MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

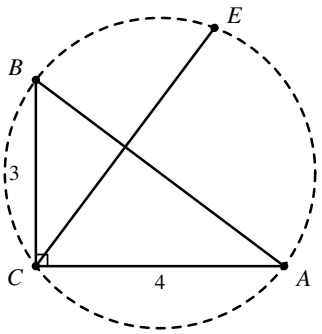
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

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1. What is the remainder when $1 + 11 + 111 + 1111 + \cdots + 1111111111$ (ten terms, the kth term consisting of k ones) is divided by 9?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

2. A right triangle has legs of lengths 3 and 4, and a circle passes through its three vertices. The chord of this circle that passes through the right-angle vertex and is perpendicular to the hypotenuse has what length?



(A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{24}{5}$ **(D)** 5 **(E)** 6

3. Let r be the larger root of the equation $x^2 - x - 1 = 0$. What is the exact value of $r^6 - 8r$? (No decimal approximation is needed.)

(A) 3 **(B)** 5 **(C)** 8 **(D)** 13 **(E)** 21

4. A sequence begins $1, 2, \ldots$ and from the third term on, each term equals the sum of *all* the previous terms. What is the 10th term of the sequence?

(A) 256 **(B)** 320 **(C)** 384 **(D)** 512 **(E)** 768

5. Five identical red balls and five identical blue balls are arranged in a row. How many of the arrangements have no three consecutive balls of the same colour?

(A) 56 **(B)** 84 **(C)** 126 **(D)** 168 **(E)** 252

6. How many real numbers x satisfy the equation $x^4 = 4x - 3$? (Count each real solution once, regardless of multiplicity.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

7. A bag contains 3 red balls and 3 blue balls. The six balls are drawn one at a time, at random and without replacement. What is the probability that the colours alternate (no two consecutive balls have the same colour)?

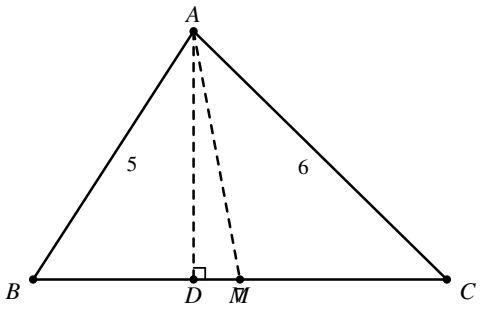
(A) $\frac{1}{10}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

8. How many of the one hundred numbers $1!, 2!, 3!, \ldots, 100!$ are perfect squares? (Recall $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

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(A) 1 **(B)** 2 **(C)** 10 **(D)** 11 **(E)** 50

9. Triangle ABC has $AB = 5$, $AC = 6$ and $BC = 7$. Let D be the foot of the altitude from A to $\overline{BC}$, and let M be the midpoint of $\overline{BC}$. What is the length of DM?



(A) $\frac{11}{14}$ **(B)** 1 **(C)** $\frac{8}{7}$ **(D)** $\frac{3}{2}$ **(E)** 2

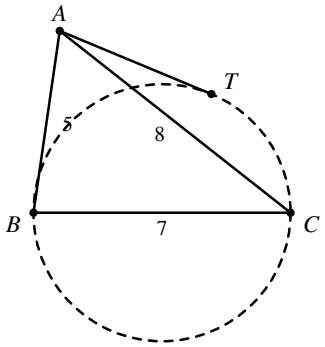
10. A right circular cone is inscribed in a sphere of radius 5: its apex and every point of the circle bounding its base lie on the sphere. The base circle has radius 4. There are two possible heights for such a cone. What is the sum of these two heights?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 16

11. How many real numbers x satisfy the equation $2^x + 3^x = 5^x$? (Negative and non-integer solutions count too.)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

12. Triangle ABC has $AB = 5$, $BC = 7$ and $CA = 8$. A circle is drawn with $\overline{BC}$ as a diameter, and a tangent line from A touches this circle at T. What is the length AT?



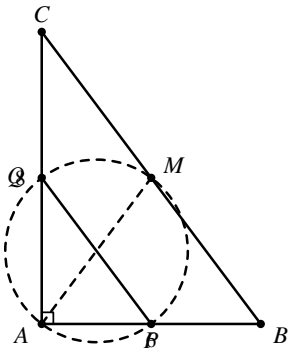
(A) 2 **(B)** $\sqrt{5}$ **(C)** $2\sqrt{5}$ **(D)** 5 **(E)** $3\sqrt{5}$

13. What is the sum of all two-digit positive integers n whose square n^2 ends in the two digits 44? (For example, $12^2 = 144$ is one such square.)

