\frac{48}{5}$ **(E)** 10

3. Euler noticed that $n^2 + n + 41$ is prime for $n = 0, 1, \ldots, 39$. For how many integers n with $1 \le n \le 2026$ is $n^2 + n + 41$ divisible by 43?

(A) 2 **(B)** 47 **(C)** 48 **(D)** 94 **(E)** 95

4. What is the shortest distance from the point $(0, 5)$ to a point on the parabola $y = \frac{x^2}{4}$?



(A) $2\sqrt{3}$ **(B)** 4 **(C)** $2\sqrt{5}$ **(D)** $\sqrt{21}$ **(E)** 5

5. A real number θ satisfies $\sin 2\theta = \frac{1}{3}$. What is the value of $\sin^6\theta + \cos^6\theta$? (The answer does not depend on which such θ is chosen.)

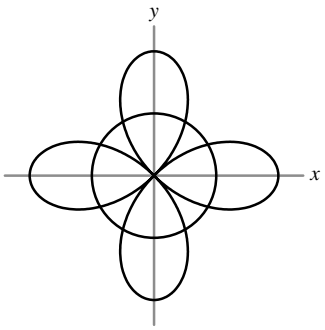
(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{11}{12}$ **(E)** $\frac{35}{36}$

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6. For a certain real number k, the equation $9^x - k \cdot 3^x + 27 = 0$ has exactly two real solutions, and one of these solutions is twice the other. What is k?

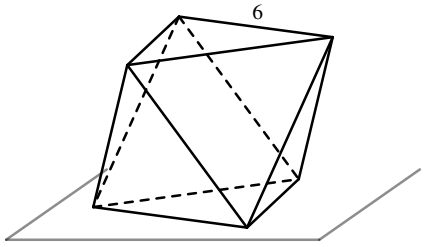
(A) 12 **(B)** 15 **(C)** 18 **(D)** 27 **(E)** 36

7. The polar curve $r = \cos 2\theta$ is the set of all points $(r\cos\theta,\ r\sin\theta)$ with $r = \cos 2\theta$, as θ ranges over all real numbers (so r is allowed to be negative); it is a rose with four petals. The circle $x^2 + y^2 = \frac{1}{4}$ meets this curve at several points, and these points are the vertices of a convex polygon. What is the area of this polygon?



(A) $\frac{1+\sqrt{3}}{4}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{3}}{2}$ **(D)** $\frac{2+\sqrt{3}}{4}$ **(E)** $\frac{1+\sqrt{3}}{2}$

8. A regular octahedron has 8 faces, each an equilateral triangle with side length 6, and 6 vertices, with four faces meeting at each vertex. It is set down on a level table so that one of its faces lies flat on the table. How high above the table is the highest point of the octahedron?



(A) $3\sqrt{2}$ **(B)** $2\sqrt{6}$ **(C)** $3\sqrt{3}$ **(D)** $4\sqrt{2}$ **(E)** $6\sqrt{2}$

9. How many integers n satisfy $(n-1)(n-4)(n-9)(n-16)\cdots(n-100) < 0$, where the product has the ten factors $n - k^2$ for $k = 1, 2, \ldots, 10$?

(A) 40 **(B)** 45 **(C)** 50 **(D)** 60 **(E)** 90

10. Two numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that there is an obtuse triangle whose three side lengths are x, y and 1?

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(A) $1 - \frac{\pi}{4}$ **(B)** $\frac{\pi}{4} - \frac{1}{2}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\pi}{2} - 1$ **(E)** $\frac{\pi}{4}$

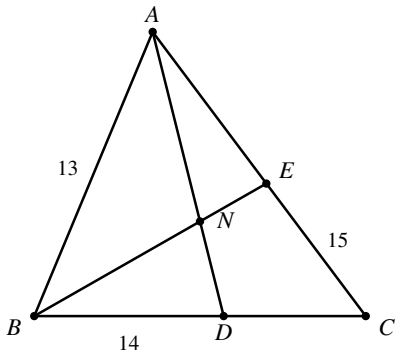
11. A regular 12-sided polygon is inscribed in a circle of radius 1. Consider every segment joining two of its vertices, counting all sides and all diagonals, each segment once. What is the sum of the squares of the lengths of all these segments?

(A) 66 **(B)** 72 **(C)** 132 **(D)** 144 **(E)** 288

12. How many real numbers x satisfy the equation $2^x = x^2 + 1$? (Count every real solution, including any that are not integers.)

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

13. A triangular field ABC has sides $AB = 13$, $BC = 14$ and $CA = 15$, and its area is 84. A straight path runs from corner A to a point D on side $\overline{BC}$, chosen so that the path splits the fence around the field into two equal lengths: $AB + BD = AC + CD$. A second straight path runs from corner B to a point E on side $\overline{CA}$ chosen the same way, so that $BA + AE = BC + CE$. The two paths cross at a point N. What is the area of triangle ABN?



(A) 32 **(B)** 36 **(C)** 40 **(D)** 42 **(E)** 48

14. What is the value of $\displaystyle\sum_{k=0}^{10} k\binom{10}{k}^2 = 0\cdot\binom{10}{0}^2 + 1\cdot\binom{10}{1}^2 + 2\cdot\binom{10}{2}^2 + \cdots + 10\cdot\binom{10}{10}^2$?

(A) 92378 **(B)** 184756 **(C)** 461890 **(D)** 923780 **(E)** 1847560

15. A string of length 6 is formed using only the characters 0, 1 and 2 (repetition allowed; for example `012210`). How many such strings contain each of the three characters 0, 1 and 2 an even number of times? (A character that does not appear at all appears 0 times, which counts as even.)

(A) 93 **(B)** 138 **(C)** 180 **(D)** 183 **(E)** 365

16. How many real numbers x with $0 < x < 2\pi$ satisfy $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$, where all three of $\tan x$, $\tan 2x$ and $\tan 3x$ must be defined?

(A) 3 **(B)** 5 **(C)** 6 **(D)** 9 **(E)** 11

17. A fair coin is flipped repeatedly. Flipping stops as soon as the number of heads so far exceeds the number

of tails so far by 4, or the number of tails so far exceeds the number of heads so far by 2. What is the expected number of flips?

(A) 8 **(B)** 9 **(C)** 16 **(D)** 20 **(E)** 36

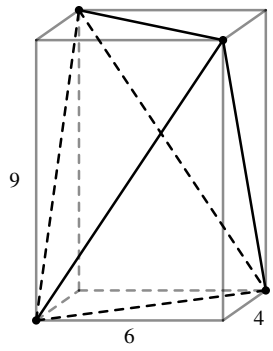
18. The equation
$x^4 - 15x^3 + ax^2 - 120x + 64 = 0$
has four real roots (counted with multiplicity) that form a geometric progression, where a is a real constant. What is a?

