MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

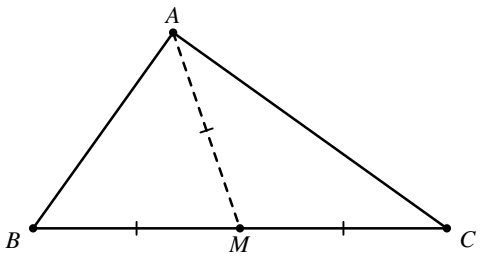
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Positive real numbers a, b and c satisfy $ab = 2$, $bc = 3$ and $ca = 6$. What is the value of the product abc?

(A) 6 **(B)** 11 **(C)** 12 **(D)** 18 **(E)** 36

2. In triangle ABC, let M be the midpoint of side $\overline{BC}$. It is given that $AM = \dfrac{1}{2}BC$. What is the measure of angle BAC, in degrees?



(A) 30 **(B)** 45 **(C)** 60 **(D)** 90 **(E)** 120

3. What is the sum of all the *distinct* real roots of the equation $(x^2-3x+2)(x^2-5x+6)(x^2-7x+12) = 0$?

(A) 6 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 20

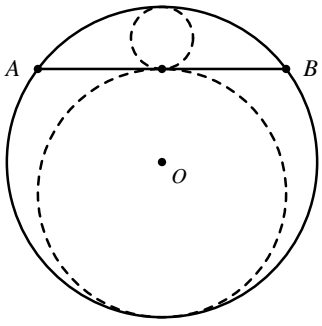
4. Let $x = \cos 36^\circ$. What is the exact value of $4x^2 - 2x$? (You may use the double-angle and product-to-sum formulas; no table of values is needed.)

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** 1 **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

5. How many positive integers less than 1000 have digits that strictly decrease from left to right? (One-digit numbers count; for example 9, 70 and 531 all qualify, but 55 and 120 do not.)

(A) 165 **(B)** 174 **(C)** 175 **(D)** 184 **(E)** 219

6. A circle of radius 5 has a chord $\overline{AB}$ of length 8. Two smaller circles are drawn, one on each side of the chord: each is tangent to $\overline{AB}$ at its midpoint and tangent to the big circle. What is the sum of the radii of the two smaller circles?



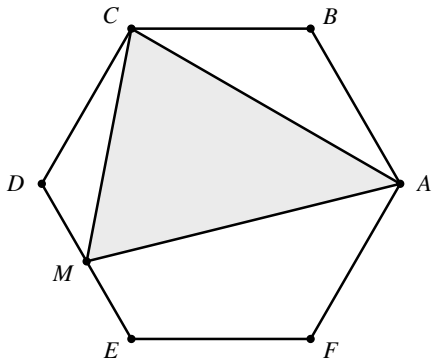
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(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

7. Seven fair coins are flipped. What is the probability that the number of heads exceeds the number of tails by at least 2?

(A) $\frac{1}{8}$ **(B)** $\frac{29}{128}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{35}{128}$ **(E)** $\frac{1}{2}$

8. Regular hexagon $ABCDEF$ has side length 1, and M is the midpoint of side $\overline{DE}$. What is the area of triangle ACM?



(A) $\frac{\sqrt{3}}{2}$ **(B)** 1 **(C)** $\frac{5\sqrt{3}}{8}$ **(D)** $\sqrt{3}$ **(E)** $\frac{3\sqrt{3}}{2}$

9. Let $f(x) = x^2 - 4x$ for every real number x. For how many real numbers x is $f(f(x)) = f(x)$, that is, applying f a second time changes nothing?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

10. How many integers n with $1 \le n \le 2026$ have the property that n and $2n$ have the same number of digits when written in base ten?

(A) 1026 **(B)** 1027 **(C)** 1471 **(D)** 1475 **(E)** 2026

11. What is the value of $\sin^2 0^\circ + \sin^2 15^\circ + \sin^2 30^\circ + \sin^2 45^\circ + \cdots + \sin^2 165^\circ$, the sum of $\sin^2(15k)^\circ$ for $k = 0, 1, \ldots, 11$?

(A) 4 **(B)** $\frac{9}{2}$ **(C)** 5 **(D)** $\frac{11}{2}$ **(E)** 6

12. For how many integers k does the equation $x^3 - 3x + k = 0$ have three distinct real roots?

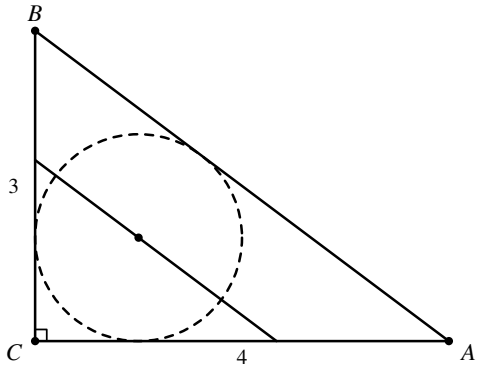
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

13. How many six-digit positive integers use each of the digits $1, 1, 2, 2, 3, 3$ exactly once (two of each) and have no two adjacent digits equal?