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(A) 100 **(B)** 150 **(C)** 188 **(D)** 200 **(E)** 238

14. Triangle ABC has a right angle at A, with $AB = 6$ and $AC = 8$. Let M be the midpoint of $\overline{BC}$. The circle with diameter $\overline{AM}$ meets side $\overline{AB}$ again at P and side $\overline{AC}$ again at Q. What is the length of PQ?



(A) 3 **(B)** 4 **(C)** $\frac{24}{5}$ **(D)** 5 **(E)** 6

15. Four red cards and four black cards are shuffled and laid out in a row. Among the 7 pairs of neighbouring cards, what is the expected number of pairs whose two cards have different colours?

(A) 3 **(B)** $\frac{7}{2}$ **(C)** $\frac{15}{4}$ **(D)** 4 **(E)** $\frac{9}{2}$

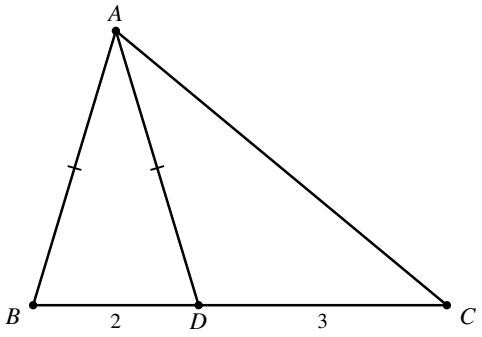
16. A sphere of radius 5 passes through all four vertices of a square whose side length is 6. What is the distance from the centre of the sphere to the plane containing the square?

(A) 2 **(B)** $\sqrt{5}$ **(C)** $\sqrt{7}$ **(D)** 3 **(E)** 4

17. A fair six-sided die is rolled repeatedly until two consecutive rolls show the same number, and then the rolling stops. What is the expected total number of rolls?

(A) 5 **(B)** 6 **(C)** $\frac{13}{2}$ **(D)** 7 **(E)** 12

18. In triangle ABC, the bisector of angle A meets side $\overline{BC}$ at D, with $BD = 2$ and $DC = 3$. It is also given that $AD = AB$. What is the length of AC?



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(A) 3 **(B)** $3\sqrt{3}$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** 9

19. How many three-digit positive integers n have the property that n and $2n$ have the same digit sum? (For example, $n = 135$ works: 135 and 270 both have digit sum 9.)

(A) 40 **(B)** 54 **(C)** 63 **(D)** 72 **(E)** 80

20. Each of the six faces of a cube is painted one of four colours, so that any two faces sharing an edge have different colours. The cube is fixed in space (two colourings are different if any face differs). How many colourings are possible?

(A) 24 **(B)** 48 **(C)** 72 **(D)** 84 **(E)** 96

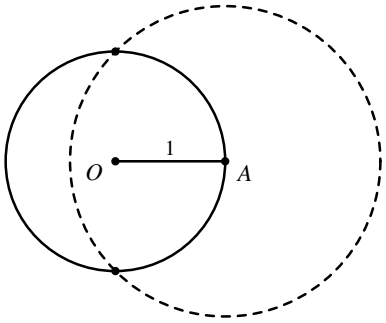
21. Real numbers a, b and c satisfy $a + b + c = 2$ and $a^2 + b^2 + c^2 = 12$. What is the largest possible value of c?

(A) 2 **(B)** $\frac{8}{3}$ **(C)** 3 **(D)** $\frac{10}{3}$ **(E)** 4

22. Call a positive integer *balanced* if its binary representation (with no leading zeros) contains exactly as many 1s as 0s. For example, $2 = 10_2$ and $12 = 1100_2$ are balanced. How many balanced integers are there with $n \le 1000$?

(A) 120 **(B)** 126 **(C)** 161 **(D)** 170 **(E)** 175

23. Point A lies on a circle ω of radius 1 with centre O. A second circle is drawn with centre A and radius $\sqrt{2}$. What is the area of the region that lies inside both circles?



(A) $\pi - 1$ **(B)** $\frac{\pi}{2} + 1$ **(C)** π **(D)** $2\pi - 2$ **(E)** $2\pi - 1$

24. A bug starts at a vertex A of a cube and walks along edges, one full edge per move, choosing any of the three edges at its current vertex (it may retrace an edge). In how many different ways can it make exactly 5 moves and end at the vertex G opposite A?

