MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

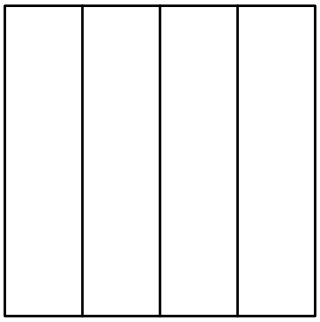
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. What is the smallest positive integer that can be written as the sum of 2026 consecutive integers? (The integers may be negative or zero; for example $-1 + 0 + 1 + 2 = 2$ is a sum of four consecutive integers.)

(A) 1013 **(B)** 2026 **(C)** 2027 **(D)** 4052 **(E)** 4053

2. A square is cut by three lines parallel to one of its sides into four congruent rectangles. Each of the four rectangles has perimeter 20. What is the area of the original square?



(A) 16 **(B)** 25 **(C)** 36 **(D)** 49 **(E)** 64

3. An ant starts at 0 on the number line and makes 2026 moves: $+1$, then -2, then $+3$, then -4, and so on, the kth move being k units to the right if k is odd and k units to the left if k is even. Where is the ant after the 2026th move?

(A) -2026 **(B)** -1013 **(C)** 0 **(D)** 1013 **(E)** 2026

4. The polynomial $x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and its four roots are exactly 1, 2, 3 and 4. What is the value of $a + b + c + d$?

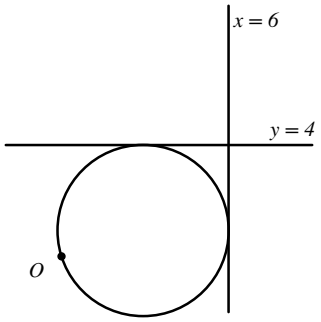
(A) -10 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 24

5. A function f defined on the positive real numbers satisfies $f(xy) = f(x) + f(y)$ for all positive x and y. It is known that $f(2) = 1$ and $f(3) = 2$. What is the value of $f(72)$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 12

6. A circle passes through the origin and is tangent to both the line $y = 4$ and the line $x = 6$. There are exactly two such circles. What is the sum of their radii?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

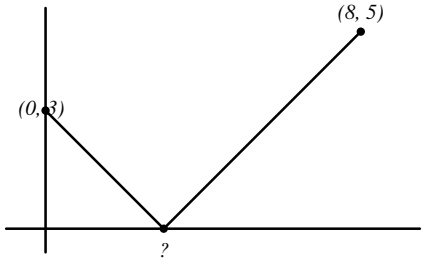


(A) 20 **(B)** 24 **(C)** 26 **(D)** 40 **(E)** 52

7. Two different numbers are chosen at random from $\{1, 2, 3, \ldots, 20\}$, all $\binom{20}{2} = 190$ pairs being equally likely. What is the probability that their product is divisible by 4?

(A) $\frac{2}{5}$ **(B)** $\frac{9}{19}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{10}{19}$ **(E)** $\frac{3}{5}$

8. A ray of light leaves the point $(0, 3)$, reflects off the x-axis (the angle of incidence equals the angle of reflection), and then passes through the point $(8, 5)$. What is the x-coordinate of the point where the ray touches the x-axis?



(A) 2 **(B)** 3 **(C)** 4 **(D)** $\frac{24}{5}$ **(E)** 5

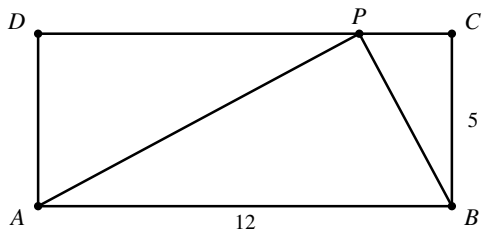
9. What is the smallest positive integer n for which $\sin(n^\circ) < 0$ and $\cos(n^\circ) > 0$ both hold?

(A) 181 **(B)** 269 **(C)** 270 **(D)** 271 **(E)** 360

10. How many positive integers $n \leq 1000$ have the property that the product $n(n+1)(n+2)$ of three consecutive integers is divisible by 8?

(A) 375 **(B)** 500 **(C)** 562 **(D)** 625 **(E)** 750

11. Rectangle $ABCD$ has $AB = 12$ and $BC = 5$. Point P lies on side $\overline{CD}$ so that $\angle APB = 90^\circ$. What is the value of $PC \cdot PD$?



(A) 11 **(B)** 24 **(C)** 25 **(D)** 30 **(E)** 36

12. Ten points are marked on a circle, and all $\binom{10}{2} = 45$ chords joining pairs of them are drawn. Two of the 45 chords are chosen at random. What is the probability that the two chords cross at a point strictly inside the circle?

