MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A positive integer N has exactly 12 positive divisors. The product of all 12 of these divisors equals N^k for some positive integer k. What is k?

(A) 3 **(B)** 6 **(C)** 11 **(D)** 12 **(E)** 24

2. A triangle has height 4 (measured from its apex to its base). A line parallel to the base cuts the triangle into two pieces of equal area. How far from the apex is this line?

(A) 2 **(B)** $\sqrt{5}$ **(C)** $\sqrt{6}$ **(D)** $\sqrt{7}$ **(E)** $2\sqrt{2}$

3. How many real numbers x satisfy the equation $|x-1|+|x-2|+|x-3|=2$? (Count every real solution.)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

4. The five complex solutions of $z^5=1$ are plotted as points in the complex plane. What is the sum of the squares of the distances from the point 2 (on the real axis) to these five points?

(A) 20 **(B)** 25 **(C)** 30 **(D)** 40 **(E)** 50

5. Nine points are arranged in a 3×3 square grid. How many triangles with positive area have all three of their vertices among these nine points?

(A) 64 **(B)** 72 **(C)** 76 **(D)** 80 **(E)** 84

6. An infinite geometric series of real numbers has sum 6, and the sum of its first two terms is $\dfrac{9}{2}$. What is the sum of all possible values of the first term?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

7. In how many ways can a 1×10 strip be completely tiled using 1×1 tiles and 1×3 tiles? (Tiles of the same size are indistinguishable; two tilings are different if the tiles cover different cells.)

(A) 21 **(B)** 28 **(C)** 41 **(D)** 55 **(E)** 89

8. Real numbers x, y and z satisfy $x+y+z=3$ and $xy+yz+zx=3$. What is the value of the product xyz?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 9

9. Three fair six-sided dice are rolled. What is the probability that the largest number showing equals the sum of the other two? (If two dice tie for the largest, take either one as "the largest".)

(A) $\frac{1}{8}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{5}{24}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

10. How many ordered triples (a,b,c) of positive integers satisfy $abc=2^5\cdot 3^2$ and $\gcd(a,b,c)=1$?

(A) 60 **(B)** 72 **(C)** 90 **(D)** 105 **(E)** 126

11. For how many positive integers $n\le 100$ is the triangular number $T_n=1+2+\cdots+n=\dfrac{n(n+1)}{2}$ a perfect square?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

12. A $3\times 3\times 3$ cube is built from 27 unit cubes. A plane passes through the centre of the big cube and is

perpendicular to one of its space diagonals. How many of the 27 unit cubes does the plane cut through their interiors?

(A) 17 **(B)** 19 **(C)** 21 **(D)** 23 **(E)** 25

13. How many ordered triples (a, b, c) of positive integers satisfy $a + b + c = 12$ and have a, b, c pairwise different (no two of them equal)?

(A) 42 **(B)** 48 **(C)** 55 **(D)** 66 **(E)** 72

14. What is the sum of all real numbers x that satisfy the equation $x^2 - 5x + 6 = |x - 2|$?

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

15. The quadratic $x^2 + bx + 2026 = 0$ has two integer roots, where b is a positive integer. What is the smallest possible value of b?

(A) 1013 **(B)** 1015 **(C)** 2027 **(D)** 2028 **(E)** 4053

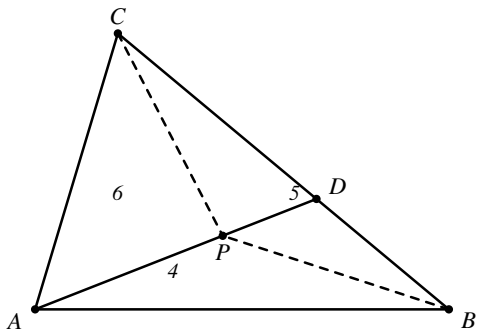
16. A polynomial $p(x)$ satisfies $p(x + 1) - p(x) = 2x$ for every real number x, and $p(0) = 1$. What is the value of $p(10)$?

(A) 45 **(B)** 55 **(C)** 90 **(D)** 91 **(E)** 101

17. What is the largest positive integer n for which $n! < 10^n$? (Here $n! = 1 \cdot 2 \cdots n$.)

(A) 20 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

18. Point P lies inside triangle ABC, and the triangles PAB, PBC and PCA have areas 4, 5 and 6 respectively. Line AP meets side $\overline{BC}$ at D. What is the ratio $\dfrac{BD}{DC}$?