(A) 30 **(B)** 36 **(C)** 48 **(D)** 54 **(E)** 60

25. How many 4×4 arrays of 0s and 1s have exactly two 1s in every row and exactly two 1s in every column?

(A) 60 **(B)** 72 **(C)** 84 **(D)** 86 **(E)** 90

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Answer Key — AMC 12 Practice Contest 15

1. (A) 1

2. (C) $\frac{24}{5}$

3. (B) 5

4. (C) 384

5. (B) 84

6. (B) 1

7. (A) $\frac{1}{10}$

8. (A) 1

9. (A) $\frac{11}{14}$

10. (C) 10

11. (B) 1

12. (C) $2\sqrt{5}$

13. (D) 200

14. (D) 5

15. (D) 4

16. (C) $\sqrt{7}$

17. (D) 7

18. (B) $3\sqrt{3}$

19. (E) 80

20. (E) 96

21. (D) $\frac{10}{3}$

22. (E) 175

23. (A) $\pi - 1$

24. (E) 60

25. (E) 90

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Solutions — AMC 12 Practice Contest 15

1. Answer: (A) 1**. The idea.** A number and its digit sum leave the same remainder modulo 9 (because $10 \equiv 1 \pmod 9$). The kth term has digit sum k, so the whole sum is congruent to $1 + 2 + \cdots + 10$.

Work it out. $1 + 2 + \cdots + 10 = 55 = 6 \cdot 9 + 1$, so the remainder is 1.

Check. The actual sum is 1234567900, whose digit sum is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 9 = 37 \equiv 1 \pmod 9$. ✓ The traps: 7 is 55 mod 12-type confusion or $1 + 6$; 2, 3, 5 come from summing only some of the terms.

2. Answer: (C) $\frac{24}{5}$**. The idea.** The hypotenuse is a *diameter* of the circle (right angle inscribed in a semicircle). A chord perpendicular to a diameter is bisected by it. So the chord is twice the distance from the right-angle vertex to the hypotenuse, i.e. twice the altitude.

Work it out. The altitude to the hypotenuse of a 3-4-5 triangle is $\dfrac{3 \cdot 4}{5} = \dfrac{12}{5}$, so the chord has length $\dfrac{24}{5}$.

Check. Intersecting chords at the foot D of the altitude: $AD \cdot DB = \frac{9}{5} \cdot \frac{16}{5} = \frac{144}{25} = \left(\frac{12}{5}\right)^2$, so the other half of the chord is also $\frac{12}{5}$. ✓ Coordinates give the same 4.8. The traps: $\frac{12}{5}$ is the altitude only; 5 is the diameter; 6 and 2 are guesses.

3. Answer: (B) 5**. The idea.** Never compute r. The equation says $r^2 = r + 1$, which lets every higher power of r be reduced to the form $mr + n$ with integers m, n (the coefficients turn out to be Fibonacci numbers).

Work it out. $r^2 = r + 1$; $r^3 = r^2 + r = 2r + 1$; $r^6 = (r^3)^2 = 4r^2 + 4r + 1 = 4(r + 1) + 4r + 1 = 8r + 5$. Hence $r^6 - 8r = 5$.

Check. $r = \frac{1+\sqrt{5}}{2} \approx 1.618$: $r^6 \approx 17.944$ and $8r \approx 12.944$, difference 5.000. ✓ (In general $r^n = F_n r + F_{n-1}$, and $F_6 = 8$, $F_5 = 5$.) The traps: 3, 8, 13, 21 are the neighbouring Fibonacci numbers.

4. Answer: (C) 384**. The idea.** If a_n is the sum of all earlier terms, then a_{n+1} is that same sum plus a_n, i.e. $a_{n+1} = 2a_n$. So from the third term on the sequence doubles every step.

Work it out. $a_3 = 1 + 2 = 3$, then 6, 12, 24, 48, 96, 192, 384: $a_{10} = 3 \cdot 2^7 = 384$.

Check. Computing the sums directly: 1, 2, 3, 6, 12, 24, 48, 96, 192, 384, and indeed $1 + 2 + 3 + 6 + 12 + 24 + 48 + 96 + 192 = 384$. ✓ The traps: 768 doubles once too often; $512 = 2^9$ forgets the factor 3; 256 and 320 are miscounts.

5. Answer: (B) 84**. The idea.** Such a row is a sequence of colour blocks of size 1 or 2, alternating in colour. The red blocks form an ordered way of writing 5 as a sum of 1s and 2s, and so do the blue blocks; there are 8 such ways for each colour (3 with three parts, 4 with four parts, 1 with five parts). Because the colours alternate, the numbers of red and blue blocks differ by at most 1.