(A) 54 **(B)** 62 **(C)** 70 **(D)** 85 **(E)** 140

19. A dial has 360 marks, numbered $0, 1, \ldots, 359$ in order around a circle. For a step size k, a pointer placed on a mark repeatedly jumps k marks clockwise. The marks the pointer can ever land on, starting from a given mark, form that mark's orbit, and the 360 marks split into $N(k)$ separate orbits. (For example, $N(1) = 1$, $N(2) = 2$, and $N(360) = 360$ because then the pointer never moves.) What is $N(1) + N(2) + \cdots + N(360)$?

(A) 1170 **(B)** 3420 **(C)** 3780 **(D)** 4320 **(E)** 8640

20. A rectangular box measures $4 \times 6 \times 9$. Its eight corners can be split into two groups of four so that no two corners in the same group are joined by an edge of the box. The four corners of each group are the vertices of a tetrahedron, so this gives two tetrahedra inside the box. What is the volume of the region that lies inside both tetrahedra?



(A) 27 **(B)** 36 **(C)** 45 **(D)** 54 **(E)** 72

21. Eight club members each write their own name on a card. The cards are shuffled and each member draws one card, so that all 8! ways of handing out the cards are equally likely. Now draw an arrow from each member to the member whose name is on the card they drew. The arrows split the members into cycles (a member who draws their own card is a cycle of length 1). What is the probability that some cycle contains at least 5 members?

(A) $\frac{307}{840}$ **(B)** $\frac{73}{168}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{107}{210}$ **(E)** $\frac{533}{840}$

22. Three complex numbers x, y, z satisfy
$x + y + z = 3, \quad x^2 + y^2 + z^2 = 5, \quad x^3 + y^3 + z^3 = 7.$
What is the value of $(x^2 + 1)(y^2 + 1)(z^2 + 1)$?

(A) $\frac{58}{9}$ **(B)** 10 **(C)** $\frac{94}{9}$ **(D)** 14 **(E)** $\frac{130}{9}$

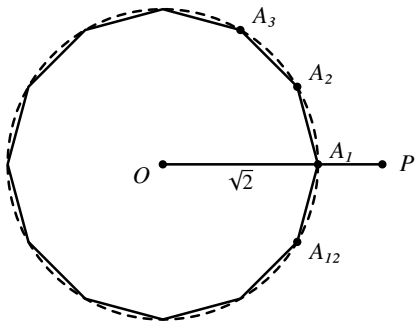
23. There are five towns. For each of the 10 pairs of towns, a fair coin is tossed, and a direct road joining those two towns is built if the coin shows heads; the 10 tosses are independent. (Roads meet only at the towns.) What is the probability that, using the roads that are built, one can travel from every town to every other town?

(A) $\frac{21}{32}$ **(B)** $\frac{43}{64}$ **(C)** $\frac{11}{16}$ **(D)** $\frac{45}{64}$ **(E)** $\frac{91}{128}$

24. The number $p = 2027$ is prime. Call an integer k a square remainder if there is an integer m such that $m^2 - k$ is divisible by 2027. For how many integers k with $1 \le k \le 2025$ are both k and $k + 1$ square remainders?

(A) 253 **(B)** 505 **(C)** 506 **(D)** 507 **(E)** 1012

25. A regular dodecagon (12-sided polygon) $A_1A_2 \ldots A_{12}$ is inscribed in a circle of radius 1 with centre O. The point P lies on ray OA_1, outside the circle, with $OP = \sqrt{2}$. What is the value of $\frac{1}{PA_1^2} + \frac{1}{PA_2^2} + \cdots + \frac{1}{PA_{12}^2}$?



(A) 4 **(B)** 6 **(C)** 12 **(D)** $\frac{256}{21}$ **(E)** $\frac{260}{21}$

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Answer Key — AMC 12 Practice Contest 2

1. (C) 30

2. (D) $\frac{48}{5}$

3. (E) 95

4. (B) 4

5. (D) $\frac{11}{12}$

6. (A) 12

7. (A) $\frac{1+\sqrt{3}}{4}$

8. (B) $2\sqrt{6}$

9. (C) 50

10. (B) $\frac{\pi}{4} - \frac{1}{2}$

11. (D) 144

12. (A) 3

13. (A) 32

14. (D) 923780

15. (D) 183

16. (B) 5

17. (A) 8

18. (C) 70

19. (C) 3780

20. (B) 36

21. (E) $\frac{533}{840}$

22. (E) $\frac{130}{9}$

23. (E) $\frac{91}{128}$

24. (C) 506

25. (E) $\frac{260}{21}$

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Solutions — AMC 12 Practice Contest 2

1. Answer: (C) 30**. The idea.** The two logarithms are reciprocals of each other: $\log_b a = \frac{1}{\log_a b}$. So the first equation is really a quadratic in the single unknown $t = \log_a b$, and once we know t we know b as a power of a.

Work it out. With $t = \log_a b$ the equation reads $t + \frac{1}{t} = \frac{10}{3}$, i.e. $3t^2 - 10t + 3 = 0$, so $(3t-1)(t-3) = 0$ and $t = 3$ or $t = \frac{1}{3}$. If $t = 3$ then $b = a^3$, and $ab = a^4 = 81$ gives $a = 3$, $b = 27$; here $a < b$, as required. If $t = \frac{1}{3}$ then $a = b^3$, and $b^4 = 81$ gives $b = 3$, $a = 27$, which breaks $a < b$. So $a = 3$, $b = 27$ and $a + b = 30$.

Check. $\log_3 27 = 3$ and $\log_{27} 3 = \frac{1}{3}$, which add to $\frac{10}{3}$; and $3 \cdot 27 = 81$. (The choice 84 comes from guessing $a = 3$, $b = 81$, which fails $ab = 81$.)

2. Answer: (D) $\frac{48}{5}$**. The idea.** Segment AP splits triangle ABC into triangles ABP and ACP. Their heights from P are exactly PX and PY, and their bases AB and AC are equal. So the two areas add up to $\frac{1}{2} \cdot 10 \cdot (PX + PY)$, which must equal the area of ABC. The position of P does not even matter.

Work it out. The altitude from A lands at the midpoint M of $\overline{BC}$, with $BM = 6$, so $AM = \sqrt{10^2 - 6^2} = 8$ and $[ABC] = \frac{1}{2} \cdot 12 \cdot 8 = 48$. Then $[ABP] + [ACP] = \frac{1}{2} \cdot AB \cdot PX + \frac{1}{2} \cdot AC \cdot PY = 5(PX + PY) = 48$, so $PX + PY = \frac{48}{5}$.