(A) $\frac{1}{6}$ **(B)** $\frac{2}{11}$ **(C)** $\frac{7}{33}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

13. Let ω be a nonreal complex number with $\omega^3 = 1$. What is the real part of $(1+\omega)^{2026}$?

(A) $-\frac{1}{2}$ **(B)** 0 **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

14. For how many integers n with $1 \le n \le 100$ does the decimal representation of 2^n begin with the digit 1? (You may use the fact that 2^{100} has 31 digits.)

(A) 10 **(B)** 20 **(C)** 29 **(D)** 30 **(E)** 31

15. A right circular cone has slant height 10. When its lateral (curved) surface is cut along a straight line from the apex to the base and unrolled flat, it forms exactly a semicircle. What is the volume of the cone?

(A) $\frac{125\sqrt{3}}{6}\pi$ **(B)** $\frac{125}{3}\pi$ **(C)** $\frac{125\sqrt{3}}{4}\pi$ **(D)** $\frac{125\sqrt{3}}{3}\pi$ **(E)** $\frac{250}{3}\pi$

16. An ant starts at a vertex of a regular tetrahedron. Each second it moves along an edge to one of the three adjacent vertices, chosen at random with equal probability. What is the expected number of seconds until the ant has visited all four vertices?

(A) $\frac{11}{2}$ **(B)** 6 **(C)** $\frac{13}{2}$ **(D)** 7 **(E)** $\frac{15}{2}$

17. For a positive integer n, let $S_n = \dfrac{1}{1\cdot 3} + \dfrac{1}{2\cdot 4} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{n(n+2)}$. What is the smallest n for which $S_n > \dfrac{37}{50}$?

(A) 49 **(B)** 74 **(C)** 97 **(D)** 98 **(E)** 99

18. Each of the 8 vertices of a cube is labelled with 1 or -1. Then each of the 6 faces is labelled with the product of the labels at its four vertices. What is the smallest possible value of the sum of all 14 labels?

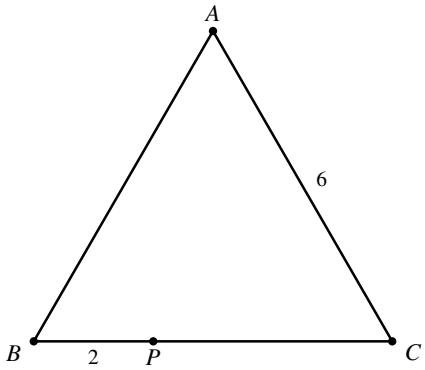
(A) -14 **(B)** -12 **(C)** -10 **(D)** -8 **(E)** -6

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

19. A fair coin is flipped 10 times and the results are written in a row. A *run* is a maximal block of consecutive identical results; for example $HHTTTHHHHH$ has 3 runs. How many of the 2^{10} possible rows have exactly 3 runs?

(A) 36 **(B)** 45 **(C)** 56 **(D)** 64 **(E)** 72

20. Equilateral triangle ABC has side length 6. Point P lies on side $\overline{BC}$ with $BP = 2$. The triangle is folded along a straight line so that vertex A lands exactly on P. What is the length of the fold line (the part of the crease inside the triangle)?



(A) 3 **(B)** $\sqrt{10}$ **(C)** $\frac{7\sqrt{21}}{10}$ **(D)** $2\sqrt{3}$ **(E)** $\frac{7}{2}$

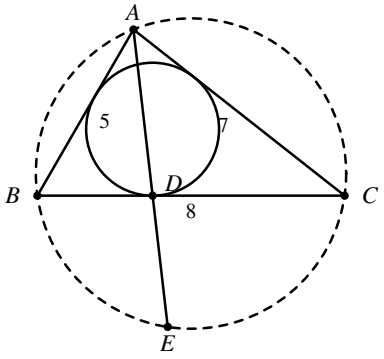
21. For a positive integer n, let $\lfloor\sqrt{n}\rfloor$ be the greatest integer less than or equal to $\sqrt{n}$. How many positive integers $n \le 1000$ have the property that $\lfloor\sqrt{n}\rfloor$ divides n?

(A) 90 **(B)** 92 **(C)** 93 **(D)** 94 **(E)** 96

22. A line through the point $(1, 2)$ meets the parabola $y = x^2$ at two distinct points A and B. The tangent lines to the parabola at A and at B meet at a point P. As the line through $(1, 2)$ varies, the point P always lies on one fixed straight line. What is the y-intercept of that fixed line?