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Work it out. Equal numbers of blocks (k each) can start with either colour: $2(3{\cdot}3{+}4{\cdot}4{+}1{\cdot}1) = 52$. One more red block than blue ($4, 3$ or $5, 4$): the row must start and end with red, order forced: $4 \cdot 3 + 1 \cdot 4 = 16$; one more blue block: another 16. Total $52 + 16 + 16 = 84$.

Check. A computer enumeration of all $\binom{10}{5} = 252$ arrangements finds 84 without a run of three. ✓ The traps: 252 ignores the restriction; 126 halves it; 56 and 168 miscount the block compositions.

6. Answer: (B) 1**. The idea.** Spot the inequality: by AM-GM, $x^4 + 1 + 1 + 1 \geq 4\sqrt[4]{x^4} = 4|x| \geq 4x$, with equality only when $x^4 = 1$ and $x \geq 0$, i.e. $x = 1$. So $x^4 - 4x + 3 \geq 0$ always, with equality exactly at $x = 1$.

Work it out. The only real solution is $x = 1$: one solution.

Check. Factoring: $x^4 - 4x + 3 = (x-1)^2(x^2 + 2x + 3)$, and $x^2 + 2x + 3 = (x+1)^2 + 2 > 0$. So $x = 1$ is a double root and the other two roots are complex. ✓ The traps: 2 counts the double root twice; 4 is the degree; 0 and 3 are guesses.

7. Answer: (A) $\frac{1}{10}$**. The idea.** Only the colour sequence matters, and every sequence of 3 R's and 3 B's is equally likely: there are $\binom{6}{3} = 20$ of them. Exactly two alternate, RBRBRB and BRBRBR.

Work it out. Probability $\dfrac{2}{20} = \dfrac{1}{10}$.

Check. Multiplying along the draws: the first ball is anything, then $\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{10}$. ✓ The traps: $\frac{1}{5}$ counts only one of the two patterns against 10; $\frac{1}{6}$ and $\frac{1}{4}$ come from $\frac{2}{12}$ or $\frac{2}{8}$; $\frac{1}{2}$ is a guess.

8. Answer: (A) 1**. The idea.** A perfect square has every prime appearing to an even power. For $n \geq 2$, look at the *largest* prime $p \leq n$: it divides $n!$ through the single factor p, and through no other factor, provided $2p > n$ (then $2p$, the next multiple of p, is beyond n). So p appears exactly once and $n!$ is not a square.

Work it out. For $2 \leq n \leq 100$ the largest prime $p \leq n$ always satisfies $2p > n$: the primes $2, 3, 5, 7, 13, 23, 43, 83, 89, 97$ (each more than half the next one) cover every n – for instance 97 covers $n = 97$ to 100, 89 covers 89 to 96, 83 covers 83 to 88, and so on down to 2 covering $n = 2, 3$. So none of $2!, \ldots, 100!$ is a square, while $1! = 1$ is. Exactly 1.

Check. $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$ – none a square; a computer check of all hundred confirms only 1!. ✓ The traps: 2 counts $0! = 1$ as well; 10 and 11 count the squares among $1, \ldots, 100$ themselves; 50 guesses "the even n".

9. Answer: (A) $\frac{11}{14}$**. The idea.** Use the two right triangles ABD and ACD with the common leg AD: $AB^2 - BD^2 = AC^2 - CD^2$, so $AC^2 - AB^2 = CD^2 - BD^2 = (CD - BD)(CD + BD) = (CD - BD) \cdot BC$. And $CD - BD$ is twice the distance from D to the midpoint M.

Work it out. $CD - BD = \dfrac{36 - 25}{7} = \dfrac{11}{7}$, so $DM = \dfrac{1}{2} \cdot \dfrac{11}{7} = \dfrac{11}{14}$.

Check. Directly, $BD = \dfrac{AB^2 + BC^2 - AC^2}{2BC} = \dfrac{25 + 49 - 36}{14} = \dfrac{19}{7}$, and $BM = \frac{7}{2}$, so $DM = \frac{7}{2} - \frac{19}{7} = \frac{11}{14}$. ✓ The traps: $\frac{8}{7} = \frac{11}{7} - \frac{3}{7}$-type slips and $\frac{11}{7}$ (not offered) forget the halving; 1, $\frac{3}{2}$ and 2 are guesses.

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10. Answer: (C) 10**. The idea.** Slice through the axis of the cone. The base circle of radius 4 is a chord-circle of the sphere, so its centre is at distance $\sqrt{5^2 - 4^2} = 3$ from the sphere's centre. The apex is on the axis, at distance 5 from the centre, on either side of the base plane.