(A) $\frac{2}{3}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{5}{6}$ **(D)** 1 **(E)** $\frac{5}{4}$

19. Consider the 25 points (x, y) with $x, y \in \{0, 1, 2, 3, 4\}$. How many squares, of any size and any orientation (tilted squares included), have all four vertices among these points?

(A) 25 **(B)** 30 **(C)** 36 **(D)** 50 **(E)** 55

20. The number $15! = 1 \cdot 2 \cdot 3 \cdots 15$ ends in a string of zeros. What is its last *nonzero* digit? (For comparison, $5! = 120$ has last nonzero digit 2.)

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

(A) 2 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

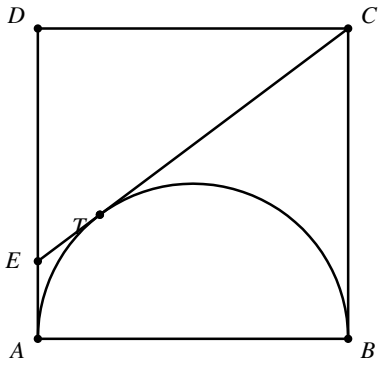
21. On a standard 8×8 chessboard, consider all rectangles made up of whole squares of the board (with sides along the grid lines). How many of these rectangles contain the square in the 5th column from the left and the 4th row from the bottom?

(A) 300 **(B)** 360 **(C)** 400 **(D)** 420 **(E)** 480

22. Real numbers x and y satisfy $x^2 + xy + y^2 = 3$. What is the largest possible value of $x + y$?

(A) 2 **(B)** $\sqrt{6}$ **(C)** 3 **(D)** $2\sqrt{3}$ **(E)** 4

23. Square $ABCD$ has side length 1. A semicircle with diameter $\overline{AB}$ is drawn inside the square. Of the two tangent lines from C to the semicircle, one is the line CB; the other touches the semicircle at T and meets side $\overline{AD}$ at E. What is the length DE?



(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{4}{5}$ **(E)** 1

24. How many strings of 10 letters, each letter A or B, contain neither two consecutive As nor three consecutive Bs?

(A) 16 **(B)** 21 **(C)** 24 **(D)** 26 **(E)** 28

25. For a positive integer m, let $d(m)$ be the number of positive divisors of m. Let $N = 2^6 \cdot 3^4 = 5184$. What is the value of the sum $\sum_{k|N} d(k)$, taken over all positive divisors k of N?

(A) 210 **(B)** 315 **(C)** 385 **(D)** 420 **(E)** 455

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — AMC 12 Practice Contest 16

1. (B) 6

2. (E) $2\sqrt{2}$

3. (A) 1

4. (B) 25

5. (C) 76

6. (E) 12

7. (B) 28

8. (A) 1

9. (C) $\frac{5}{24}$

10. (C) 90

11. (C) 3

12. (B) 19

13. (A) 42

14. (D) 6

15. (B) 1015

16. (D) 91

17. (D) 24

18. (A) $\frac{2}{3}$

19. (D) 50

20. (E) 8

21. (C) 400

22. (A) 2

23. (C) $\frac{3}{4}$

24. (E) 28

25. (D) 420

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — AMC 12 Practice Contest 16

1. Answer: (B) 6**. The idea.** Pair each divisor d with $\dfrac{N}{d}$, which is also a divisor. Each pair multiplies to N, and the 12 divisors form 6 such pairs (N is not a perfect square, since a square has an odd number of divisors, so no divisor pairs with itself).

Work it out. The product is N^6, so $k = 6$.

Check. $N = 60$ has divisors $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$; the pairs $(1, 60)$, $(2, 30)$, $(3, 20)$, $(4, 15)$, $(5, 12)$, (each multiply to 60, product 60^6. ✓ The traps: 12 multiplies N by itself once per divisor; 11 excludes N itself; 3 and 24 halve or double wrongly.

2. Answer: (E) $2\sqrt{2}$**. The idea.** The piece at the apex is a smaller triangle similar to the whole one. Similar figures have areas in the ratio of the *square* of their linear ratio, so to get half the area the small triangle must have $\dfrac{1}{\sqrt{2}}$ of the height, not half of it.