Check. Directly: $\sin B = \frac{AM}{AB} = \frac{8}{10}$, so $PX = PB \sin B = 5 \cdot \frac{4}{5} = 4$ and $PY = PC \sin C = 7 \cdot \frac{4}{5} = \frac{28}{5}$. The sum is $\frac{20+28}{5} = \frac{48}{5}$. It also equals the altitude from B to AC (put P at B), which is $\frac{2 \cdot 48}{10}$.

3. Answer: (E) 95**. The idea.** Divisibility by 43 only depends on n mod 43, so first find which residues work. Solve the quadratic congruence the way you solve a quadratic equation: complete the square. Because 43 is prime, a square can be $\equiv 9$ only if the number itself is $\equiv \pm 3$, so there are exactly two good residues.

Work it out. Multiply by 4: $4(n^2+n+41) = (2n+1)^2 + 163$, and $163 = 4 \cdot 43 - 9$. So $43 \mid n^2+n+41$ exactly when $(2n+1)^2 \equiv 9 \pmod{43}$ (multiplying by 4 changes nothing, since $\gcd(4, 43) = 1$). That means $43 \mid (2n + 1 - 3)(2n + 1 + 3)$, and since 43 is prime, $2n + 1 \equiv 3$ or $2n + 1 \equiv -3$. The first gives $2n \equiv 2$, so $n \equiv 1$; the second gives $2n \equiv -4$, so $n \equiv -2 \equiv 41 \pmod{43}$. Now count. Since $43 \cdot 47 = 2021$: the numbers $n \equiv 1$ are $1, 44, \ldots, 2022$, which is 48 numbers; the numbers $n \equiv 41$ are $41, 84, \ldots, 2019$, which is 47 numbers (the next one, 2062, is too big). Total $48 + 47 = 95$.

Check. $n = 1$: $1 + 1 + 41 = 43$. $n = 41$: $1681 + 41 + 41 = 1763 = 43 \cdot 41$. The two residues also add to $42 \equiv -1$, as Vieta's formula for $n^2 + n + 41$ predicts. Counting 47 in both classes gives 94; counting only one class gives 47 or 48; and 2 is just the number of good residues, not the number of n.

4. Answer: (B) 4**. The idea.** The obvious guess, the vertex $(0, 0)$, is 5 away, but the parabola is wide near its bottom, so points off to the side can be closer. Write the squared distance to a general point $\left(x, \frac{x^2}{4}\right)$; it only involves x^2, so with $t = x^2$ it is a quadratic in t, and we minimise that quadratic over $t \geq 0$.

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Work it out. The squared distance is $x^2 + \left(\frac{x^2}{4} - 5\right)^2 = t + \frac{t^2}{16} - \frac{5t}{2} + 25 = \frac{t^2}{16} - \frac{3t}{2} + 25$. This parabola in t has its lowest point at $t = \frac{3/2}{2/16} = 12$, which is allowed since $12 \geq 0$. There the squared distance is $\frac{144}{16} - 18 + 25 = 16$, so the shortest distance is 4, reached at $\left(\pm 2\sqrt{3}, 3\right)$.

Check. The point $(2\sqrt{3}, 3)$ is on the parabola since $\frac{12}{4} = 3$, and its distance from $(0, 5)$ is $\sqrt{12 + 4} = 4 < 5$. The slope of the segment to it is $\frac{3-5}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$ and the tangent slope there is $\frac{x}{2} = \sqrt{3}$; their product is -1, so the segment meets the parabola at a right angle, as a shortest segment must. (Answering $2\sqrt{3}$ reports x instead of the distance; $2\sqrt{5}$ is the distance to $(2, 1)$.)

5. Answer: (D) $\frac{11}{12}$. The idea. Let $a = \sin^2\theta$ and $b = \cos^2\theta$. Then $a + b = 1$, and the target is $a^3 + b^3$, a symmetric expression, so it depends only on $a + b$ and ab. And $ab = (\sin\theta\cos\theta)^2$ is controlled by $\sin 2\theta$.

Work it out. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 1 - 3ab$. Since $\sin 2\theta = 2\sin\theta\cos\theta$, we have $\sin\theta\cos\theta = \frac{1}{6}$ and $ab = \frac{1}{36}$. So $\sin^6\theta + \cos^6\theta = 1 - \frac{3}{36} = 1 - \frac{1}{12} = \frac{11}{12}$.

Check. The general identity is $\sin^6\theta + \cos^6\theta = 1 - \frac{3}{4}\sin^2 2\theta$; at $\theta = 0$ it gives 1 (correct: $0 + 1$) and at $\theta = 45°$ it gives $\frac{1}{4}$ (correct: $2 \cdot \frac{1}{8}$). With $\sin^2 2\theta = \frac{1}{9}$ it gives $1 - \frac{1}{12} = \frac{11}{12}$.

6. Answer: (A) 12. The idea. Substitute $t = 3^x$. The equation becomes the quadratic $t^2 - kt + 27 = 0$, and each positive root t gives exactly one solution $x = \log_3 t$. The product of the two roots is 27, and a product of powers of 3 is a power of 3 with the exponents *added*: that pins down the sum of the two solutions.

Work it out. Let the solutions be x_1 and x_2, with roots $t_1 = 3^{x_1}$, $t_2 = 3^{x_2}$. By Vieta, $t_1 t_2 = 27$, so $3^{x_1 + x_2} = 3^3$ and $x_1 + x_2 = 3$. With $x_1 = 2x_2$ we get $3x_2 = 3$, so $x_2 = 1$ and $x_1 = 2$. Then $t_2 = 3$, $t_1 = 9$ and $k = t_1 + t_2 = 12$.

Check. For $k = 12$: $t^2 - 12t + 27 = (t - 3)(t - 9)$, so $3^x = 3$ or $3^x = 9$, i.e. $x = 1$ or $x = 2$, and indeed $2 = 2 \cdot 1$. Both roots are positive, so neither is lost.

7. Answer: (A) $\frac{1+\sqrt{3}}{4}$. The idea. A negative r means stepping backwards: the point lies at distance $|r|$ in the direction $\theta + 180°$. Since $\cos 2(\theta + 180°) = \cos 2\theta$, a point at true distance ρ and true direction φ lies on the rose exactly when $\rho = |\cos 2\varphi|$. So the petals from negative r are just as real as the others, and the intersection points are where $|\cos 2\varphi| = \frac{1}{2}$.