Work it out. Apex on the far side of the centre: height $5 + 3 = 8$. Apex on the near side: height $5 - 3 = 2$. Sum $8 + 2 = 10$.

Check. In the axial cross-section, the apex and the two ends of a base diameter form a triangle inscribed in a circle of radius 5 with base 8; the two positions of the apex are the ends of the diameter perpendicular to that chord, and the two heights add up to the full diameter, 10, no matter where the chord is. ✓ The traps: 8 and 6 are one height or the base diameter minus 2; 12 and 16 add the radius twice.

11. Answer: (B) 1**. The idea.** Divide through by 5^x: the equation becomes $\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x = 1$. Both terms are strictly decreasing functions of x (bases less than 1), so the left side is strictly decreasing: it crosses the value 1 at most once.

Work it out. It does cross: at $x = 1$ the left side is $\frac{2}{5} + \frac{3}{5} = 1$. As $x \to -\infty$ it tends to $+\infty$ and as $x \to +\infty$ to 0, so $x = 1$ is the unique solution.

Check. $x = 0$: $1+1 = 2 \neq 1$ (left side too big, as expected for $x < 1$); $x = 2$: $0.16+0.36 = 0.52 < 1$. ✓ The traps: 2 expects a second crossing for negative x; 0 overlooks $x = 1$; 3 and 4 are guesses.

12. Answer: (C) $2\sqrt{5}$**. The idea.** The tangent length from a point to a circle satisfies $AT^2 = AM^2 - r^2$, where M is the centre and r the radius (right angle at T). Here M is the midpoint of BC and $r = \frac{7}{2}$, so everything hinges on the median AM, which the median-length formula gives from the three sides.

Work it out. $AM^2 = \dfrac{2AB^2 + 2AC^2 - BC^2}{4} = \dfrac{50 + 128 - 49}{4} = \dfrac{129}{4}$. Then $AT^2 = \dfrac{129}{4} - \dfrac{49}{4} = 20$, so $AT = 2\sqrt{5}$.

Check. Coordinates $B = (0, 0)$, $C = (7, 0)$, $A = \left(\frac{5}{7}, \frac{\sqrt{1176}}{7}\right)$ (from $AB = 5$, $AC = 8$): $AM^2 = \left(\frac{5}{7} - \frac{7}{2}\right)^2 + \frac{1176}{49} = \frac{129}{4}$. ✓ (Equivalently $AT^2 = AB \cdot AC \cos A$, since AT^2 is the power of A, and $\cos A = \frac{25+64-49}{80} = \frac{1}{2}$: $AT^2 = 40 \cdot \frac{1}{2} = 20$.) The traps: $\sqrt{5}$ halves the answer; 5 and 2 come from $AM^2 - r^2$ with $r = 7$ or with the median formula missing the factor 2; $3\sqrt{5}$ is $\sqrt{45}$ from $\frac{129-49}{4} + 25$.

13. Answer: (D) 200**. The idea.** $n^2 \equiv 44 \pmod{100}$ splits into $n^2 \equiv 0 \pmod 4$ and $n^2 \equiv 19 \pmod{25}$. The first says n is even. For the second, $19 \equiv 144 = 12^2 \pmod{25}$, so $n \equiv \pm 12 \pmod{25}$, i.e. $n \equiv 12$ or $13 \pmod{25}$.

Work it out. Even numbers below 100 that are 12 or 13 (mod 25): from the class 12: 12, 37, 62, 87, keep 12 and 62; from the class 13: 13, 38, 63, 88, keep 38 and 88. Sum: $12 + 38 + 62 + 88 = 200$.

Check. $38^2 = 1444$, $62^2 = 3844$, $88^2 = 7744$. ✓ And e.g. $37^2 = 1369$ fails (odd). The pairs $12 + 88$ and $38 + 62$ each sum to 100, as the symmetry $n \to 100 - n$ predicts. The traps: 100 and 150 take one pair or miss a class; 188 and 238 include 87 or 37.

14. Answer: (D) 5**. The idea.** Any point on a circle with diameter AM sees AM at a right angle. So $\angle APM = 90°$: P is the foot of the perpendicular from M to AB. Since M is the midpoint of

BC and $MP \parallel AC$, P is the midpoint of AB. Likewise Q is the midpoint of AC.

Work it out. $AP = 3$, $AQ = 4$, and $\angle PAQ = 90°$, so $PQ = \sqrt{9+16} = 5$.