Work it out. Distance from the apex $= 4 \cdot \dfrac{1}{\sqrt{2}} = 2\sqrt{2} \approx 2.83$.

Check. The small triangle has height $2\sqrt{2}$ and base $\frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$ of the original base, so its area is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$ of the whole. ✓ The traps: 2 halves the height; $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ are nearby guesses between 2 and $2\sqrt{2}$.

3. Answer: (A) 1**. The idea.** The left side is the total distance from x to the three points 1, 2, 3 on the number line. The outer two distances $|x - 1| + |x - 3|$ are at least the gap 2 between 1 and 3, with equality exactly when $1 \le x \le 3$; and the middle distance $|x - 2|$ is at least 0, with equality only at $x = 2$.

Work it out. So the left side is always at least 2, and it equals 2 only when both equalities hold, i.e. $x = 2$. Exactly one solution.

Check. At $x = 2$: $1 + 0 + 1 = 2$. ✓ At $x = 1.5$: $0.5 + 0.5 + 1.5 = 2.5$; at $x = 0$: 6. The function is V-shaped with its minimum 2 at $x = 2$, so no other x reaches 2. The traps: 2 expects a symmetric pair of solutions as for a typical absolute-value equation; 3, 4, 6 count the break points or cases.

4. Answer: (B) 25**. The idea.** For a point z on the unit circle, $|2 - z|^2 = (2 - z)(2 - \bar{z}) = 4 - 2(z + \bar{z}) + |z|^2 = 5 - 4\,\mathrm{Re}\, z$. So the sum only needs $\sum \mathrm{Re}\, z$ over the five roots, and the five roots of $z^5 = 1$ add up to 0 (the coefficient of z^4 in $z^5 - 1$ is 0).

Work it out. $\sum |2 - z|^2 = 5 \cdot 5 - 4 \cdot 0 = 25$.

Check. The real parts are $1, \cos 72°, \cos 72°, \cos 144°, \cos 144° \approx 1, 0.309, 0.309, -0.809, -0.809$, summing to 0.000; the five squared distances are 1, 3.76, 3.76, 8.24, 8.24, total 25.0. ✓ The traps: $20 = 5 \cdot 4$ forgets $|z|^2$; 30 and 50 add the real parts wrongly; 40 doubles.

5. Answer: (C) 76**. The idea.** Every choice of three points gives a triangle unless the three are collinear, so count all triples and subtract the collinear ones. On a 3×3 grid a line through three of the points must be a row, a column or a main diagonal.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Work it out. $\binom{9}{3} = 84$ triples; collinear triples: 3 rows +3 columns +2 diagonals = 8. Triangles: $84 - 8 = 76$.

Check. No other line hits three grid points (a line of slope $\frac{1}{2}$, for instance, passes through only two of them). A computer check of all 84 triples confirms 76 non-degenerate. ✓ The traps: 84 forgets collinearity; 80 misses the diagonals; 72 and 64 subtract too much.

6. Answer: (E) 12**. The idea.** With first term a and ratio r ($|r| < 1$): $\dfrac{a}{1-r} = 6$ and $a(1+r) = \dfrac{9}{2}$. Multiply: $\dfrac{a}{1-r} \cdot a(1+r)$ is awkward, but dividing the second by the first gives $(1-r)(1+r) = \dfrac{9/2}{6} = \dfrac{3}{4}$, which pins down r^2 without a.

Work it out. $1 - r^2 = \frac{3}{4}$, so $r = \pm\frac{1}{2}$. For $r = \frac{1}{2}$: $a = 6(1-r) = 3$. For $r = -\frac{1}{2}$: $a = 6 \cdot \frac{3}{2} = 9$. Both ratios are allowed, so the possible first terms are 3 and 9, with sum 12.

Check. $r = \frac{1}{2}$, $a = 3$: series $3 + \frac{3}{2} + \cdots = 6$, first two terms $\frac{9}{2}$. ✓ $r = -\frac{1}{2}$, $a = 9$: $9 - \frac{9}{2} + \cdots = \dfrac{9}{3/2} = 6$, first two terms $\frac{9}{2}$. ✓ The traps: 3 and 9 keep only one ratio; 6 is the sum of the series; 8 is a miscount.