Work it out. On the circle $\rho = \frac{1}{2}$, so we need $\cos 2\varphi = \pm\frac{1}{2}$ with $0° \leq \varphi < 360°$. $\cos 2\varphi = \frac{1}{2}$ gives $\varphi = 30°, 150°, 210°, 330°$; $\cos 2\varphi = -\frac{1}{2}$ gives $\varphi = 60°, 120°, 240°, 300°$. So there are 8 points on the circle of radius $\frac{1}{2}$, and going around, the central angles between neighbours alternate $30°, 60°, 30°, 60°, \ldots$. Cut the octagon into 8 triangles from the centre; a triangle with two sides $\frac{1}{2}$ and angle α has area $\frac{1}{2} \cdot \frac{1}{4}\sin\alpha$. The total is $\frac{1}{8}(4\sin 30° + 4\sin 60°) = \frac{1}{8}(2 + 2\sqrt{3}) = \frac{1+\sqrt{3}}{4}$.

Check. The points are $\left(\pm\frac{\sqrt{3}}{4}, \pm\frac{1}{4}\right)$ and $\left(\pm\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right)$. The first four alone (all you get from $r = +\frac{1}{2}$) form a $\frac{\sqrt{3}}{2} \times \frac{1}{2}$ rectangle of area $\frac{\sqrt{3}}{4}$, and the other four cap its two long sides (top and bottom) with two trapezoids. The top one has parallel sides $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ and height $\frac{\sqrt{3}}{4} - \frac{1}{4}$, so its area is $\frac{1}{2}\left(\frac{\sqrt{3}+1}{2}\right)\frac{\sqrt{3}-1}{4} = \frac{2}{16} = \frac{1}{8}$. Total: $\frac{\sqrt{3}}{4} + 2 \cdot \frac{1}{8} = \frac{1+\sqrt{3}}{4}$. Guessing a regular octagon would give $\frac{\sqrt{2}}{2}$ instead.

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8. Answer: (B) $2\sqrt{6}$**. The idea.** Place the octahedron with its vertices on the coordinate axes. Then every face lies in a plane $\pm x \pm y \pm z = c$, the face opposite the bottom face is parallel to it, and the height is simply the distance between two parallel planes.

Work it out. Put the vertices at $(\pm a, 0, 0)$, $(0, \pm a, 0)$, $(0, 0, \pm a)$. Neighbouring vertices such as $(a, 0, 0)$ and $(0, a, 0)$ are $a\sqrt{2}$ apart, so $a\sqrt{2} = 6$ and $a = 3\sqrt{2}$. The face with vertices $(a, 0, 0)$, $(0, a, 0)$, $(0, 0, a)$ lies in the plane $x+y+z = a$, and the opposite face lies in $x+y+z = -a$. These planes are parallel, and every vertex lies between them, so when one of them is the table, the other is the top. Their distance apart is $\frac{2a}{\sqrt{3}} = \frac{6\sqrt{2}}{\sqrt{3}} = 2\sqrt{6}$.

Check. By volume: the octahedron is two square pyramids with base side 6 and height $3\sqrt{2}$, so its volume is $2 \cdot \frac{1}{3} \cdot 36 \cdot 3\sqrt{2} = 72\sqrt{2}$. Its insphere touches all 8 faces, of total area $8 \cdot 9\sqrt{3} = 72\sqrt{3}$, so its inradius is $\frac{3 \cdot 72\sqrt{2}}{72\sqrt{3}} = \sqrt{6}$, and the height between opposite faces is twice that, $2\sqrt{6} \approx 4.9$. (Standing on a vertex it would be $6\sqrt{2}$ tall instead.)

9. Answer: (C) 50**. The idea.** Don't expand anything: the sign of a product is decided by how many factors are negative. Walk n down the number line from far to the right, where every factor is positive; each time n passes below a square k^2, one more factor turns negative and the sign flips.

Work it out. If n equals some k^2 the product is 0, so ignore those. For $n > 100$ no factor is negative, so the product is positive. For $81 < n < 100$ exactly one factor $(n - 100)$ is negative: product negative. For $64 < n < 81$ two are negative: positive. The signs keep alternating, so the product is negative exactly on $(81, 100)$, $(49, 64)$, $(25, 36)$, $(9, 16)$, $(1, 4)$, and for $n < 1$ all ten factors are negative, which makes the product positive. The integers strictly inside these gaps number $18 + 14 + 10 + 6 + 2 = 50$ (strictly between k^2 and $(k+1)^2$ lie $2k$ integers).

Check. $n = 2$: factors $1, -2, -7, \ldots, -98$, one positive and nine negative, so the product is negative. The other gaps $(4, 9)$, $(16, 25)$, $(36, 49)$, $(64, 81)$ hold $4 + 8 + 12 + 16 = 40$ integers, the wrong-parity answer; 90 counts every non-square from 1 to 100.

10. Answer: (B) $\frac{\pi}{4} - \frac{1}{2}$**. The idea.** Think of (x, y) as a random point of the unit square, so a probability is an area. Since $x, y \le 1$, the side 1 is the longest, so the triangle exists when $x + y > 1$ and is obtuse exactly when the angle opposite the side 1 is obtuse, i.e. when $x^2 + y^2 < 1$ (law of cosines).

Work it out. The condition $x + y > 1$ keeps the part of the square above the line from $(1, 0)$ to $(0, 1)$. The condition $x^2 + y^2 < 1$ keeps the part inside the quarter circle of radius 1 centred at the origin. The region satisfying both lies between the chord and the arc: its area is (quarter disc) minus (triangle under the chord) $= \frac{\pi}{4} - \frac{1}{2}$.

Check. $\frac{\pi}{4} - \frac{1}{2} \approx 0.285$. The complementary cases also add up: no triangle has probability $\frac{1}{2}$, and an acute (or right) triangle has probability $1 - \frac{\pi}{4} \approx 0.215$; $0.5 + 0.285 + 0.215 = 1$.

11. Answer: (D) 144**. The idea.** Fix one vertex. The squared distance to the vertex k steps away is, by the law of cosines, $1 + 1 - 2\cos(30k^\circ) = 2 - 2\cos(30k^\circ)$. Summing over all the other vertices, the cosines almost cancel, because the cosines of the twelve equally spaced angles $0^\circ, 30^\circ, \ldots, 330^\circ$ add to 0.

Work it out. From one vertex, the total is $\sum_{k=1}^{11}(2 - 2\cos 30k^\circ) = 22 - 2\sum_{k=1}^{11}\cos 30k^\circ$. Since $\sum_{k=0}^{11}\cos 30k^\circ = 0$ and the $k = 0$ term is 1, the sum from $k = 1$ to 11 is -1. So each vertex

contributes $22+2=24$. Adding over all 12 vertices counts every segment twice (once from each end), so the answer is $\frac{12\cdot 24}{2}=144$.