Check. $APMQ$ is a rectangle (three right angles), so $PQ = AM$, and the median to the hypotenuse is half of $BC = 10$: $AM = 5$. ✓ Coordinates confirm: the circle with diameter from $(0,0)$ to $(3,4)$ meets the axes again at $(3,0)$ and $(0,4)$. The traps: $\frac{24}{5}$ is the altitude from A; 3 and 4 are AP and AQ; 6 is AB.

15. Answer: (D) 4**. The idea.** Linearity of expectation: the expected number is 7 times the probability that one particular neighbouring pair has different colours, and every pair of positions behaves the same way as any other.

Work it out. For a fixed pair of positions, the first card is red or black and the second must be of the other colour: $\dfrac{4}{7}$ of the remaining 7 cards. So the expected number is $7 \cdot \dfrac{4}{7} = 4$.

Check. Averaging over all $\binom{8}{4} = 70$ colour patterns by computer gives exactly 4. ✓ (Equivalently, there are on average 5 maximal colour blocks.) The traps: $\frac{7}{2}$ uses probability $\frac{1}{2}$; $\frac{15}{4}$ and $\frac{9}{2}$ miscount the remaining cards; 3 is a guess.

16. Answer: (C) $\sqrt{7}$**. The idea.** The four vertices lie on the sphere and on a plane, so they lie on a circle: the plane cuts the sphere in a circle, and that circle is the circumscribed circle of the square. Its centre is the foot of the perpendicular from the sphere's centre to the plane, and its radius is half the square's diagonal.

Work it out. Half the diagonal is $\dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$. If d is the required distance, the right triangle formed by the sphere's centre, the circle's centre and a vertex gives $d^2 + (3\sqrt{2})^2 = 5^2$, so $d^2 = 25 - 18 = 7$ and $d = \sqrt{7}$.

Check. Coordinates: vertices $(\pm 3, \pm 3, 0)$ and centre $(0, 0, \sqrt{7})$: distance $\sqrt{9+9+7} = 5$. ✓ The traps: 4 uses the half-side 3 instead of the half-diagonal ($\sqrt{25-9}$); 3 and 2 are guesses; $\sqrt{5}$ uses a radius of $\sqrt{20}$.

17. Answer: (D) 7**. The idea.** Whatever the previous roll was, the next roll matches it with probability $\frac{1}{6}$, independently of the past. So after the first roll, the number of further rolls until a match is a geometric random variable with success probability $\frac{1}{6}$, whose expectation is 6.

Work it out. Expected total = 1 + 6 = 7.

Check. Directly: the game ends after exactly n rolls with probability $\left(\frac{5}{6}\right)^{n-2} \cdot \frac{1}{6}$ for $n \geq 2$, and $\sum_{n \geq 2} n \left(\frac{5}{6}\right)^{n-2} \frac{1}{6} = 7$. ✓ The traps: 6 forgets the first roll; 12 doubles; 5 and $\frac{13}{2}$ come from using $\frac{5}{6}$ or averaging wrongly.

18. Answer: (B) $3\sqrt{3}$**. The idea.** Two facts about the bisector. The angle bisector theorem gives $AB : AC = BD : DC = 2 : 3$, so $AB = 2k$, $AC = 3k$. And extending AD to meet the circumcircle at E: triangles ABD and AEC are similar (equal angles at A, and $\angle ABD = \angle AEC$ on the same arc), giving $AB \cdot AC = AD \cdot AE$, while the intersecting-chords relation at D gives $BD \cdot DC = AD \cdot DE$. Subtracting: $AD^2 = AB \cdot AC - BD \cdot DC$.

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Work it out. With $AD = AB = 2k$: $4k^2 = 6k^2 - 6$, so $k^2 = 3$, $k = \sqrt{3}$, and $AC = 3\sqrt{3}$.

Check. Coordinates ($B = (0,0)$, $C = (5,0)$, $D = (2,0)$, with $AB : AC = 2 : 3$ and $AD = AB$) give $AC = 5.196 = 3\sqrt{3}$. ✓ Also $AB = 2\sqrt{3} \approx 3.46$, and $2\sqrt{3} + 3\sqrt{3} > 5$, a genuine triangle. The traps: 3 uses $AC = DC$; 6 and 9 take $k = 2$ or $k = 3$; $4\sqrt{3}$ is $2AB$.

19. Answer: (E) 80**. The idea.** Doubling a number digit by digit doubles the digit sum, except that each digit 5 or more produces a carry, and a carry replaces a 10 in one place by a 1 in the next: it lowers the digit sum by 9. So $S(2n) = 2S(n) - 9c$, where c is the number of digits of n that are at least 5. The condition $S(2n) = S(n)$ becomes $S(n) = 9c$.