7. Answer: (B) 28**. The idea.** Look at the last tile: it covers either 1 cell or 3 cells. So the number of tilings t_n of a strip of length n satisfies $t_n = t_{n-1} + t_{n-3}$, with $t_0 = 1$ (the empty tiling), $t_1 = 1$, $t_2 = 1$.

Work it out. $t_3 = 2$, $t_4 = 3$, $t_5 = 4$, $t_6 = 6$, $t_7 = 9$, $t_8 = 13$, $t_9 = 19$, $t_{10} = 28$.

Check. By the number of 3-tiles: none (1 way), one (8 positions among 8 tiles: $\binom{8}{1} = 8$), two ($\binom{6}{2} = 15$), three ($\binom{4}{3} = 4$): $1 + 8 + 15 + 4 = 28$. ✓ The traps: 55 and 89 are Fibonacci numbers from the 1-and-2 tile problem; 41 uses $t_n = t_{n-1} + t_{n-2} + \ldots$-type slips; 21 is $t_9 + 2$.

8. Answer: (A) 1**. The idea.** Compare the sum of squares with the sum of products: $x^2+y^2+z^2 = (x+y+z)^2 - 2(xy+yz+zx) = 9-6 = 3$, which equals $xy+yz+zx$. But $x^2+y^2+z^2-(xy+yz+zx) = \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]$, and that vanishes only when $x = y = z$.

Work it out. So $x = y = z = 1$ and $xyz = 1$.

Check. $1 + 1 + 1 = 3$ and $1 + 1 + 1 = 3$. ✓ No other real triple works, since equality in $x^2 + y^2 + z^2 \geq xy + yz + zx$ forces all three equal. The traps: 3 and 9 repeat the given numbers; 2 and 6 assume a different cubic.

9. Answer: (C) $\frac{5}{24}$**. The idea.** Count by the multiset of values. We need values a, b, c with $c = a + b$ (so c is automatically the largest) and $c \leq 6$. List the pairs $\{a, b\}$ with $a + b \leq 6$, then count the orderings of the three dice.

Work it out. Unordered pairs with $a < b$ and $a+b \leq 6$: $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}$: six multisets $\{a, b, a+b\}$ of three distinct values, each with $3! = 6$ orderings: 36. Pairs with $a = b$ and $2a \leq 6$: $a = 1, 2, 3$, giving multisets $\{a, a, 2a\}$ with 3 orderings each: 9. Favourable outcomes $36 + 9 = 45$ out of 216: probability $\dfrac{45}{216} = \dfrac{5}{24}$.

Check. Direct enumeration of the 216 outcomes finds 45. ✓ (Note the "tie" clause never matters: if $c = a + b$ with $a, b \geq 1$ then c is strictly largest.) The traps: $\frac{1}{6} = \frac{36}{216}$ forgets the $\{a, a, 2a\}$ cases; $\frac{1}{4}$ and $\frac{1}{3}$ over-count orderings; $\frac{5}{36}$ uses two dice.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

10. Answer: (C) 90**. The idea.** Distribute the exponents of each prime among a, b, c separately. For the prime 2, the exponents (x, y, z) are nonnegative integers with $x + y + z = 5$; for the prime 3, (u, v, w) with $u + v + w = 2$. The condition $\gcd(a, b, c) = 1$ says no prime divides all three: for each prime, at least one of its exponents is 0.

Work it out. Prime 2: all distributions $\binom{7}{2} = 21$, minus those with all exponents ≥ 1 ($x' + y' + z' = 2$: $\binom{4}{2} = 6$): 15. Prime 3: $\binom{4}{2} = 6$ distributions, and none has all three exponents ≥ 1 (that would need a sum of at least 3): 6. Total $15 \cdot 6 = 90$.

Check. A computer search over all divisor triples finds 90. ✓ The traps: $126 = 21 \cdot 6$ ignores the gcd condition; 105 and 72 apply the correction to the wrong prime; 60 over-corrects.

11. Answer: (C) 3**. The idea.** $n(n + 1) = 2m^2$ with n and $n + 1$ coprime: two coprime numbers whose product is twice a square must be a square and twice a square, in some order. So we are looking for a perfect square and twice a perfect square that are *consecutive* integers, both at most 101.