(A) -4 **(B)** -3 **(C)** $-\frac{5}{2}$ **(D)** -2 **(E)** -1

23. Triangle ABC has $AB = 5$, $AC = 7$ and $BC = 8$. The inscribed circle touches side $\overline{BC}$ at D, and line AD meets the circumscribed circle of the triangle again at E. What is the length of $\overline{DE}$?



(A) $\frac{15\sqrt{19}}{19}$ **(B)** $\frac{15}{4}$ **(C)** $\sqrt{19}$ **(D)** $\frac{3\sqrt{19}}{2}$ **(E)** $\frac{15}{2}$

24. Three of the eight vertices of a cube are chosen at random, all $\binom{8}{3} = 56$ choices being equally likely. What is the probability that the plane through the three chosen vertices divides the cube into two solids of equal volume?

(A) $\frac{1}{7}$ **(B)** $\frac{3}{14}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{5}{14}$ **(E)** $\frac{3}{7}$

25. The equation $x^3 + x^2 - 2x - 1 = 0$ has three real roots $r > s > t$. What is the value of $r^2 - s$?

(A) 1 **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{5}{2}$ **(E)** 3

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — AMC 12 Practice Contest 13

1. (A) 1013

2. (E) 64

3. (B) -1013

4. (B) -1

5. (B) 7

6. (A) 20

7. (C) $\frac{1}{2}$

8. (B) 3

9. (D) 271

10. (D) 625

11. (C) 25

12. (C) $\frac{7}{33}$

13. (A) $-\frac{1}{2}$

14. (D) 30

15. (D) $\frac{125\sqrt{3}}{3}\pi$

16. (A) $\frac{11}{2}$

17. (E) 99

18. (C) -10

19. (E) 72

20. (C) $\frac{7\sqrt{21}}{10}$

21. (B) 92

22. (D) -2

23. (A) $\frac{15\sqrt{19}}{19}$

24. (E) $\frac{3}{7}$

25. (C) 2

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — AMC 12 Practice Contest 13

1. Answer: (A) 1013**. The idea.** Write the integers as $a, a+1, \ldots, a+2025$. Their sum is $2026a + (0+1+\cdots+2025) = 2026a + \dfrac{2025 \cdot 2026}{2} = 1013\,(2a+2025)$. Whatever a is, $2a+2025$ is *odd*, so the sum is always an odd multiple of 1013.

Work it out. The smallest positive odd multiple of 1013 is 1013 itself, and it is attained: $2a + 2025 = 1$ gives $a = -1012$, so the integers from -1012 to 1013 work (everything from -1012 to 1012 cancels, leaving 1013).

Check. Summing $-1012, \ldots, 1013$ by computer gives 1013, and no choice of a gives a positive sum below 1013. The traps: 2026 is the number of terms; 2027 and 4053 are not multiples of 1013 at all; $4052 = 4 \cdot 1013$ is an even multiple, impossible.

2. Answer: (E) 64**. The idea.** If the square has side s, each strip is s by $\dfrac{s}{4}$, so one strip's perimeter is $2\left(s + \dfrac{s}{4}\right) = \dfrac{5s}{2}$. The perimeter of a strip pins down s with no further information.

Work it out. $\dfrac{5s}{2} = 20$ gives $s = 8$, and the area is 64.

Check. Strips 8×2 have perimeter $2(8+2) = 20$. ✓ The traps: 25 and 36 come from reading the strip as 5×5 or 6×4; 16 and 49 use $s = 4$ or $s = 7$ from a wrong perimeter equation.

3. Answer: (B) -1013**. The idea.** Group the moves in pairs: $(+1-2) + (+3-4) + \cdots$. Every pair moves the ant exactly 1 unit to the left.

Work it out. 2026 moves make 1013 pairs, so the ant ends at -1013.

Check. After 2 moves: -1; after 4: -2; after 6: -3. ✓ Summing $\sum_{k=1}^{2026}(-1)^{k+1}k$ by computer gives -1013. The traps: $+1013$ pairs with the wrong sign; 0 assumes cancellation; ± 2026 forget to halve.

4. Answer: (B) -1**. The idea.** $a+b+c+d$ is the sum of the coefficients *except* the leading 1, and the sum of all the coefficients of a polynomial P is $P(1)$. Here $P(x) = (x-1)(x-2)(x-3)(x-4)$ and 1 is a root, so $P(1) = 0$.

Work it out. $P(1) = 1 + a + b + c + d = 0$, so $a + b + c + d = -1$.