Work it out. $c = 1$: one digit $d \geq 5$ and two digits at most 4 with sum $9 - d$. For $d = 5, 6, 7, 8, 9$ the number of ordered pairs of small digits is $5, 4, 3, 2, 1$ (sum 15), and the big digit can sit in any of 3 positions: 45 strings, minus the 10 with a leading 0 (small digit 0 in front, then d and $9 - d$ in either order, 5 choices of d): 35. $c = 2$: two digits ≥ 5 with sum t and one small digit $18 - t$: for $t = 14, \ldots, 18$ there are $5, 4, 3, 2, 1$ ordered pairs (sum 15), times 3 positions for the small digit: 45, minus the 1 string 099 with a leading zero: 44. $c = 3$: only 999. Total $35 + 44 + 1 = 80$.

Check. $n = 135$: $c = 1$, $S = 9$. ✓ $n = 189$: $c = 2$, $S = 18$, and 378 has digit sum 18. ✓ A computer count over 100–999 gives 80. The traps: 72 and 63 forget one of the leading-zero corrections or the $c = 3$ case; 54 and 40 count only one of the cases $c = 1, 2$.

20. Answer: (E) 96**. The idea.** Only opposite faces may share a colour (every other pair of faces shares an edge). Organise by how many of the three opposite pairs are monochromatic. If a pair is split into two different colours, those two colours are both forbidden on all four remaining faces.

Work it out. *All three pairs monochromatic:* three distinct colours for three pairs, $4 \cdot 3 \cdot 2 = 24$. *Exactly two pairs monochromatic:* choose the split pair (3 ways), colour the two solid pairs ($4 \cdot 3 = 12$), then the split pair's two faces get the remaining two colours in some order (2 ways): 72. *Fewer than two monochromatic pairs:* two split pairs would need four pairwise different colours plus a colour for the solid pair, five in all – impossible with four colours. Total $24 + 72 = 96$.

Check. A computer enumeration of the $4^6 = 4096$ colourings finds 96 valid ones. Sanity check with three colours: only the all-monochromatic case survives, giving $3! = 6$, which matches the obvious count. The traps: 24 and 72 are single cases; 48 and 84 miscount the split-pair orderings.

21. Answer: (D) $\frac{10}{3}$**. The idea.** Eliminate a and b by what they must satisfy: $a + b = 2 - c$ and $a^2 + b^2 = 12 - c^2$. Two real numbers with a given sum have $a^2 + b^2 \geq \dfrac{(a+b)^2}{2}$ (equality when $a = b$), and that inequality is the only constraint linking c to reality.

Work it out. $12 - c^2 \geq \dfrac{(2-c)^2}{2}$ gives $24 - 2c^2 \geq 4 - 4c + c^2$, i.e. $3c^2 - 4c - 20 \leq 0$, i.e. $(3c - 10)(c + 2) \leq 0$, so $-2 \leq c \leq \dfrac{10}{3}$. The maximum $c = \dfrac{10}{3}$ is attained with $a = b = \dfrac{2 - 10/3}{2} = -\dfrac{2}{3}$.

Check. $-\frac{2}{3} - \frac{2}{3} + \frac{10}{3} = 2$ and $\frac{4}{9} + \frac{4}{9} + \frac{100}{9} = \frac{108}{9} = 12$. ✓ The traps: 4 and 3 ignore the sum constraint or use $a^2 + b^2 \geq 0$ only; $\frac{8}{3}$ and 2 solve the quadratic with a sign error.

22. Answer: (E) 175**. The idea.** A balanced number has an even number of binary digits, $2k$, with

exactly k ones, one of which is the leading 1. So the count with $2k$ digits is the number of ways to place the other $k-1$ ones among the remaining $2k-1$ positions: $\binom{2k-1}{k-1}$.

Work it out. 2 digits: $\binom{1}{1}=1$; 4 digits: $\binom{3}{1}=3$; 6 digits: $\binom{5}{2}=10$; 8 digits: $\binom{7}{3}=35$; 10 digits: $\binom{9}{4}=126$. Ten-digit numbers run from 512 to 1023, so we must drop any balanced ones above 1000. But $1001 = 1111101001_2$ and every number from 1001 to 1023 starts with five 1s, hence has at least five 1s, with equality only for $1111100000_2 = 992$; so none of them is balanced. Total $1+3+10+35+126 = 175$.