Work it out. Squares up to 101: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. Twice a square up to 101: $2, 8, 18, 32, 50, 72, 98$. Consecutive pairs: $(1, 2)$, $(8, 9)$, $(49, 50)$. These give $n = 1, 8, 49$: three values.

Check. $T_1 = 1$, $T_8 = 36$, $T_{49} = 1225 = 35^2$. ✓ The next one is $T_{288} = 41616 = 204^2$, far beyond 100. A computer check confirms exactly three. The traps: 2 forgets $n = 1$; 4 and 5 include $n = 288$-type values or T_{50}; 1 stops at the first.

12. Answer: (B) 19**. The idea.** Put the big cube at $0 \leq x, y, z \leq 3$. The plane perpendicular to the diagonal through the centre is $x + y + z = \frac{9}{2}$. A unit cube with lowest corner (a, b, c) (each in $\{0, 1, 2\}$) occupies values of $x + y + z$ from $a + b + c$ to $a + b + c + 3$, so the plane cuts its interior exactly when $a + b + c < \frac{9}{2} < a + b + c + 3$, i.e. $a + b + c \in \{2, 3, 4\}$.

Work it out. The number of triples (a, b, c) from $\{0, 1, 2\}$ with sum 2, 3, 4 is 6, 7, 6 respectively (sum 3: the six permutations of $(0, 1, 2)$ plus $(1, 1, 1)$). Total $6 + 7 + 6 = 19$.

Check. The untouched cubes are the 4 with sum ≤ 1 (the corner cube and its three neighbours) and the 4 with sum ≥ 5 at the opposite corner: $27 - 8 = 19$. ✓ The traps: 21 and 23 count cubes merely touched at a face or edge; 17 drops $(1, 1, 1)$ or two others; 25 leaves only one corner cube on each side.

13. Answer: (A) 42**. The idea.** Count all ordered triples with sum 12, then throw away those with a repeated value by inclusion-exclusion on the three events $a = b$, $b = c$, $c = a$.

Work it out. All positive solutions of $a + b + c = 12$: $\binom{11}{2} = 55$. Solutions with $a = b$: $2a + c = 12$ with $a, c \geq 1$, so $a = 1, \ldots, 5$: 5; the same for each of the three pairs, 15 in total. Solutions with all three equal: $a = b = c = 4$, one triple, which was counted three times among the 15. So the number with at least one repeat is $15 - 3 + 1 = 13$, and the answer is $55 - 13 = 42$.

Check. Directly: the sets $\{a, b, c\}$ of three different positive integers with sum 12 are $\{1, 2, 9\}$, $\{1, 3, 8\}$, $\{1, 4, 7\}$, $\{$ – seven sets, each with $3! = 6$ orderings: 42. ✓ The traps: 55 ignores the distinctness; $48 = 55 - 7$ subtracts sets instead of triples; 66 and 72 miscount.

14. Answer: (D) 6**. The idea.** Factor the left side: $x^2 - 5x + 6 = (x - 2)(x - 3)$. Then the equation reads $(x - 2)(x - 3) = |x - 2|$, and either $x = 2$ or we may divide by $|x - 2|$, which leaves

$\pm(x-3) = 1$ depending on the sign of $x-2$.

Work it out. $x = 2$ works ($0 = 0$). If $x > 2$: $x - 3 = 1$, so $x = 4$. If $x < 2$: $-(x-3) = 1$, so $x = 2$, which is not less than 2: no solution there. The solutions are 2 and 4, with sum 6.

Check. $x = 4$: $16 - 20 + 6 = 2 = |4 - 2|$. ✓ $x = 1$ (a candidate someone might guess): $1 - 5 + 6 = 2 \neq 1$. The traps: 7 adds the spurious root $x = 2$ from the second case as if it were $x = 1$; 4 and 2 keep a single solution; 5 is the sum of the roots of the quadratic alone.

15. Answer: (B) 1015**. The idea.** By Vieta the roots r, s satisfy $rs = 2026$ and $r + s = -b$. Since $b > 0$ the roots are both negative, and $b = |r| + |s|$ where $|r| \cdot |s| = 2026 = 2 \cdot 1013$ with 1013 prime. So b is the sum of a pair of complementary divisors of 2026, and we want the pair with the smallest sum.