Check. With vectors: for unit vectors $v_1,\dots,v_{12}$ summing to 0, $\sum_{i<j}|v_i-v_j|^2 = 12\sum|v_i|^2 - |\sum v_i|^2 = 12\cdot 12-0=144$. Forgetting the $k=0$ correction gives 132, and counting each segment from both ends gives 288.

12. Answer: (A) 3**. The idea.** Two solutions are easy to spot, $x=0$ and $x=1$. The danger is stopping there: 2^x eventually outgrows any polynomial, and just after $x=1$ it is *below* x^2+1, so the graphs must cross once more. To be sure there are no others, look at $f(x)=2^x-x^2-1$: its third derivative is $(\ln 2)^3 2^x>0$, and a function whose third derivative is always positive has at most three real zeros.

Work it out. Check f at a few points: $f(0)=0$, $f(\frac{1}{2})=\sqrt{2}-\frac{5}{4}>0$, $f(1)=0$, $f(2)=4-5=-1$, $f(4)=16-17=-1$, $f(5)=32-26=6$. So besides 0 and 1 there is a zero between 4 and 5, three zeros so far. Why no more: if f had four zeros, Rolle's theorem would give three zeros of f' between them, then two zeros of f'', then one zero of $f^{(3)}$. But $f^{(3)}(x)=(\ln 2)^3 2^x$ is never 0. So there are exactly 3 solutions. (For $x<0$ there are none anyway: $2^x<1\le x^2+1$.)

Check. Numerically the third solution is about $x\approx 4.26$: $2^{4.26}\approx 19.2$ and $4.26^2+1\approx 19.1$. Its existence is the whole point of the problem: at $x=4$ the parabola is still ahead, 17 against 16, but by $x=5$ the exponential leads 32 to 26.

13. Answer: (A) 32**. The idea.** Each path cuts the perimeter $2s=42$ into two halves of length $s=21$, so it lands on the side at a distance from each end that is one of the numbers $s-a$, $s-b$, $s-c$. Once the three segments on each side are known, mass points (or areas) give how N divides the paths.

Work it out. Here $s=21$, and $s-a=7$, $s-b=6$, $s-c=8$ for $a=BC$, $b=CA$, $c=AB$. From $AB+BD=21$ we get $BD=8=s-c$ and $DC=6=s-b$. From $BA+AE=21$ we get $AE=8=s-c$ and $EC=7=s-a$.

Put masses x,y,z at A,B,C so that D and E are balance points: on $\overline{BC}$, $y\cdot BD=z\cdot DC$ gives $8y=6z$; on $\overline{CA}$, $z\cdot CE=x\cdot EA$ gives $7z=8x$. Taking $z=8$ gives $x=7$, $y=6$, i.e. masses $7:6:8$, which are exactly $s-a:s-b:s-c$. The point N is the centre of mass of these masses, and the area of the triangle ABN opposite C is the fraction $\frac{z}{x+y+z}$ of the whole: $\frac{8}{21}\cdot 84=32$.

Check. Coordinates: $B=(0,0)$, $C=(14,0)$, $A=(5,12)$. Then $D=(8,0)$ and $E=C+\frac{7}{15}(A-C)=(\frac{49}{5},\frac{28}{5})$. Line AD is $x=8-\frac{y}{4}$, line BE is $y=\frac{4}{7}x$; they meet at $N=(7,4)$. The area of ABN is $\frac{1}{2}|5\cdot 4-12\cdot 7|=32$. (The three perimeter-splitting paths all pass through N, the Nagel point.)

14. Answer: (D) 923780**. The idea.** Read the sum as a count. A club has 10 boys and 10 girls; choose a committee with equally many boys and girls, and name one of its boys captain. With k boys and k girls there are $\binom{10}{k}\binom{10}{k}$ committees and k choices of captain, so the sum counts exactly these captained committees. Now count them in another order: captain first.

Work it out. Pick the captain: 10 ways. The rest of the committee is some $k-1$ of the other 9 boys and k of the 10 girls. Replace "the k girls chosen" by "the $10-k$ girls left out": then we choose $k-1$ from the 9 remaining boys and $10-k$ from the girls, a total of $(k-1)+(10-k)=9$ people from a pool of 19. Every choice of 9 people from those 19 arises exactly once (the number of boys among them fixes k). So the count is

$10\binom{19}{9} = 10 \cdot 92378 = 923780$.

Check. Pairing k with $10-k$: the terms $k\binom{10}{k}^2$ and $(10-k)\binom{10}{10-k}^2$ add to $10\binom{10}{k}^2$, so the sum is half of $10\sum\binom{10}{k}^2 = 10\binom{20}{10}$, i.e. $5 \cdot 184756 = 923780$. Small case $n=2$: $1\cdot 4 + 2\cdot 1 = 6 = 2\binom{3}{1}$. Forgetting the factor 10 gives 92378; dropping the weights k gives $\binom{20}{10} = 184756$; forgetting to halve in the pairing gives 1847560.

15. Answer: (D) 183**. The idea.** The counts of 0s, 1s and 2s are three even numbers adding up to 6, so there are only a few possible count patterns. List them and count the arrangements of each. (A slicker route uses ± 1 signs to filter the parities.)

Work it out. The even counts (n_0, n_1, n_2) with $n_0+n_1+n_2 = 6$ are, up to order, $(6,0,0)$, $(4,2,0)$ and $(2,2,2)$.

$(6,0,0)$: choose which character fills the string: 3 strings.

$(4,2,0)$: choose the character used 4 times and the one used twice: $3 \cdot 2 = 6$ ways; then place the pair among the six spots: $\binom{6}{2} = 15$. That is 90 strings.

$(2,2,2)$: arrange `001122`: $\frac{6!}{2!\,2!\,2!} = 90$ strings.

Total: $3 + 90 + 90 = 183$.

Check. Parity filter: the number of strings with all three counts even is $\frac{1}{8}\sum(s_0+s_1+s_2)^6$ over the eight sign choices $s_i = \pm 1$, since each such sum weights a string by $s_0^{n_0}s_1^{n_1}s_2^{n_2}$, and averaging over signs keeps exactly the all-even strings. The sums $s_0+s_1+s_2$ are ± 3 once each and ± 1 three times each, giving $\frac{2\cdot 729+6\cdot 1}{8} = \frac{1464}{8} = 183$. Forgetting the $(2,2,2)$ case gives 93; forgetting the three constant strings gives 180; choosing the $(4,2)$ characters as an unordered pair gives 138; 365 counts strings with just an even number of 0s.