Check. Expanding, $(x-1)(x-2)(x-3)(x-4) = x^4 - 10x^3 + 35x^2 - 50x + 24$, and $-10+35-50+24 = -1$. ✓ The traps: 0 forgets the leading coefficient; 24 and -10 are single coefficients; 1 has the sign wrong.

5. Answer: (B) 7**. The idea.** The rule turns a product into a sum, so factor $72 = 2^3 \cdot 3^2$ and apply the rule repeatedly: $f(x^n) = nf(x)$ follows by applying it $n-1$ times, so $f(2^3) = 3f(2)$ and $f(3^2) = 2f(3)$.

Work it out. $f(72) = f(2^3) + f(3^2) = 3 \cdot 1 + 2 \cdot 2 = 7$.

Check. The two given values determine f on every number of the form $2^a 3^b$: $f(2^a 3^b) = a + 2b$, and this is consistent with the rule since exponents add when such numbers are multiplied. $f(72) = 3 + 4 = 7$. ✓ The traps: $6 = f(8) + f(3)$ forgets one factor 3; 8 and 9 add the exponents wrongly; 12 multiplies $f(8) \cdot f(9)$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

6. Answer: (A) 20**. The idea.** A circle tangent to $x = 6$ and $y = 4$ and passing through the origin (which lies below and to the left of both lines) has its centre at $(6 - r, 4 - r)$, where r is the radius. The condition "passes through the origin" is then a single equation in r, a quadratic, and the question asks for the sum of its roots: Vieta, no need to solve.

Work it out. $(6 - r)^2 + (4 - r)^2 = r^2$ expands to $r^2 - 20r + 52 = 0$. Both roots are positive (product $52 > 0$, sum $20 > 0$, discriminant $400 - 208 > 0$), so both circles exist and the radii add up to 20.

Check. The roots are $10 \pm 4\sqrt{3} \approx 3.07$ and 16.93; the small circle sits snugly in the corner, the big one wraps around the origin from the far side. ✓ The traps: 24 and 26 come from a wrong expansion; 40 and 52 are twice the sum or the product of the roots.

7. Answer: (C) $\frac{1}{2}$**. The idea.** Count the pairs whose product is *not* divisible by 4. That happens only when both numbers are odd, or when one is odd and the other is twice an odd number $(2, 6, 10, 14, 18)$. Any pair containing a multiple of 4, or two even numbers, is fine.

Work it out. Both odd: $\binom{10}{2} = 45$. One odd and one of $\{2, 6, 10, 14, 18\}$: $10 \cdot 5 = 50$. Bad pairs: 95, so good pairs: $190 - 95 = 95$, and the probability is $\dfrac{95}{190} = \dfrac{1}{2}$.

Check. Direct enumeration of the 190 pairs finds 95 with product divisible by 4. ✓ The traps: $\frac{9}{19}$ and $\frac{10}{19}$ miscount by one pair; $\frac{2}{5}$ and $\frac{3}{5}$ forget the "two evens" case or the "odd times 2" case.

8. Answer: (B) 3**. The idea.** Reflect the starting point across the mirror. The reflected ray from $(0, 3)$ is the straight continuation of the ray from $(0, -3)$: the equal-angle law says exactly that the incoming and outgoing legs make the same angle with the axis, which is what a straight line through $(0, -3)$ and $(8, 5)$ does. So the touch point is where the line through $(0, -3)$ and $(8, 5)$ crosses the x-axis.

Work it out. That line has slope $\dfrac{5 - (-3)}{8 - 0} = 1$, so it is $y = x - 3$, which meets $y = 0$ at $x = 3$.

Check. From $(0, 3)$ to $(3, 0)$ the leg has slope -1; from $(3, 0)$ to $(8, 5)$ the leg has slope $+1$: equal angles of 45° with the axis. ✓ Equivalently the touch point splits the horizontal run 8 in the ratio of the heights, $3 : 5$. The traps: 5 uses the ratio backwards; 4 takes the midpoint; $\frac{24}{5}$ comes from $8 \cdot \frac{3}{5}$ instead of $8 \cdot \frac{3}{3+5}$; 2 is a guess.

9. Answer: (D) 271**. The idea.** Think of the unit circle: sin is the y-coordinate and cos the x-coordinate. Negative y and positive x is the fourth quadrant, the open range of angles strictly between 270° and 360°.

Work it out. The smallest integer strictly greater than 270 is 271. Check the boundary: at 270° the cosine is 0, not positive, so 270 fails.