Work it out. The divisor pairs are $(1, 2026)$ and $(2, 1013)$, with sums 2027 and 1015. The smallest b is 1015, from the roots -2 and -1013.

Check. $(x+2)(x+1013) = x^2 + 1015x + 2026$. ✓ For a pair of positive numbers with a fixed product, the sum is smallest when they are closest together, and 2 and 1013 are as close as divisors of 2026 get. The traps: 1013 forgets to add the 2; 2027 uses the pair $(1, 2026)$; 2028 and 4053 add wrongly.

16. Answer: (D) 91**. The idea.** The condition tells us each step from $p(k)$ to $p(k+1)$: it adds $2k$. So $p(10)$ is $p(0)$ plus the sum of the steps at $k = 0, 1, \ldots, 9$ – a telescoping sum, with no need to find the polynomial.

Work it out. $p(10) = p(0) + \sum_{k=0}^{9} 2k = 1 + 2 \cdot 45 = 91$.

Check. The polynomial is $p(x) = x^2 - x + 1$: indeed $(x+1)^2 - (x+1) - x^2 + x = 2x$ and $p(0) = 1$; then $p(10) = 100 - 10 + 1 = 91$. ✓ The traps: 90 forgets $p(0)$; $101 = 10^2 + 1$ guesses $p(x) = x^2 + 1$; 45 and 55 are $\sum k$ with or without the 10.

17. Answer: (D) 24**. The idea.** Follow the ratio $r_n = \dfrac{n!}{10^n}$: going from $n-1$ to n multiplies it by $\dfrac{n}{10}$. So r_n decreases while $n < 10$, bottoms out at $n = 10$, and then increases for ever; it crosses 1 exactly once, and the question is where. We need r_{10} and then the product of the factors $\dfrac{11}{10}, \dfrac{12}{10}, \ldots$

Work it out. $10! = 3628800$, so $r_{10} \approx 3.63 \times 10^{-4}$. Pair the next factors from the outside in: $(1.1 \cdot 2.4)(1.2 \cdot 2.3)(1.3 \cdot 2.2)(1.4 \cdot 2.1)(1.5 \cdot 2.0)(1.6 \cdot 1.9)(1.7 \cdot 1.8) \approx 2.64 \cdot 2.76 \cdot 2.86 \cdot 2.94 \cdot 3.00 \cdot 3.04 \cdot 3.06 \approx 1710$. So $r_{24} \approx 1710 \cdot 3.63 \times 10^{-4} \approx 0.62 < 1$, while $r_{25} = 2.5\, r_{24} \approx 1.55 > 1$. The largest n is 24.

Check. Exactly, $24! = 620448401733239439360000 \approx 6.2 \times 10^{23} < 10^{24}$ and $25! \approx 1.55 \times 10^{25} > 10^{25}$. ✓ The traps: 25 is the first failure; 23, 22, 20 underestimate the product of the factors.

18. Answer: (A) $\frac{2}{3}$**. The idea.** Triangles sharing the vertex A with bases BD and DC on one line have areas in the ratio $BD : DC$; the same is true for the triangles with vertex P. So $\dfrac{BD}{DC} =$

$\dfrac{[ABD]}{[ADC]} = \dfrac{[PBD]}{[PDC]}$, and subtracting the second pair from the first leaves the triangles PAB and PCA.

Work it out. $\dfrac{BD}{DC} = \dfrac{[ABD]-[PBD]}{[ADC]-[PDC]} = \dfrac{[PAB]}{[PCA]} = \dfrac{4}{6} = \dfrac{2}{3}$.

Check. In coordinates with $P = \dfrac{5A+6B+4C}{15}$ (the areas opposite A, B, C are $5, 6, 4$), the line AP meets BC at $D = \dfrac{6B+4C}{10}$, so $BD : DC = 4 : 6 = 2 : 3$. ✓ The traps: $\frac{4}{5}$ and $\frac{5}{6}$ use the wrong pair of areas; $\frac{5}{4}$ inverts the ratio with the wrong areas; 1 assumes D is the midpoint.