16. Answer: (B) 5**. The idea.** "Sum equals product" for three tangents is exactly the numerator of the tangent addition formula for $\tan(A+B+C)$. Clearing denominators makes this precise: multiplying by $\cos x \cos 2x \cos 3x$ turns the left side minus the right side into $\sin(x+2x+3x) = \sin 6x$. The real work is then remembering that the cosines must not vanish.

Work it out. Write $c_k = \cos kx$, $s_k = \sin kx$. Multiplying $\tan x + \tan 2x + \tan 3x - \tan x \tan 2x \tan 3x$ by $c_1c_2c_3$ gives

$$s_1c_2c_3 + c_1s_2c_3 + c_1c_2s_3 - s_1s_2s_3,$$

which is the expansion of $\sin(x+2x+3x) = \sin 6x$ (expand $\sin((x+2x)+3x)$ and then $\sin 3x$, $\cos 3x$ of the inner sum). Where all three tangents are defined the cosines are nonzero, so the equation is equivalent to $\sin 6x = 0$, i.e. $x = \frac{k\pi}{6}$ with $k = 1, 2, \ldots, 11$.

Now discard the values where a tangent is undefined. If k is odd, $3x = \frac{k\pi}{2}$ is an odd multiple of $\frac{\pi}{2}$, so $\tan 3x$ is undefined: this removes $k = 1, 3, 5, 7, 9, 11$. If $k = 2m$ is even, $x = \frac{m\pi}{3}$, and x, $2x$, $3x$ are multiples of $\frac{\pi}{3}$, none of which is an odd multiple of $\frac{\pi}{2}$; so all three tangents exist. That leaves $x = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$: five solutions.

Check. At $x = \frac{\pi}{3}$: $\tan x = \sqrt{3}$, $\tan 2x = -\sqrt{3}$, $\tan 3x = 0$; the sum is 0 and so is the product. At $x = \pi$ all three tangents are 0. A second route: since $\tan 3x = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$, we have $\tan x + \tan 2x = \tan 3x - \tan x \tan 2x \tan 3x$, and the equation collapses to $\tan x + \tan 2x = 0$, so $\tan 2x = \tan(-x)$ and $3x$ is a multiple of π: again $x = \frac{m\pi}{3}$. Keeping all eleven values $\frac{k\pi}{6}$ gives 11; removing only the two where $\tan x$ itself is undefined ($\frac{\pi}{2}$, $\frac{3\pi}{2}$) gives 9; counting $x = 0$ as well gives 6; working on $(0, \pi)$ only gives 3.

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17. Answer: (A) 8**. The idea.** Only the lead d = (heads) − (tails) matters. It starts at 0, moves up or down by 1 with each flip, and flipping stops when d reaches 4 or −2. Let E_d be the expected number of further flips from lead d. Each flip costs 1 and moves to $d \pm 1$ with equal chance, so $E_d = 1 + \frac{1}{2}(E_{d-1} + E_{d+1})$, i.e. the second difference $E_{d+1} - 2E_d + E_{d-1} = -2$ is constant. A sequence with constant second difference −2 is a quadratic with leading coefficient −1.

Work it out. So $E_d = -d^2 + pd + q$, and it must vanish at both stopping values $d = 4$ and $d = -2$. A quadratic with leading coefficient −1 and roots 4 and −2 is $E_d = -(d-4)(d+2) = (4-d)(d+2)$. At the start, $E_0 = 4 \cdot 2 = 8$.

Check. Solve the equations directly for the states $d = -1, 0, 1, 2, 3$: the formula gives $E_{-1} = 5$, $E_0 = 8$, $E_1 = 9$, $E_2 = 8$, $E_3 = 5$, and indeed $8 = 1 + \frac{5+9}{2}$, $9 = 1 + \frac{8+8}{2}$, $5 = 1 + \frac{8+0}{2}$. Note the answer is the product of the distances to the two stopping values, 4×2. The symmetric game (stop at a lead of 3 either way) would give $3 \times 3 = 9$. Adding the squares of the distances, $4^2 + 2^2$, gives 20; squaring the total width, $(4+2)^2$, gives 36; using only the farther stop, 4^2, gives 16.

18. Answer: (C) 70**. The idea.** The reciprocals of a geometric progression form another geometric progression, and the sum of the reciprocals of the roots is $\frac{e_3}{e_4}$. Comparing it with the sum e_1 of the roots pins down the middle of the progression; then the sum alone fixes the ratio.

Work it out. Let the roots be t, tq, tq^2, tq^3. Vieta gives $e_1 = 15$, $e_3 = 120$, $e_4 = 64$. The sum of reciprocals is $\frac{1}{tq^3}(q^3 + q^2 + q + 1) = \frac{e_1}{t^2q^3}$, and it also equals $\frac{e_3}{e_4}$. So $\frac{15}{t^2q^3} = \frac{120}{64}$, giving $t^2q^3 = 8$. In particular $q > 0$, and $t = \pm\frac{2\sqrt{2}}{q^{3/2}}$. Put $u = \sqrt{q} + \frac{1}{\sqrt{q}} \geq 2$. Then

$$15 = t(1 + q + q^2 + q^3) = \pm 2\sqrt{2}\left(q^{-3/2} + q^{-1/2} + q^{1/2} + q^{3/2}\right) = \pm 2\sqrt{2}\,(u^3 - 2u).$$

Since $u^3 - 2u > 0$ for $u \geq 2$, the sign is +, and $u^3 - 2u = \frac{15}{2\sqrt{2}}$. The left side increases for $u \geq 2$, so there is one solution, and $u = \frac{3}{\sqrt{2}}$ works: $\frac{27}{2\sqrt{2}} - \frac{6}{\sqrt{2}} = \frac{15}{2\sqrt{2}}$. Then $\sqrt{q} = \sqrt{2}$ or $\frac{1}{\sqrt{2}}$, so $q = 2$ (or $\frac{1}{2}$, the same progression read backwards), $t = 1$, and the roots are $1, 2, 4, 8$. Finally

$$a = e_2 = 1 \cdot 2 + 1 \cdot 4 + 1 \cdot 8 + 2 \cdot 4 + 2 \cdot 8 + 4 \cdot 8 = 2 + 4 + 8 + 8 + 16 + 32 = 70.$$

Check. $(x-1)(x-2)(x-4)(x-8) = (x^2 - 9x + 8)(x^2 - 6x + 8) = x^4 - 15x^3 + 70x^2 - 120x + 64$. Also $e_2 = \frac{e_1^2 - \sum r^2}{2} = \frac{225-85}{2} = 70$; forgetting to halve gives 140, and confusing e_2 with $\sum r^2$ gives 85. Dropping a product such as $2 \cdot 8$ or $1 \cdot 8$ gives 54 or 62.