Check. $12 = 1100_2$ and $10 = 1010_2$ and $9 = 1001_2$ are the three 4-digit balanced numbers. ✓ A computer count over $n \le 1000$ gives 175. The traps: 126 counts only the ten-digit block; 170 and 161 wrongly drop some ten-digit numbers; 120 miscounts the smaller blocks.

23. Answer: (A) $\pi-1$**. The idea.** Locate the intersection points first. Put $O=(0,0)$ and $A=(1,0)$; subtracting $x^2+y^2=1$ from $(x-1)^2+y^2=2$ gives $-2x+1=1$, so $x=0$: the circles meet at $B=(0,1)$ and $C=(0,-1)$, the ends of a *diameter* of ω perpendicular to OA. So the common chord splits the lens into a half-disc of ω and a circular segment of the big circle.

Work it out. Right of the chord ($x \ge 0$): the half-disc of ω, area $\dfrac{\pi}{2}$, which lies inside the big circle. Left of the chord: the segment of the circle centred A cut off by chord BC. Since $AB = AC = \sqrt{2}$ and $BC = 2$, $\angle BAC = 90°$, so the segment is a quarter-disc minus a right triangle: $\dfrac{\pi\cdot 2}{4} - \dfrac{1}{2}\cdot\sqrt{2}\cdot\sqrt{2} = \dfrac{\pi}{2}-1$. Total area $\dfrac{\pi}{2}+\dfrac{\pi}{2}-1 = \pi-1$.

Check. Numerically $\pi-1 \approx 2.14$, about 68% of ω's area π; a grid count of points inside both discs agrees to two decimals, and exact integration gives $\pi-1$. ✓ The traps: $\frac{\pi}{2}+1$ adds the triangle instead of subtracting; π forgets the triangle; $2\pi-2$ doubles; $2\pi-1$ uses the whole disc of ω.

24. Answer: (E) 60**. The idea.** Track only the bug's *distance* from A (the number of edges in a shortest path): 0, 1, 2 or 3. From distance 1 there is 1 edge back to distance 0 and 2 edges to distance 2; from distance 2, 2 edges to distance 1 and 1 edge to distance 3 (which is G); from G, all 3 edges lead to distance 2. Let $w_n(d)$ count walks of n moves ending at distance d.

Work it out. $w_0 = (1,0,0,0)$. $w_1 = (0,3,0,0)$. $w_2 = (3,0,6,0)$. $w_3 = (0,\ 3\cdot 3+6\cdot 2,\ 0,\ 6\cdot 1) = (0,21,0,6)$. $w_4 = (21,0,\ 21\cdot 2+6\cdot 3,\ 0) = (21,0,60,0)$. $w_5 = (0,\ 21\cdot 3+60\cdot 2,\ 0,\ 60\cdot 1) = (0,183,0,60)$. The only vertex at distance 3 is G, so there are 60 walks.

Check. Total walks of length 5 are $3^5 = 243 = 183+60$. ✓ Parity makes sense: after an odd number of moves the bug is at odd distance. A direct enumeration of all 243 walks finds 60 ending at G. The traps: $36 = 6\cdot 6$ and 54 come from counting shortest paths times detours incorrectly; 30 halves; 48 forgets the walks that return through A.

25. Answer: (E) 90**. The idea.** Each row is a choice of 2 of the 4 columns. Fix the first row's columns, say $\{1,2\}$ (by symmetry every first row behaves the same, and there are $\binom{4}{2}=6$ of them), and split by how the second row overlaps it.

Work it out. *Second row equal to* $\{1,2\}$: columns 1, 2 are full, so rows 3 and 4 must both be $\{3,4\}$: 1 array. *Second row disjoint,* $\{3,4\}$: every column still needs exactly one more 1 from rows 3, 4, so row 3 can be any 2-set (6 ways) and row 4 is forced to be its complement: 6 arrays. *Second row sharing one column,* e.g. $\{1,3\}$ (4 such rows): column 1 is full, column 4 needs two

more 1s and columns $2, 3$ one each, so rows $3, 4$ both contain column 4 and split columns $2, 3$ between them: 2 arrays each, 8 in all. Per first row: $1 + 6 + 8 = 15$; total $6 \cdot 15 = 90$.

Check. A computer enumeration of all $6^4 = 1296$ row choices finds 90 with every column sum 2. (These arrays are exactly the 2-regular bipartite graphs on two sets of four labelled vertices.) The traps: 72 and 84 miss the "disjoint" or the "equal" case; 60 and 86 miscount the overlap case.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.