19. Answer: (D) 50**. The idea.** Every square with lattice vertices is inscribed in a unique axis-parallel square (its "bounding box") of some side k, with the tilted square's vertices sitting at the same offset j along each side of the box ($0 \le j < k$; $j = 0$ is the box itself). So count boxes of side k in the 4×4 grid of unit cells and multiply by the k possible offsets.

Work it out. Boxes of side k number $(5-k)^2$. Total: $\sum_{k=1}^{4} k\,(5-k)^2 = 1\cdot 16 + 2\cdot 9 + 3\cdot 4 + 4\cdot 1 = 16 + 18 + 12 + 4 = 50$.

Check. Axis-parallel squares alone: $16+9+4+1 = 30$; tilted squares: 9 (in 2×2 boxes) +8 (two per 3×3 box, four boxes) +3 (three in the 4×4 box) = 20; total 50. ✓ A computer enumeration agrees. The traps: 30 counts only axis-parallel squares; 36 and 55 miscount tilted ones; 25 is the number of points.

20. Answer: (E) 8**. The idea.** The trailing zeros come from pairing factors 2 and 5. Factor 15! into primes, remove one 2 for every 5 (that strips exactly the trailing zeros), and reduce what remains modulo 10. Since what remains is odd-or-even but never a multiple of 5, its last digit is the last nonzero digit of 15!.

Work it out. $15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$ (the exponent of 2 is $7+3+1 = 11$; of 3 is $5+1 = 6$). Removing $2^3 \cdot 5^3 = 1000$ leaves $2^8 \cdot 3^6 \cdot 7^2 \cdot 11 \cdot 13$. Modulo 10: $2^8 = 256 \to 6$; $3^6 = 729 \to 9$; $7^2 = 49 \to 9$; $11 \to 1$; $13 \to 3$. Product $6 \cdot 9 \cdot 9 \cdot 1 \cdot 3 = 1458 \to 8$.

Check. $15! = 1307674368000$, last nonzero digit 8. ✓ The traps: 5 and 6 forget to strip the zeros properly (a leftover 5) or use 2^{11}; 2 is the answer for 5!; 4 is 2^8 mod 10 with the rest ignored.

21. Answer: (C) 400**. The idea.** A rectangle is determined by its range of columns and its range of rows, independently. It contains the marked square exactly when its column range contains column 5 and its row range contains row 4. So count the column ranges containing 5, count the row ranges containing 4, and multiply.

Work it out. Column ranges $[c_1, c_2]$ with $c_1 \le 5 \le c_2$: $c_1 \in \{1, \ldots, 5\}$ and $c_2 \in \{5, \ldots, 8\}$, so $5 \cdot 4 = 20$. Row ranges $[r_1, r_2]$ with $r_1 \le 4 \le r_2$: $4 \cdot 5 = 20$. Rectangles: $20 \cdot 20 = 400$.

Check. For a corner square the same count is $(1 \cdot 8)^2 = 64$, which is right: a rectangle containing the corner is determined by its opposite corner. For a 2×2 board and a given square, $(1\cdot 2)(1\cdot 2) = 4$: the square itself, two dominoes and the whole board. ✓ A direct enumeration of all 1296 rectangles finds 400 containing the marked square. The traps: 360 and 480 use 18 or 24 ranges in one direction; 300 and 420 miscount $5 \cdot 4$ as 15 or 21.

22. Answer: (A) 2**. The idea.** Write the constraint in terms of $s = x + y$ and $p = xy$: $x^2 + xy + y^2 =$

$s^2 - p = 3$, so $p = s^2 - 3$. Real x, y with sum s and product p exist exactly when $s^2 \geq 4p$ (the discriminant of $t^2 - st + p$).

Work it out. $s^2 \geq 4(s^2 - 3)$ gives $3s^2 \leq 12$, so $s \leq 2$. Equality needs $s^2 = 4p$, i.e. $x = y$: with $x = y = 1$ we get $1 + 1 + 1 = 3$. ✓ The maximum is 2.

Check. Try $x = 2, y = -1$: $4 - 2 + 1 = 3$, sum $1 < 2$. Try $x = \sqrt{3}, y = 0$: sum $1.73 < 2$. Lagrange multipliers also give the extremum at $x = y = 1$. The traps: $\sqrt{6}$ and $2\sqrt{3}$ come from $s^2 = 6$ or 12 (dropping the factor); 3 and 4 treat the constraint as $x^2 + y^2 = 3$ or guess.