Check. $\sin 271^\circ \approx -0.9998 < 0$ and $\cos 271^\circ \approx 0.0175 > 0$. ✓ For every n from 1 to 270, either $\sin n^\circ \geq 0$ ($n \leq 180$) or $\cos n^\circ \leq 0$ ($180 < n \leq 270$). The traps: 270 sits on the axis; 181 has negative sine but also negative cosine; 269 is third quadrant; 360 has sine 0.

10. Answer: (D) 625**. The idea.** Split by the parity of n. If n is even, then n and $n+2$ are consecutive even numbers, so one of them is a multiple of 4 and the other a multiple of 2: the product already has a factor 8. If n is odd, then n and $n + 2$ are odd and contribute nothing, so all three factors of 2 must come from $n + 1$: we need $8 \mid n + 1$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Work it out. Even $n \le 1000$: 500 values. Odd n with $n \equiv 7 \pmod 8$: $7, 15, \ldots, 999$, which is $\frac{999-7}{8} + 1 = 125$ values. Total $500 + 125 = 625$.

Check. $n = 2$: $2 \cdot 3 \cdot 4 = 24$. ✓ $n = 7$: $7 \cdot 8 \cdot 9 = 504 = 8 \cdot 63$. ✓ $n = 3$: 60, not divisible. ✓ A computer count gives 625. The traps: 500 counts only the even n; 750 assumes three in four work (as for divisibility by 4); 375 and 562 come from miscounting the residues modulo 8.

11. Answer: (C) 25**. The idea.** $\angle APB = 90°$ means P lies on the circle with diameter AB. That circle has centre the midpoint M of AB and radius 6, and it crosses line CD (which is 5 from M) at two points. Coordinates show that the product $PC \cdot PD$ comes out without ever locating P, and that it is the same for both positions.

Work it out. Put $A = (0,0)$, $B = (12,0)$, $D = (0,5)$, $C = (12,5)$ and $P = (t,5)$. Then $\overrightarrow{PA} \cdot \overrightarrow{PB} = (-t)(12-t) + 25 = 0$, so $t(12-t) = 25$. But $PD = t$ and $PC = 12 - t$, so $PC \cdot PD = 25$ – no need to find t at all.

Check. Solving, $t = 6 \pm \sqrt{11}$, both in $(0, 12)$, so two valid positions exist and $(6+\sqrt{11})(6-\sqrt{11}) = 36 - 11 = 25 = BC^2$. (In general $PC \cdot PD = BC^2$: triangles APD and PBC are similar.) The traps: 36 is MP^2, the square of the radius; 11 is $36 - 25$, the square of the half-distance between the two positions; 24 and 30 are $2 \cdot 12$ and $6 \cdot 5$, products of given lengths.

12. Answer: (C) $\frac{7}{33}$**. The idea.** Count crossing pairs by their endpoints instead of by the chords. Two chords that cross inside the circle have four distinct endpoints, and conversely any four points on the circle determine exactly *one* crossing pair: of the three ways to pair the four points into two chords, only the pairing of opposite points (the "diagonals" of the quadrilateral) crosses.

Work it out. Crossing pairs: $\binom{10}{4} = 210$. All pairs of chords: $\binom{45}{2} = 990$. Probability $\dfrac{210}{990} = \dfrac{7}{33}$.

Check. A direct count by computer over all 990 pairs finds 210 crossing pairs. Note that pairs sharing an endpoint ($10 \cdot \binom{9}{2} = 360$ of them) never cross inside, and the remaining 630 pairs with four distinct endpoints split as 210 crossing, 420 not, matching "one in three". The traps: $\frac{1}{3}$ is the probability *given* four distinct endpoints; $\frac{1}{4}$, $\frac{2}{11}$ and $\frac{1}{6}$ come from miscounting the chord pairs or using $\binom{10}{4}$ against the wrong total.

13. Answer: (A) $-\frac{1}{2}$**. The idea.** Since $\omega \neq 1$ and $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$, we have $1 + \omega + \omega^2 = 0$, so $1 + \omega = -\omega^2$. That turns the awkward base into a pure power of ω.

Work it out. $(1+\omega)^{2026} = (-\omega^2)^{2026} = \omega^{4052}$. Since $4052 = 3 \cdot 1350 + 2$, this is ω^2, the *other* nonreal cube root of unity, whose real part is $\cos 240° = -\frac{1}{2}$ (both nonreal cube roots have real part $-\frac{1}{2}$).

Check. With $\omega = e^{2\pi i/3}$, $1 + \omega = e^{i\pi/3}$ and $(e^{i\pi/3})^{2026} = e^{i \cdot 2026\pi/3} = e^{i \cdot 4\pi/3}$ (since $2026 = 6 \cdot 337 + 4$), with real part $-\frac{1}{2}$. ✓ The traps: $\frac{1}{2}$ has the sign wrong; 1 and 2 treat $1 + \omega$ as real; 0 assumes the real part vanishes.