(A) 24 **(B)** 30 **(C)** 36 **(D)** 48 **(E)** 90

14. A right triangle has legs of lengths 3 and 4, and a circle is inscribed in it. The line through the centre of this circle parallel to the hypotenuse cuts off a smaller right triangle at the right-angle vertex. What is the area of that smaller triangle?

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(A) $\frac{49}{24}$ **(B)** $\frac{9}{4}$ **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** $\frac{7}{2}$

15. A regular octahedron has edge length 2. Each of its eight faces is parallel to exactly one other face. What is the distance between two such opposite (parallel) faces?

(A) 1 **(B)** $\frac{2\sqrt{3}}{3}$ **(C)** $\sqrt{2}$ **(D)** $\frac{2\sqrt{6}}{3}$ **(E)** 2

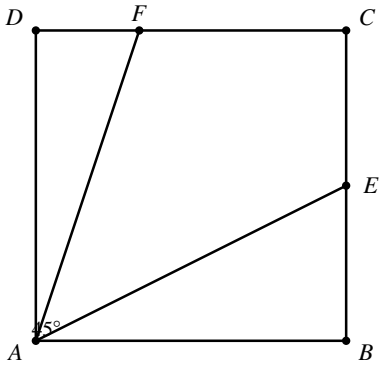
16. How many ordered pairs (x, y) of integers (positive, negative or zero) satisfy the equation $x^2+xy+y^2 = 7$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 24

17. For how many integers n with $1 \le n \le 100$ is $n!$ divisible by 2^{n-1}? (For example $n = 4$ qualifies, since $4! = 24$ is divisible by 2^3.)

(A) 1 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 50

18. Square $ABCD$ has side length 1. Point E is the midpoint of side $\overline{BC}$, and point F lies on side $\overline{CD}$ so that $\angle EAF = 45°$. What is the length of DF?



(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

19. How many ordered triples (a, b, c) of nonnegative integers satisfy $a + b + c = 20$ and have the property that a, b and c are the side lengths of a triangle with positive area?

(A) 20 **(B)** 24 **(C)** 30 **(D)** 33 **(E)** 36

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20. In how many ways can the numbers $1, 2, 3, 4, 5, 6$ be arranged in a row so that, reading from left to right, the numbers first strictly increase and then strictly decrease, with the number 6 somewhere strictly between the two ends? (For example, 2 4 6 5 3 1 is such an arrangement.)

(A) 15 **(B)** 16 **(C)** 24 **(D)** 28 **(E)** 30

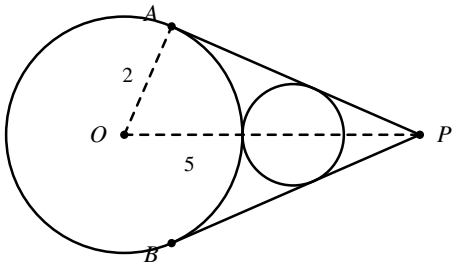
21. Consider the graph of $y = |x^2 - 4|$ and the line $y = x + k$. For most values of k the two graphs meet in 0, 2 or 4 points. What is the sum of all real numbers k for which they meet in *exactly* 3 points?

(A) 2 **(B)** $\frac{17}{4}$ **(C)** $\frac{25}{4}$ **(D)** 8 **(E)** 10

22. A polynomial $P(x)$ with integer coefficients satisfies $P(1) = 5$ and $P(5) = 1$. It is also known that $P(k) = k$ for some integer k. What is k?

(A) -3 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

23. A circle of radius 2 has centre O, and a point P lies at distance 5 from O. The two tangent lines from P touch the circle at A and B. A second, smaller circle is tangent to segment $\overline{PA}$, to segment $\overline{PB}$, and to the minor arc AB of the first circle (it sits in the region bounded by those three). What is the radius of the smaller circle?



(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{6}{7}$ **(E)** 1

24. Real numbers x and y satisfy $x^2 + y^2 = 4$. What is the largest possible value of $x^3y - xy^3$?

(A) 1 **(B)** 2 **(C)** $2\sqrt{2}$ **(D)** $2\sqrt{3}$ **(E)** 4

25. A *descent* in an arrangement $p_1, p_2, \ldots, p_6$ of the numbers 1 through 6 is a position i with $p_i > p_{i+1}$. How many arrangements of $1, 2, 3, 4, 5, 6$ have exactly one descent?