23. Answer: (C) $\frac{3}{4}$**. The idea.** Equal tangent lengths. From C, the tangents to the semicircle are CB (touching at B) and CT, so $CT = CB = 1$. From E, the tangents are EA (side AD touches the semicircle at A) and ET, so $ET = EA$. Hence $CE = CT + TE = 1 + AE$, and triangle CDE is right-angled at D.

Work it out. Let $AE = e$, so $DE = 1 - e$ and $CE = 1 + e$. Pythagoras in triangle CDE: $(1 + e)^2 = 1^2 + (1 - e)^2$, so $1 + 2e + e^2 = 2 - 2e + e^2$, $4e = 1$, $e = \frac{1}{4}$, and $DE = \frac{3}{4}$.

Check. $CE = \frac{5}{4}, CD = 1, DE = \frac{3}{4}$: a 3-4-5 triangle. ✓ In coordinates the tangent from $C = (1, 1)$ with slope $\frac{3}{4}$ is at distance $\frac{1}{2}$ from the centre $(\frac{1}{2}, 0)$, and it meets $x = 0$ at $y = \frac{1}{4}$. ✓ The traps: $\frac{1}{2}$ assumes E is the midpoint; $\frac{2}{3}$ and $\frac{4}{5}$ come from arithmetic slips in the Pythagorean step; 1 puts E at A.

24. Answer: (E) 28**. The idea.** Build strings letter by letter and keep track of how they end. Let a_n, b_n, c_n be the numbers of valid strings of length n ending in A, in a single B (preceded by A or nothing), and in BB. An A may follow any string not ending in A; a B may follow a string ending in A (giving a single B) or in a single B (giving BB).

Work it out. $a_n = b_{n-1} + c_{n-1}$, $b_n = a_{n-1}$, $c_n = b_{n-1}$, with $(a_1, b_1, c_1) = (1, 1, 0)$. Then: $n = 2$: $(1, 1, 1)$; 3: $(2, 1, 1)$; 4: $(2, 2, 1)$; 5: $(3, 2, 2)$; 6: $(4, 3, 2)$; 7: $(5, 4, 3)$; 8: $(7, 5, 4)$; 9: $(9, 7, 5)$; 10: $(12, 9, 7)$. Total for $n = 10$: $12 + 9 + 7 = 28$.

Check. The totals $2, 3, 4, 5, 7, 9, 12, 16, 21, 28$ satisfy $t_n = t_{n-2} + t_{n-3}$ (a Padovan-type recurrence), and $t_{10} = t_8 + t_7 = 16 + 12 = 28$. ✓ For $n = 3$ the four strings are ABA, ABB, BAB, BBA. A computer enumeration of all 1024 strings finds 28. The traps: 21 and 16 are t_9 and t_8; 24 and 26 come from a wrong recurrence.

25. Answer: (D) 420**. The idea.** The sum counts pairs: $\sum_{k|N} d(k)$ is the number of pairs (j, k) of positive integers with $j \mid k \mid N$. Such a pair is a pair of exponent choices for each prime separately: for the prime 2 we need $0 \leq \alpha \leq \beta \leq 6$, and for the prime 3, $0 \leq \gamma \leq \delta \leq 4$. The choices for the two primes are independent, so the count is a product.

Work it out. Pairs $0 \leq \alpha \leq \beta \leq 6$: that is choosing a multiset of size 2 from $\{0, \ldots, 6\}$, $\binom{7+1}{2} = 28$. Pairs $0 \leq \gamma \leq \delta \leq 4$: $\binom{5+1}{2} = 15$. Total $28 \cdot 15 = 420$.

Check. For the prime 2 alone, the sum over $k \mid 2^6$ of $d(k)$ is $1 + 2 + \cdots + 7 = 28$. ✓ For $k \mid 3^4$: $1 + 2 + 3 + 4 + 5 = 15$. ✓ And d is multiplicative on the 35 divisors $2^\beta 3^\delta$, so the double sum factors. A computer summation over the 35 divisors of 5184 gives 420. The traps: $385 = 35 \cdot 11$ and 455 come from $d(N)\cdot$(something); 210 halves; $315 = 21 \cdot 15$ uses $\binom{7}{2}$ instead of $\binom{8}{2}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.