14. Answer: (D) 30**. The idea.** Doubling a number either keeps its number of digits or adds one, and it adds one exactly when the result starts with 1: if 2^{n-1} starts with $5, 6, 7, 8$ or 9, then 2^n gains a digit and starts with 1; if 2^{n-1} starts with $1, 2, 3$ or 4, then 2^n keeps its digit count and starts with 2 to 9. So "2^n starts with 1" happens exactly when the digit count jumps at step n.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Work it out. From $2^1 = 2$ (1 digit) to 2^{100} (31 digits) the digit count increases 30 times, each time at a different $n \le 100$. So exactly 30 of the powers $2^1, \dots, 2^{100}$ start with 1.

Check. The first few are $2^4 = 16$, $2^7 = 128$, $2^{10} = 1024$, $2^{14} = 16384$, one per digit length. A computer check counts 30. ✓ The traps: 31 counts $2^0 = 1$ or the digit count itself; 29 forgets the final jump; 10 and 20 are guesses from "one in ten".

15. Answer: (D) $\frac{125\sqrt{3}}{3}\pi$**. The idea.** The unrolled surface is a sector whose radius is the slant height and whose arc is the circumference of the base. A semicircle of radius 10 has arc 10π, so the base circumference is 10π and the base radius is 5. The height then comes from the right triangle formed by the radius, height and slant height.

Work it out. $r = 5$, $h = \sqrt{10^2 - 5^2} = 5\sqrt{3}$. Volume $= \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi \cdot 25 \cdot 5\sqrt{3} = \dfrac{125\sqrt{3}}{3}\pi$.

Check. The apex angle of the cross-section is 60° on each side of the axis ($\sin = 5/10$), so the cross-section is an equilateral triangle of side 10 with height $5\sqrt{3}$. ✓ The traps: $\frac{125}{3}\pi$ uses $h = 5$; $\frac{250}{3}\pi$ uses $h = 10$ (the slant height as the height); $\frac{125\sqrt{3}}{6}\pi$ halves the volume; $\frac{125\sqrt{3}}{4}\pi$ uses $\frac{1}{4}$ in place of $\frac{1}{3}$.

16. Answer: (A) $\frac{11}{2}$**. The idea.** Every vertex of a tetrahedron is adjacent to every other, so the only thing that matters is how many vertices have been visited so far: the ant is always at a visited vertex, and from there each unvisited vertex is one step away with probability $\frac{1}{3}$. Count the expected waiting time to reach each new vertex, as in collecting coupons.

Work it out. The first move always reaches a new vertex: 1 second, and 2 vertices visited. With 2 visited, each move lands on an unvisited vertex with probability $\frac{2}{3}$ (two of the three neighbours are new), so the expected wait is $\frac{3}{2}$ seconds. With 3 visited, each move reaches the last vertex with probability $\frac{1}{3}$, so the expected wait is 3 seconds. Total: $1 + \frac{3}{2} + 3 = \frac{11}{2}$.

Check. Why the stages are memoryless: when the ant steps onto an already-visited vertex, the picture from the new vertex is the same as before (the same number of unvisited neighbours), because every vertex sees all the others. Solving the full chain on the 32 states (current vertex, set visited) exactly also gives $\frac{11}{2}$. The traps: $6 = 3 + 3$ and $\frac{15}{2}$ treat the middle stage like the last one; $\frac{13}{2}$ and 7 add an extra move for the starting vertex or for "returning".

17. Answer: (E) 99**. The idea.** Each term splits as $\dfrac{1}{k(k+2)} = \dfrac{1}{2}\left(\dfrac{1}{k} - \dfrac{1}{k+2}\right)$, and the sum telescopes two steps at a time, leaving the first two positive terms and the last two negative ones: $S_n = \dfrac{1}{2}\left(1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right) = \dfrac{3}{4} - \dfrac{2n+3}{2(n+1)(n+2)}$. So $S_n > \dfrac{37}{50}$ means the tail $\dfrac{2n+3}{2(n+1)(n+2)}$ is less than $\dfrac{3}{4} - \dfrac{37}{50} = \dfrac{1}{100}$.

Work it out. The condition is $100(2n+3) < 2(n+1)(n+2)$, i.e. $200n + 300 < 2n^2 + 6n + 4$, i.e. $n^2 - 97n - 148 > 0$. The positive root of $n^2 - 97n - 148 = 0$ is $\dfrac{97 + \sqrt{9409 + 592}}{2} = \dfrac{97 + \sqrt{10001}}{2} \approx \dfrac{97 + 100.005}{2} \approx 98.5$, so the smallest integer n is 99.