(A) 57 **(B)** 60 **(C)** 62 **(D)** 64 **(E)** 120

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Answer Key — AMC 12 Practice Contest 14

1. (A) 6

2. (D) 90

3. (B) 10

4. (C) 1

5. (B) 174

6. (C) 5

7. (B) $\frac{29}{128}$

8. (C) $\frac{5\sqrt{3}}{8}$

9. (D) 4

10. (C) 1471

11. (E) 6

12. (A) 3

13. (B) 30

14. (A) $\frac{49}{24}$

15. (D) $\frac{2\sqrt{6}}{3}$

16. (A) 12

17. (D) 7

18. (B) $\frac{1}{3}$

19. (E) 36

20. (E) 30

21. (C) $\frac{25}{4}$

22. (E) 3

23. (D) $\frac{6}{7}$

24. (E) 4

25. (A) 57

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Solutions — AMC 12 Practice Contest 14

1. Answer: (A) 6**. The idea.** Do not solve for a, b, c one at a time. Multiplying the three equations together gives $(ab)(bc)(ca) = a^2b^2c^2 = (abc)^2$, which is exactly the square of what is asked.

Work it out. $(abc)^2 = 2 \cdot 3 \cdot 6 = 36$, so $abc = 6$ (positive, since all three numbers are positive).

Check. Dividing $abc = 6$ by each equation: $c = 3$, $a = 2$, $b = 1$, and indeed $ab = 2$, $bc = 3$, $ca = 6$. ✓ The traps: 36 forgets the square root; $11 = 2+3+6$ adds; 12 and 18 multiply only two of the equations.

2. Answer: (D) 90**. The idea.** $AM = \frac{1}{2}BC = BM = CM$, so A, B and C are all at the same distance from M: they lie on a circle centred at M with $\overline{BC}$ as a diameter. An angle inscribed in a semicircle is a right angle.

Work it out. $\angle BAC = 90°$.

Check. In the triangles ABM and ACM, both isosceles, $\angle BAM = \angle ABM$ and $\angle CAM = \angle ACM$; adding, $\angle BAC = \angle B + \angle C$, and since the three angles sum to 180°, $\angle BAC = 90°$. ✓ The traps: 60 and 30 assume an equilateral or 30-60-90 picture; 45 and 120 are guesses.

3. Answer: (B) 10**. The idea.** Factor each quadratic: $x^2 - 3x + 2 = (x-1)(x-2)$, $x^2 - 5x + 6 = (x-2)(x-3)$, $x^2 - 7x + 12 = (x-3)(x-4)$. The roots 2 and 3 each appear twice, so Vieta on the degree-6 polynomial would overcount.

Work it out. The distinct roots are $1, 2, 3, 4$, with sum 10.

Check. Vieta gives the sum *with* multiplicity: $3+5+7 = 15 = 1+2+2+3+3+4$, and removing the repeated 2 and 3 leaves 10. ✓ The traps: 15 counts multiplicity; 20 and 12 miscount; 6 stops at the first two factors.

4. Answer: (C) 1**. The idea.** $4x^2 - 2x = 2(2x^2 - 1) - 2x + 2 = 2\cos 72° - 2\cos 36° + 2$ by the double-angle formula, so the question is really about $\cos 36° - \cos 72°$. That difference is a classic: multiply by $2\sin 36°$ and use product-to-sum.

Work it out. Let $D = \cos 36° - \cos 72°$. Then $2\sin 36°\, D = 2\sin 36°\cos 36° - 2\sin 36°\cos 72° = \sin 72° - (\sin 108° - \sin 36°) = \sin 36°$, since $\sin 108° = \sin 72°$. So $D = \frac{1}{2}$, and $4x^2 - 2x = 2 - 2D = 1$.

Check. $\cos 36° = \dfrac{1+\sqrt{5}}{4} \approx 0.809$: $4(0.809)^2 - 2(0.809) = 2.618 - 1.618 = 1.000$. ✓ (Indeed $x = \frac{1+\sqrt{5}}{4}$ is a root of $4x^2 - 2x - 1 = 0$.) The traps: $\frac{\sqrt{5}-1}{2}$ and $\frac{1+\sqrt{5}}{2}$ are $2\cos 72°$ and $2\cos 36°$ themselves; $\frac{1}{2}$ is D; 2 forgets the $-2D$.

5. Answer: (B) 174**. The idea.** A strictly decreasing string of digits is determined by *which* digits it uses: any set of distinct digits can be arranged in decreasing order in exactly one way, and the leading digit is automatically the largest, hence nonzero (unless the set is $\{0\}$ alone).

Work it out. One-digit numbers: 1 through 9, that is 9 (the digit 0 alone is not a positive integer). Two-digit: choose 2 of the 10 digits, $\binom{10}{2} = 45$ (sets like $\{1, 0\}$ give 10, which is fine). Three-digit: $\binom{10}{3} = 120$. Total $9 + 45 + 120 = 174$.

Check. A computer count of the integers below 1000 with strictly decreasing digits gives 174. The traps: 175 counts 0; 165 forgets the one-digit numbers; 184 uses $\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + 9$; 219 adds $\binom{10}{2}$ twice or includes four-digit cases.

6. Answer: (C) 5. The idea. By symmetry each small circle is centred on the diameter perpendicular to the chord, so it touches the chord at its midpoint N and touches the big circle at the end of that diameter. Hence its diameter is exactly the distance from N to the big circle along the perpendicular, i.e. the height of the circular segment on its side.