19. Answer: (C) 3780**. The idea.** With step k, the pointer visits exactly the marks congruent to its start modulo $d = \gcd(k, 360)$, so $N(k) = \gcd(k, 360)$. Adding up $\gcd(k, 360)$ is easiest by grouping the k according to the value of the gcd.

Work it out. For a divisor d of $n = 360$, the k in $1, \ldots, n$ with $\gcd(k, n) = d$ are $k = dj$ with $\gcd(j, \frac{n}{d}) = 1$, and there are $\varphi(\frac{n}{d})$ of them. So the sum is $\sum_{d|n} d\,\varphi(\frac{n}{d})$. This is a multiplicative function of n, and for a prime power it equals $\sum_{i=0}^{a} p^i \varphi(p^{a-i}) = (a+1)p^a - ap^{a-1}$ (each of the a terms with $i < a$ is $p^a - p^{a-1}$, plus p^a for $i = a$). With $360 = 2^3 \cdot 3^2 \cdot 5$:

2^3: $4 \cdot 8 - 3 \cdot 4 = 20$; $\quad 3^2$: $3 \cdot 9 - 2 \cdot 3 = 21$; $\quad 5$: $2 \cdot 5 - 1 = 9$.

The total is $20 \cdot 21 \cdot 9 = 3780$.

Check. For a dial of 4 marks, $N(1), \ldots, N(4) = 1, 2, 1, 4$, total 8, and the formula gives $3 \cdot 4 - 2 \cdot 2 = 8$. Forgetting the step $k = 360$ gives 3420; adding the divisors of 360 instead gives $\sigma(360) = 1170$.

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20. Answer: (B) 36**. The idea.** Stretching the box along its three edge directions multiplies every volume by the same factor, so it is enough to solve the problem in a unit cube. There the overlap turns out to be the octahedron whose six vertices are the centres of the cube's faces.

Work it out. Take the unit cube $0 \le x, y, z \le 1$ and put $u = x - \frac{1}{2}$, $v = y - \frac{1}{2}$, $w = z - \frac{1}{2}$. The tetrahedron on $(0,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ is cut out by the four face planes $x+y+z \le 2$, $x-y-z \le 0$, $-x+y-z \le 0$, $-x-y+z \le 0$; the other one, on $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$, by $x+y+z \ge 1$, $x-y-z \ge -1$, $-x+y-z \ge -1$, $-x-y+z \ge -1$. In the centred coordinates the eight conditions together say exactly

$$|u+v+w| \le \tfrac{1}{2},\ |u-v-w| \le \tfrac{1}{2},\ |-u+v-w| \le \tfrac{1}{2},\ |-u-v+w| \le \tfrac{1}{2},$$

which is the same as $|u| + |v| + |w| \le \frac{1}{2}$ (the largest of the four expressions $|\pm u \pm v \pm w|$ is $|u|+|v|+|w|$). That is an octahedron made of two square pyramids, of volume $\frac{4}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{6}$ of the cube. Stretching back, the overlap in the box has volume $\frac{1}{6} \cdot 4 \cdot 6 \cdot 9 = 36$.

Check. Each tetrahedron is the box minus four corner pieces of volume $\frac{1}{6}abc$ each, so it has volume $\frac{1}{3}abc = 72$. The overlap 36 is half of that, and the union has volume $72 + 72 - 36 = 108$, half the box. An octahedron with perpendicular diagonals 4, 6, 9 has volume $\frac{4 \cdot 6 \cdot 9}{6} = 36$, in agreement.

21. Answer: (E) $\frac{533}{840}$**. The idea.** A cycle of length $k \ge 5$ uses more than half of the 8 members, so there can be at most one such cycle. The events "there is a cycle of length 5", "of length 6", "7", "8" are therefore disjoint, and each has a strikingly simple probability: $\frac{1}{k}$.

Work it out. Count the permutations of 8 with a cycle of length k, where $k > 4$ (so the cycle is unique): choose its k members in $\binom{8}{k}$ ways, arrange them in a cycle in $(k-1)!$ ways, and permute the other $8-k$ members arbitrarily in $(8-k)!$ ways. The product is $\frac{8!}{k!(8-k)!}(k-1)!(8-k)! = \frac{8!}{k}$, so the probability is $\frac{1}{k}$. Adding,

$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} = \tfrac{168+140+120+105}{840} = \tfrac{533}{840}.$$

Check. For $k \le 4$ the count $\frac{8!}{k}$ would double count permutations with two such cycles (for example two 4-cycles), which is why the argument only works above half. A complete enumeration of all 40320 permutations gives 25584 with a cycle of length at least 5, and $\frac{25584}{40320} = \frac{533}{840} \approx 0.63$.

22. Answer: (E) $\frac{130}{9}$**. The idea.** The power sums determine the elementary symmetric functions $e_1 = x+y+z$, $e_2 = xy+yz+zx$, $e_3 = xyz$, hence the monic cubic $p(t) = (t-x)(t-y)(t-z)$. Then $(x^2+1) = (x-i)(x+i)$, so the requested product is $p(i)\,p(-i)$.

Work it out. From $x^2+y^2+z^2 = e_1^2 - 2e_2$: $5 = 9 - 2e_2$, so $e_2 = 2$. From $x^3+y^3+z^3 = e_1^3 - 3e_1e_2 + 3e_3$: $7 = 27 - 18 + 3e_3$, so $e_3 = -\frac{2}{3}$. Thus $p(t) = t^3 - 3t^2 + 2t + \frac{2}{3}$. Now $\prod(x^2+1) = \prod(i-x)(-i-x) = p(i)\,p(-i)$, and $p(i) = -i + 3 + 2i + \frac{2}{3} = \frac{11}{3} + i$, $p(-i) = \frac{11}{3} - i$. The product is $\frac{121}{9} + 1 = \frac{130}{9}$.

Check. Expand directly: $\prod(1+x^2) = 1 + \sum x^2 + \sum x^2y^2 + x^2y^2z^2$, with $\sum x^2y^2 = e_2^2 - 2e_1e_3 = 4 + 4 = 8$ and $x^2y^2z^2 = e_3^2 = \frac{4}{9}$. The total is $1 + 5 + 8 + \frac{4}{9} = \frac{130}{9}$. (A sign slip $e_3 = +\frac{2}{3}$ gives $\frac{58}{9}$; dropping e_3^2 gives 14.)