Check. $n = 98$: $98^2 - 97 \cdot 98 - 148 = 98 - 148 < 0$, so $S_{98} \le \frac{37}{50}$; $n = 99$: $99^2 - 97 \cdot 99 - 148 = 198 - 148 > 0$. ✓ Direct summation gives $S_{98} = 0.73995\ldots$ and $S_{99} = 0.74005\ldots$ The traps:

98 and 97 are off-by-one readings of the root; 49 solves the inequality with the tail $\frac{1}{n+1}$ only; 74 is $0.74 \cdot 100$.

18. Answer: (C) -10**. The idea.** A face label is -1 exactly when the face has an odd number of -1 vertices. To make the total small we want many -1 vertices *and* many "odd" faces, and these pull against each other: with all 8 vertices equal to -1, every face has four -1s and all six face labels are $+1$. Organise by the number k of -1 vertices.

Work it out. $k = 8$: vertex sum -8, faces all $+1$: total -2. $k = 7$: the lone $+1$ vertex lies on 3 faces, which are odd (three -1s); the other 3 faces are even: total $-6 + (-3 + 3) = -6$. $k = 6$: the two $+1$ vertices either share a face or are opposite ends of a space diagonal; if opposite, every face contains exactly one $+1$ (three -1s, odd), so all six faces are -1: total $-4 - 6 = -10$. If they share a face, two faces are even and the total is only -2. $k \le 5$: the vertex sum is at least -2 and the faces at least -6, so the total is at least -8. The minimum is -10.

Check. A computer search over all $2^8 = 256$ labellings finds minimum -10 (and maximum 14, all labels $+1$). The traps: -14 is unattainable, since six -1 faces force every face to contain a $+1$ vertex; -12 and -8 miscount a case; -6 stops at $k = 7$.

19. Answer: (E) 72**. The idea.** A row with exactly 3 runs is determined by the colour of the first run (2 choices) and by where the two colour changes happen: two of the 9 gaps between consecutive positions. Choosing the gaps is $\binom{9}{2}$, and the gaps cannot coincide, so every choice gives a valid row.

Work it out. $2 \cdot \binom{9}{2} = 2 \cdot 36 = 72$.

Check. A computer count over all 1024 rows finds 72 with exactly 3 runs. In general the number of rows with exactly k runs is $2\binom{9}{k-1}$, and $\sum_k 2\binom{9}{k-1} = 2 \cdot 2^9 = 1024$. ✓ The traps: 36 forgets the choice of the starting colour; $45 = \binom{10}{2}$ chooses from 10 gaps instead of 9; $56 = \binom{8}{3}$ and $64 = 2^6$ come from counting the runs' lengths as compositions in the wrong way.

20. Answer: (C) $\frac{7\sqrt{21}}{10}$**. The idea.** Folding A onto P reflects A across the crease, so the crease is the perpendicular bisector of $\overline{AP}$. It meets $\overline{AB}$ at a point X with $XA = XP$ and $\overline{AC}$ at a point Y with $YA = YP$; the law of cosines in triangles XBP and YCP (with 60° angles) locates X and Y, and then XY follows from triangle AXY.

Work it out. Let $XA = x$, so $XB = 6 - x$ and $XP = x$. In triangle XBP: $x^2 = (6-x)^2 + 2^2 - 2(6-x)(2)\cos 60^\circ = 36 - 12x + x^2 + 4 - 12 + 2x$, so $10x = 28$ and $x = \frac{14}{5}$. Let $YA = y$, so $YC = 6 - y$ and $PC = 4$: $y^2 = (6-y)^2 + 16 - (6-y) \cdot 4 = 36 - 12y + y^2 + 16 - 24 + 4y$, so $8y = 28$ and $y = \frac{7}{2}$. In triangle AXY with $\angle A = 60^\circ$: $XY^2 = \frac{196}{25} + \frac{49}{4} - \frac{14}{5} \cdot \frac{7}{2} = \frac{784+1225-980}{100} = \frac{1029}{100}$, so $XY = \dfrac{\sqrt{1029}}{10} = \dfrac{7\sqrt{21}}{10}$.