Work it out. The chord is at distance $\sqrt{5^2 - 4^2} = 3$ from the centre, so the two heights are $5 - 3 = 2$ and $5 + 3 = 8$; the radii are 1 and 4, with sum 5.

Check. The two diameters add up to the full diameter 10 regardless of where the chord is, so the radii always add to 5. ✓ The traps: 4 and 6 miscount one height; 3 is the distance to the chord; 8 is the larger diameter.

7. Answer: (B) $\frac{29}{128}$. The idea. With 7 flips the difference heads minus tails is odd, so "at least 2 more heads" means at least 3 more, i.e. at least 5 heads. Count those outcomes.

Work it out. $\binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 21 + 7 + 1 = 29$ outcomes out of $2^7 = 128$: probability $\dfrac{29}{128}$.

Check. By symmetry the probability of "at least 2 more tails" is also $\frac{29}{128}$, and the remaining outcomes, exactly 4-3 either way, number $2\binom{7}{4} = 70$; indeed $29+29+70 = 128$. ✓ The traps: $\frac{35}{128}$ counts 4 heads as well ($\binom{7}{4} = 35$) and then drops a term; $\frac{1}{4}$ and $\frac{1}{8}$ are rough guesses; $\frac{1}{2}$ is "more heads than tails".

8. Answer: (C) $\frac{5\sqrt{3}}{8}$. The idea. As a point P slides along segment $\overline{DE}$, the area of triangle ACP changes linearly (the base AC is fixed and the height varies linearly). So the area at the midpoint M is the average of the areas at D and at E: $[ACM] = \frac{1}{2}\big([ACD] + [ACE]\big)$.

Work it out. AD is a diameter of the circumscribed circle (length 2), so $\angle ACD = 90^\circ$ with $CD = 1$ and $AC = \sqrt{3}$: $[ACD] = \frac{1}{2} \cdot \sqrt{3} \cdot 1 = \dfrac{\sqrt{3}}{2}$. Triangle ACE is equilateral with side $\sqrt{3}$: $[ACE] = \dfrac{\sqrt{3}}{4} \cdot 3 = \dfrac{3\sqrt{3}}{4}$. Average: $\frac{1}{2}\left(\dfrac{\sqrt{3}}{2} + \dfrac{3\sqrt{3}}{4}\right) = \dfrac{5\sqrt{3}}{8}$.

Check. Coordinates with the centre at the origin: $A = (1, 0)$, $C = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $M = (-\frac{3}{4}, -\frac{\sqrt{3}}{4})$; the shoelace formula gives $\frac{5\sqrt{3}}{8} \approx 1.083$. ✓ The traps: $\frac{\sqrt{3}}{2}$ and $\frac{3\sqrt{3}}{2}$ are $[ACD]$ and the hexagon's area; $\sqrt{3}$ and 1 are guesses.

9. Answer: (D) 4. The idea. Put $y = f(x)$. The condition says $f(y) = y$, so y must be a fixed point of f: $y^2 - 4y = y$ gives $y(y - 5) = 0$, so $y = 0$ or $y = 5$. Now count the x with $f(x) = 0$ and with $f(x) = 5$.

Work it out. $f(x) = 0$: $x(x - 4) = 0$, so $x = 0$ or 4. $f(x) = 5$: $x^2 - 4x - 5 = (x - 5)(x + 1) = 0$, so $x = 5$ or -1. Four values in all: $-1, 0, 4, 5$.

Check. $f(-1) = 5$, $f(5) = 5$. ✓ $f(4) = 0$, $f(0) = 0$. ✓ Expanding $f(f(x)) - f(x)$ gives a quartic $x(x - 4)(x - 5)(x + 1)$, with exactly these four real roots. The traps: 2 counts only the fixed points of f; 3 misses $x = -1$; 5 and 1 are miscounts.

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10. Answer: (C) 1471**. The idea.** Doubling keeps the number of digits exactly when $2n$ stays below the next power of ten: for a k-digit n (so $10^{k-1} \le n < 10^k$), $2n < 10^k$ means $n < 5 \cdot 10^{k-1}$. In other words, n keeps its digit count under doubling exactly when its leading digit is $1, 2, 3$ or 4.

Work it out. One digit: $1, 2, 3, 4$ (4 numbers). Two digits: 10 to 49 (40). Three digits: 100 to 499 (400). Four digits up to 2026: 1000 to 2026 all have leading digit 1 or 2, and indeed $2 \cdot 2026 = 4052$ still has four digits: 1027 numbers. Total $4 + 40 + 400 + 1027 = 1471$.

Check. $n = 49$: 98, two digits. ✓ $n = 50$: 100, three digits. ✓ A computer count gives 1471. The traps: 1027 counts only the four-digit block; 1026 is an off-by-one of that; 1475 counts 444 three-digit numbers plus 4; 2026 assumes doubling never adds a digit.