23. Answer: (E) $\frac{91}{128}$**. The idea.** All $2^{10} = 1024$ road networks are equally likely, so count the connected ones. Counting them directly is hard, but a disconnected network is easy to describe: look at the group of towns that can be reached from town 1 (its component). If that group has

$k < n$ towns, the network is a connected network on those k towns, plus no road leaving the group, plus anything at all among the other $n-k$ towns. This gives a recursion for c_n, the number of connected networks on n labelled towns.

Work it out. Choosing the other $k-1$ towns of town 1's component in $\binom{n-1}{k-1}$ ways,

$c_n = 2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} c_k \, 2^{\binom{n-k}{2}}$.

Start with $c_1 = c_2 = 1$. Then $c_3 = 8-(1\cdot1\cdot2+2\cdot1\cdot1) = 4$, and $c_4 = 64-(1\cdot1\cdot8+3\cdot1\cdot2+3\cdot4\cdot1) = 64-26 = 38$. Finally

$c_5 = 1024 - (1\cdot1\cdot64 + 4\cdot1\cdot8 + 6\cdot4\cdot2 + 4\cdot38\cdot1) = 1024 - (64+32+48+152) = 728.$

The probability is $\frac{728}{1024} = \frac{91}{128}$.

Check. Count $c_4 = 38$ directly: a connected network on 4 towns needs at least 3 roads; with 3 roads it is a tree, and there are $4^{4-2} = 16$ labelled trees (the other 4 choices of 3 roads are triangles, which leave a town cut off); every choice of 4, 5 or 6 roads is connected ($15+6+1$), since cutting off a group of towns forbids at least 3 of the 6 roads. Total $16 + 22 = 38$. Subtracting $5 \cdot 64$ for "some town is isolated" without inclusion–exclusion gives $\frac{11}{16}$. A computer check of all 1024 networks finds exactly 728 connected ones.

24. Answer: (C) 506**. The idea.** Use the Legendre symbol $\chi(k)$, equal to 1 if k is a nonzero square remainder, -1 if it is not, and $\chi(0) = 0$. It is multiplicative, and $\frac{1+\chi(k)}{2}$ is 1 or 0 according as k is a nonzero square remainder or not. So the count is a sum of products of these indicators, and it reduces to a few character sums that can be evaluated exactly.

Work it out. Exactly half of $1, \dots, p-1$ are square remainders, so $\sum_{k=1}^{p-1} \chi(k) = 0$. Since $p = 2027 \equiv 3 \pmod 4$, -1 is not a square remainder, so $\chi(-1) = \chi(p-1) = -1$. The count is

$N = \sum_{k=1}^{p-2} \frac{(1+\chi(k))(1+\chi(k+1))}{4} = \frac{1}{4}\left[(p-2) + \sum_{k=1}^{p-2}\chi(k) + \sum_{k=1}^{p-2}\chi(k+1) + \sum_{k=1}^{p-2}\chi(k(k+1))\right].$

The first sum is $0-\chi(p-1) = 1$. The second is $\sum_{j=2}^{p-1}\chi(j) = 0-\chi(1) = -1$. For the third, write $k(k+1) = k^2(1+k^{-1})$ (inverse mod p), so $\chi(k(k+1)) = \chi(1+k^{-1})$. As k runs over $1, \dots, p-2$, k^{-1} runs over every nonzero residue except -1, so $1+k^{-1}$ runs over every residue except 1 and 0. Hence the third sum is $0-\chi(1)-\chi(0) = -1$. Therefore $N = \frac{1}{4}(p-2+1-1-1) = \frac{p-3}{4} = \frac{2024}{4} = 506$.

Check. For $p = 7$ (also 3 mod 4) the square remainders are $1, 2, 4$; among $k = 1, \dots, 5$ only $k = 1$ works, and $\frac{7-3}{4} = 1$. For $p = 11$ they are $1, 3, 4, 5, 9$; the pairs are $(3, 4)$ and $(4, 5)$, and $\frac{11-3}{4} = 2$. A direct computer count for 2027 gives 506 (the naive estimate $\frac{2025}{4} \approx 506.25$ is close but cannot decide between 506 and 507).

25. Answer: (E) $\frac{260}{21}$**. The idea.** Put the dodecagon at the 12th roots of unity z in the complex plane and P at the real number $d = \sqrt{2}$. Each term is $\frac{1}{(d-z)(d-\bar{z})}$ with $\bar{z} = \frac{1}{z}$; partial fractions turn this into two sums of the form $\sum \frac{1}{c-z}$, and such a sum over all roots of $x^{12}-1$ is the logarithmic derivative $\frac{12c^{11}}{c^{12}-1}$.

Work it out. For $|z| = 1$, $|d-z|^2 = (d-z)(d-\frac{1}{z}) = \frac{(d-z)(dz-1)}{z}$, and partial fractions give

$\frac{z}{(d-z)(dz-1)} = \frac{1}{d^2-1}\left(\frac{d}{d-z} + \frac{1}{dz-1}\right).$

From $x^{12}-1 = \prod(x-z)$ we get $\sum_z \frac{1}{c-z} = \frac{12c^{11}}{c^{12}-1}$. So $\sum_z \frac{d}{d-z} = \frac{12d^{12}}{d^{12}-1}$, and $\sum_z \frac{1}{dz-1} = -\frac{1}{d}\sum_z \frac{1}{\frac{1}{d}-z} = -\frac{1}{d}\cdot\frac{12d^{-11}}{d^{-12}-1} = \frac{12}{d^{12}-1}$. Altogether

$\sum_{k=1}^{12} \frac{1}{PA_k^2} = \frac{12(d^{12}+1)}{(d^2-1)(d^{12}-1)}$.

With $d^2 = 2$ and $d^{12} = 64$ this is $\frac{12\cdot 65}{1\cdot 63} = \frac{260}{21}$.

Check. The same formula for a "2-gon" (the points ± 1) with $d = \sqrt{2}$ gives $\frac{2\cdot 3}{1\cdot 1} = 6$, and directly $\frac{1}{(\sqrt{2}-1)^2} + \frac{1}{(\sqrt{2}+1)^2} = (3 + 2\sqrt{2}) + (3 - 2\sqrt{2}) = 6$. For large d the formula behaves like $\frac{12}{d^2}$, as it should. Summing the twelve terms $\frac{1}{3-2\sqrt{2}\cos(30^\circ k)}$ numerically gives $12.381 \approx \frac{260}{21}$; the tempting guess $\frac{12}{3} = 4$ (pretending the cosines average out inside the reciprocal) is far off.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.