11. Answer: (E) 6**. The idea.** Pair each angle with its complement: $\sin^2\theta + \sin^2(90^\circ - \theta) = \sin^2\theta + \cos^2\theta = 1$. The twelve angles $0^\circ, 15^\circ, \ldots, 165^\circ$ split into such pairs, provided we handle the angles beyond 90° (use $\sin(180^\circ - \theta) = \sin\theta$ to fold them back).

Work it out. Folding, the twelve values $\sin^2(15k)^\circ$ are: $\sin^2 0^\circ$, then $\sin^2 15^\circ, \sin^2 30^\circ, \ldots, \sin^2 75^\circ$ each *twice* (once from θ, once from $180^\circ - \theta$), then $\sin^2 90^\circ$. So the sum is $0+1+2\left(\sin^2 15^\circ + \sin^2 30^\circ + \sin^2 45^\circ + \text{si}\right.$ In the bracket, 15° pairs with 75° and 30° with 60° (each pair sums to 1), and $\sin^2 45^\circ = \frac{1}{2}$: bracket $= 2 + \frac{1}{2} = \frac{5}{2}$. Total $1 + 5 = 6$.

Check. Via $\sin^2\theta = \frac{1-\cos 2\theta}{2}$: the sum is $\frac{12}{2} - \frac{1}{2}\sum_{k=0}^{11}\cos(30k)^\circ = 6 - 0 = 6$, because the twelve cosines of equally spaced angles around a full turn cancel. ✓ The traps: 5 and $\frac{11}{2}$ forget $\sin^2 90^\circ$ or $\sin^2 45^\circ$; 4 pairs only four pairs; $\frac{9}{2}$ halves wrongly.

12. Answer: (A) 3**. The idea.** The graph of $y = x^3 - 3x$ has a local maximum at $x = -1$ (value 2) and a local minimum at $x = 1$ (value -2). The equation $x^3 - 3x = -k$ has three distinct real roots exactly when the horizontal line $y = -k$ cuts the graph three times, i.e. strictly between the two extreme values.

Work it out. Three distinct real roots $\iff -2 < -k < 2 \iff -2 < k < 2$. The integers are $k = -1, 0, 1$: three of them.

Check. $k = 0$: roots $0, \pm\sqrt{3}$. ✓ $k = 2$: $x^3 - 3x + 2 = (x-1)^2(x+2)$, a repeated root, so not distinct. ✓ $k = 3$: only one real root. The traps: 5 includes $k = \pm 2$; 4 includes one of them; 6 and 7 count all $|k| \le 3$.

13. Answer: (B) 30**. The idea.** Start from all $\dfrac{6!}{2!\,2!\,2!} = 90$ arrangements and remove those with some pair of equal digits adjacent, by inclusion-exclusion over which digits form an adjacent pair. Gluing a pair into a single block shortens the word.

Work it out. One specified pair glued (say 11 as a block): arrangements of the block plus 2, 2, 3, 3: $\dfrac{5!}{2!\,2!} = 30$; three choices of pair, 90 total. Two pairs glued: $\dfrac{4!}{2!} = 12$, three choices, 36. All three glued: $3! = 6$. Bad arrangements: $90 - 36 + 6 = 60$. Good: $90 - 60 = 30$.

Check. Direct enumeration by computer finds 30. By symmetry there are 10 good words starting with each digit; for instance the ones starting with 12 are 121323, 123123, 123132, 123213 and 123231 (five of them), and the ones starting with 13 are their mirror images with 2 and 3 swapped. The traps: 90 ignores the restriction; 36 and 24 stop inclusion-exclusion early; 48 double-counts.

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14. Answer: (A) $\frac{49}{24}$**. The idea.** The small triangle is similar to the big one (parallel hypotenuses), so its area is 6 times the square of the scale factor. The scale factor is the ratio of distances from the right-angle vertex to the two parallel lines, and the incenter's line is easy to locate: the incenter of a right triangle with legs on the axes is (r, r) with $r = \dfrac{3+4-5}{2} = 1$.

Work it out. The hypotenuse is $3x + 4y = 12$; the parallel line through $(1, 1)$ is $3x + 4y = 7$. The distances from the origin to these lines are in the ratio $7 : 12$, so the small triangle has area $6 \cdot \left(\dfrac{7}{12}\right)^2 = \dfrac{49}{24}$.

Check. The small triangle has intercepts $\frac{7}{3}$ and $\frac{7}{4}$: area $\frac{1}{2} \cdot \frac{7}{3} \cdot \frac{7}{4} = \frac{49}{24}$. ✓ The traps: $\frac{9}{4}$ and $\frac{5}{2}$ come from a wrong inradius ($r = \frac{3}{2}$ or $r = \frac{6}{5}$-type slips); 3 is half the triangle; $\frac{7}{2}$ uses the ratio $7 : 12$ without squaring.

15. Answer: (D) $\frac{2\sqrt{6}}{3}$**. The idea.** Use volume. The octahedron is two square pyramids glued along a square of side 2; its inscribed sphere touches all eight faces at their centres, so the volume also equals $\frac{1}{3}r \cdot$ (surface area). The distance between opposite faces is $2r$.

Work it out. Each pyramid has base area 4 and height equal to half the diagonal of the square, $\sqrt{2}$, so the volume is $2 \cdot \frac{1}{3} \cdot 4 \cdot \sqrt{2} = \dfrac{8\sqrt{2}}{3}$. The surface is 8 equilateral triangles of side 2: $8 \cdot \sqrt{3} = 8\sqrt{3}$. Then $\dfrac{8\sqrt{2}}{3} = \frac{1}{3}r \cdot 8\sqrt{3}$ gives $r = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$, and the distance between opposite faces is $2r = \dfrac{2\sqrt{6}}{3}$.

Check. Coordinates: vertices $(\pm\sqrt{2}, 0, 0)$, $(0, \pm\sqrt{2}, 0)$, $(0, 0, \pm\sqrt{2})$ (edge 2); the face through the three positive vertices is the plane $x + y + z = \sqrt{2}$, at distance $\sqrt{2}/\sqrt{3}$ from the origin, and the opposite face is at the same distance on the other side: total $\frac{2\sqrt{6}}{3} \approx 1.63$. ✓ The traps: $\sqrt{2}$ is the height of one pyramid; $\frac{2\sqrt{3}}{3}$ is the circumradius of a face; 2 and 1 are the edge and half-edge.

16. Answer: (A) 12**. The idea.** Complete the square to bound one variable: $4(x^2 + xy + y^2) = (2x + y)^2 + 3y^2$, so $(2x + y)^2 + 3y^2 = 28$. Then $3y^2 \le 28$ forces $|y| \le 3$, and for each y the value $(2x + y)^2 = 28 - 3y^2$ must be a perfect square.

Work it out. $y = 0$: 28, not a square. $y = \pm 1$: 25, so $2x + y = \pm 5$, giving $x = 2, -3$ (for $y = 1$) and $x = 3, -2$ (for $y = -1$). $y = \pm 2$: 16, so $2x + y = \pm 4$: $x = 1, -3$ ($y = 2$) and $x = 3, -1$ ($y = -2$). $y = \pm 3$: 1, so $2x + y = \pm 1$: $x = -1, -2$ ($y = 3$) and $x = 2, 1$ ($y = -3$). That is 2 solutions for each of the six values $y = \pm 1, \pm 2, \pm 3$: 12 ordered pairs.

Check. $(2, 1)$: $4 + 2 + 1 = 7$. ✓ $(-3, 1)$: $9 - 3 + 1 = 7$. ✓ $(-1, 3)$: $1 - 3 + 9 = 7$. ✓ A computer search over $|x|, |y| \le 10$ finds exactly 12. The traps: 24 double-counts by also swapping signs that were already included; 16 and 18 include $y = 0$ or a false square; 14 is a miscount.

17. Answer: (D) 7**. The idea.** Count the factors of 2 in $n!$ by Legendre's formula, $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \cdots$, which is at most $\frac{n}{2} + \frac{n}{4} + \cdots = n$, with a deficit that is exactly the number of 1s in the binary representation of n: $v_2(n!) = n - s_2(n)$. (Each floor loses the fractional part, and adding up those losses over all the binary digits gives $s_2(n)$.)

Work it out. $v_2(n!) \ge n - 1$ means $s_2(n) \le 1$, i.e. n is a power of 2: $1, 2, 4, 8, 16, 32, 64$. Seven values.

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Check. $v_2(8!) = 4 + 2 + 1 = 7 = 8 - 1$. ✓ $v_2(6!) = 3 + 1 = 4 < 5$. ✓ A computer check over $n \le 100$ confirms exactly the seven powers of 2. The traps: 6 forgets $n = 1$ ($1! = 1$ is divisible by 2^0); 5 forgets $n = 1, 2$; 50 guesses "the even numbers"; 1 assumes only $n = 1$.

18. Answer: (B) $\frac{1}{3}$**. The idea.** Measure angles at A with tangents. Since $\angle BAD = 90°$ and $\angle EAF = 45°$, the two leftover angles $\angle BAE$ and $\angle DAF$ add to 45°. We know $\tan \angle BAE = \dfrac{BE}{AB} = \dfrac{1}{2}$, so $\tan \angle DAF = \tan(45° - \angle BAE)$ follows from the subtraction formula.

Work it out. $\tan(45° - \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta} = \dfrac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \dfrac{1}{3}$. Since $\tan \angle DAF = \dfrac{DF}{AD} = DF$, we get $DF = \dfrac{1}{3}$.

Check. With A at the origin, $E = (1, \frac{1}{2})$ and $F = (\frac{1}{3}, 1)$: the dot product $AE \cdot AF = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$, while $|AE|\,|AF| = \frac{\sqrt{5}}{2} \cdot \frac{\sqrt{10}}{3} = \frac{5\sqrt{2}}{6}$, so $\cos \angle EAF = \frac{5/6}{5\sqrt{2}/6} = \frac{1}{\sqrt{2}}$. ✓ (A classic alternative: rotating triangle ABE about A by 90° shows $EF = BE + DF$, and then the Pythagorean theorem in triangle ECF gives $DF = \frac{1}{3}$.) The traps: $\frac{1}{2}$ assumes F mirrors E; $\frac{1}{4}$, $\frac{2}{5}$ and $\frac{2}{3}$ come from mis-applying the subtraction formula.

19. Answer: (E) 36**. The idea.** With $a + b + c = 20$, the triangle inequality $a < b + c$ is the same as $a < 20 - a$, i.e. $a < 10$. So the condition is simply that all three numbers are at most 9 (and then none can be 0 either: if $a = 0$, then $b + c = 20$ forces one of them to be at least 10). Count solutions of $a + b + c = 20$ with each variable in $\{0, 1, \ldots, 9\}$ by inclusion-exclusion.

Work it out. All nonnegative solutions: $\binom{22}{2} = 231$. Solutions with $a \ge 10$: substitute $a = a' + 10$, giving $\binom{12}{2} = 66$; same for b and c: 198. Two variables ≥ 10: $a' + b' + c = 0$, one solution, three pairs: 3. Three variables ≥ 10: impossible. Count: $231 - 198 + 3 = 36$.

Check. Equivalently, set $a = 10 - a'$ etc. with $a', b', c' \ge 1$ and $a' + b' + c' = 10$: $\binom{9}{2} = 36$. ✓ Direct enumeration confirms. The traps: $33 = 36 - 3$ forgets to add back the pairs; 30 and 24 come from requiring each side ≤ 8 or counting unordered triples; 20 is a guess.

20. Answer: (E) 30**. The idea.** The 6 sits at the peak. Everything to its left must be increasing and everything to its right decreasing, so an arrangement is completely determined by *which* of the numbers 1 through 5 go to the left of the 6: those are then written in increasing order, and the rest in decreasing order.

Work it out. There are $2^5 = 32$ subsets of $\{1, 2, 3, 4, 5\}$. The empty subset puts 6 first and the full subset puts 6 last; both are excluded. That leaves $32 - 2 = 30$ arrangements.

Check. For the numbers 1, 2, 3 the same reasoning gives $2^2 - 2 = 2$ arrangements, namely 1 3 2 and 2 3 1. ✓ Direct enumeration of the 720 permutations of 1 to 6 finds 30. The traps: $16 = 2^4$ and 15 forget one of the five numbers or count only subsets of a certain size; 24 and 28 are miscounts of the excluded cases.

21. Answer: (C) $\frac{25}{4}$**. The idea.** The graph of $y = |x^2 - 4|$ is the parabola $y = x^2 - 4$ with the part below the axis ($-2 < x < 2$) flipped up into the arch $y = 4 - x^2$, which has corners at $(\pm 2, 0)$ and a peak at $(0, 4)$. As the line $y = x + k$ slides upward, the number of intersection points changes only when the line passes through a corner or becomes tangent to the arch. An odd count, 3, happens exactly at such transitions.

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Work it out. *Tangency to the arch:* $4 - x^2 = x + k$ has a double root when $x^2 + x + (k-4)$ has discriminant 0: $1 - 4(k-4) = 0$, so $k = \frac{17}{4}$, tangent at $x = -\frac{1}{2}$ (inside the arch). The line also meets the two outer branches, so there are 3 points. *Through the corner* $(-2, 0)$*:* $k = 2$. The line $y = x + 2$ meets the arch again at $x = 1$ and the right branch at $x = 3$: 3 points. *Through the corner* $(2, 0)$*:* $k = -2$. The line $y = x - 2$ meets the arch only at the corner itself and the branches nowhere else: just 1 point. In between: for $k < -2$ the line misses the graph entirely, for $-2 < k < 2$ there are 2 points, for $2 < k < \frac{17}{4}$ there are 4, and for $k > \frac{17}{4}$ there are 2. So exactly 3 points occur for $k = 2$ and $k = \frac{17}{4}$, with sum $\dfrac{25}{4}$.

Check. $k = 3$: arch $x^2 + x - 1 = 0$ gives $x \approx 0.62, -1.62$ (both inside); branches $x^2 - x - 7 = 0$ gives $x \approx 3.19, -2.19$ (both outside): 4 points. ✓ $k = 5$: arch has no real roots; branches give 2 points. ✓ A symbolic count of intersections at each candidate k confirms $\{2, \frac{17}{4}\}$. The traps: $\frac{17}{4}$ and 2 each take only one transition; 8 and 10 include $k = -2$ with the wrong sign or double a value.

22. Answer: (E) 3**. The idea.** For a polynomial with integer coefficients, $a - b$ divides $P(a) - P(b)$ whenever a and b are integers (each term $c_j(a^j - b^j)$ is divisible by $a - b$). Apply this to the pairs $(k, 1)$ and $(k, 5)$.

Work it out. $k - 1$ divides $P(k) - P(1) = k - 5$, and $k - 5$ divides $P(k) - P(5) = k - 1$. Two integers that divide each other have the same absolute value: $|k-1| = |k-5|$. Since $k - 1 \neq k - 5$, we need $k - 1 = -(k - 5)$, i.e. $k = 3$. (Note $k = 1$ and $k = 5$ are impossible since $P(1) \neq 1$ and $P(5) \neq 5$.)

Check. $k = 3$ really can occur: $P(x) = 6 - x$ has $P(1) = 5$, $P(5) = 1$ and $P(3) = 3$. ✓ For any other candidate, say $k = 2$: $k - 1 = 1$ divides $k - 5 = -3$, but -3 does not divide 1. A computer check of the two divisibility conditions over $|k| \leq 500$ finds only $k = 3$. The traps: 0, 1, 2 and -3 each satisfy one of the two divisibilities but not both.

23. Answer: (D) $\frac{6}{7}$**. The idea.** Everything is symmetric about line OP, so the small circle's centre Q lies on $\overline{OP}$. Two facts pin it down: (1) Q is at distance r from line PA, and since PA makes angle θ with OP where $\sin\theta = \dfrac{OA}{OP} = \dfrac{2}{5}$, that distance is $PQ \sin\theta$; (2) the small circle is externally tangent to the big one, so $OQ = 2 + r$.

Work it out. Let $PQ = d$. Then $r = \dfrac{2d}{5}$ and $OQ = 5 - d = 2 + r = 2 + \dfrac{2d}{5}$. So $3 = \dfrac{7d}{5}$, $d = \dfrac{15}{7}$ and $r = \dfrac{2}{5} \cdot \dfrac{15}{7} = \dfrac{6}{7}$.

Check. $OQ = 5 - \frac{15}{7} = \frac{20}{7} = 2 + \frac{6}{7}$. ✓ Coordinates ($O$ at the origin, $P = (5, 0)$, $A = (\frac{4}{5}, \frac{2\sqrt{21}}{5})$): the distance from $Q = (\frac{20}{7}, 0)$ to line PA is $\frac{15}{7} \cdot \frac{2}{5} = \frac{6}{7}$. ✓ The traps: 1 and $\frac{2}{3}$ use $OQ = 2 - r$ (internal tangency) or drop the $\sin\theta$; $\frac{3}{5}$ and $\frac{1}{2}$ use $PQ = 3$ or halve the gap $OP - 2 = 3$ by eye.

24. Answer: (E) 4**. The idea.** Factor: $x^3y - xy^3 = xy\,(x^2 - y^2)$. The two factors are linked by the constraint in a hidden way: $(2xy)^2 + (x^2 - y^2)^2 = (x^2 + y^2)^2 = 16$. So if $u = 2xy$ and $v = x^2 - y^2$, the point (u, v) lies on a circle of radius 4, and we want to maximise $\dfrac{uv}{2}$.

Work it out. By AM-GM, $uv \leq \dfrac{u^2 + v^2}{2} = 8$, with equality when $u = v = 2\sqrt{2}$. So $x^3y - xy^3 =$

$\frac{uv}{2} \le 4$, and the value 4 is attained (take $x^2 - y^2 = 2\sqrt{2}$ and $2xy = 2\sqrt{2}$, which does have real solutions, e.g. $x = 2\cos 22.5°$, $y = 2\sin 22.5°$).

Check. With $x = 2\cos\theta$, $y = 2\sin\theta$ the expression becomes $16\sin\theta\cos\theta\cos 2\theta = 4\sin 4\theta$, whose maximum is 4. ✓ A numerical sweep agrees. The traps: 2 and $2\sqrt{2}$ forget the factor 2 in u or maximise only uv with the wrong radius; $2\sqrt{3}$ and 1 are guesses.

25. Answer: (A) 57**. The idea.** An arrangement with exactly one descent is two increasing runs placed side by side: the first run is some subset A of $\{1, \ldots, 6\}$ written in increasing order, and the second run is the complement, also increasing. So each arrangement corresponds to a subset A, as long as the join really is a descent.

Work it out. Every subset A with $A \neq \emptyset$ and $A \neq \{1, \ldots, 6\}$ gives a well-defined arrangement with *at most* one descent (at the join). The join fails to be a descent exactly when the largest element of A is smaller than the smallest element of the complement, i.e. when $A = \{1, 2, \ldots, j\}$ for some j; those 5 subsets ($j = 1, \ldots, 5$) produce the identity arrangement, with no descent. Also, distinct subsets other than those give distinct arrangements (the first run's length and contents are visible). Count: $2^6 - 2 - 5 = 57$.

Check. For $n = 3$ the same count gives $2^3 - 2 - 2 = 4$: the arrangements 132, 213, 231, 312. ✓ (In general this is $2^n - n - 1$, the Eulerian number $\left\langle {n \atop 1} \right\rangle$.) Direct enumeration of the 720 arrangements of 1 to 6 finds 57. The traps: $62 = 2^6 - 2$ forgets to remove the subsets that give the identity; 64 forgets both corrections; 60 and 120 are guesses from 5! and 6!/12.

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