Check. Coordinates $B = (0,0)$, $C = (6,0)$, $A = (3, 3\sqrt{3})$, $P = (2,0)$: the perpendicular bisector of AP meets AB at $\frac{8}{15}$ of the way from B and AC at $\frac{5}{12}$ of the way from C, and the distance between those points is $3.2078\ldots = \frac{7\sqrt{21}}{10}$. ✓ The traps: 3 is the crease when A folds onto the midpoint of BC; $\sqrt{10}$, $2\sqrt{3}$ and $\frac{7}{2}$ come from using $XA = YA$ or dropping a term in the law of cosines.

21. Answer: (B) 92**. The idea.** Group the integers by $k = \lfloor \sqrt{n} \rfloor$: those are exactly $n = k^2, k^2 +$

$1, \ldots, k^2 + 2k$ (the next square is $k^2 + 2k + 1$). Writing $n = k^2 + r$ with $0 \le r \le 2k$, k divides n precisely when k divides r, and in the range $0 \le r \le 2k$ that happens for $r = 0$, k and $2k$ only.

Work it out. So every k contributes exactly three values: k^2, $k^2 + k$ and $k^2 + 2k$. For $k = 1, 2, \ldots, 30$ all three lie below $31^2 = 961 \le 1000$, giving 90 values. For $k = 31$ the candidates are 961, 992 and 1023; only the first two are at most 1000. Total $90 + 2 = 92$.

Check. Small cases: $k = 1$ gives $1, 2, 3$ (all divisible by 1); $k = 2$ gives $4, 6, 8$ (and indeed 5 and 7 fail). A computer check over $n \le 1000$ finds 92. The traps: 90 forgets the partial block $k = 31$; 93 includes 1023; 94 and 96 miscount the partial block or the value $r = 0$.

22. Answer: (D) -2. The idea. The tangent to $y = x^2$ at (a, a^2) is $y = 2ax - a^2$ (the line through the point with slope $2a$ meets the parabola only once). Two such tangents, at (a, a^2) and (b, b^2), meet where $2ax - a^2 = 2bx - b^2$, i.e. at $x = \dfrac{a+b}{2}$, $y = ab$. So P is governed by the sum and product of the two x-coordinates – exactly what Vieta extracts from the chord's equation.

Work it out. A chord of slope m through $(1, 2)$ is $y = m(x - 1) + 2$; its intersections with $y = x^2$ satisfy $x^2 - mx + (m - 2) = 0$, so $a + b = m$ and $ab = m - 2$. Then $P = \left(\dfrac{m}{2}, m - 2\right)$: writing $x = \frac{m}{2}$, $y = m - 2 = 2x - 2$. So P always lies on the line $y = 2x - 2$, whose y-intercept is -2.

Check. The chord of slope 1 meets the parabola at $x^2 - x - 1 = 0$, $a + b = 1$, $ab = -1$, so $P = (\frac{1}{2}, -1)$, and indeed $2 \cdot \frac{1}{2} - 2 = -1$. For slope 3: $P = (\frac{3}{2}, 1) = (\frac{3}{2}, 2 \cdot \frac{3}{2} - 2)$. ✓ (The fixed line is the *polar* of $(1, 2)$.) The traps: -1 and -4 are the y-coordinates of P for the slopes 1 and -2; -3 and $-\frac{5}{2}$ come from $ab = m - 3$ or from halving the wrong coordinate.

23. Answer: (A) $\frac{15\sqrt{19}}{19}$. The idea. Two tools meet at D. Power of a point: since A, E and B, C are two chords through D, $DA \cdot DE = DB \cdot DC$. And the touch point is pinned by the tangent lengths: $DB = s - AC$, $DC = s - AB$. So $DE = \dfrac{DB \cdot DC}{DA}$, and DA comes from Stewart's theorem.

Work it out. $s = 10$, so $DB = 10 - 7 = 3$ and $DC = 10 - 5 = 5$. Stewart: $AB^2 \cdot DC + AC^2 \cdot DB = BC\,(AD^2 + DB \cdot DC)$, i.e. $25 \cdot 5 + 49 \cdot 3 = 8(AD^2 + 15)$, so $272 = 8AD^2 + 120$ and $AD^2 = 19$. Then $DE = \dfrac{3 \cdot 5}{\sqrt{19}} = \dfrac{15\sqrt{19}}{19}$.

Check. Coordinates: $B = (0, 0)$, $C = (8, 0)$, $A = \left(\frac{5}{4}, \frac{5\sqrt{15}}{4}\right)$, $D = (3, 0)$; intersecting line AD with the circumcircle gives E with $DE = 3.441\ldots = \frac{15}{\sqrt{19}}$. The traps: $\sqrt{19}$ is AD; $\frac{15}{4}$ and $\frac{15}{2}$ use $AD = 4$ or 2; $\frac{3\sqrt{19}}{2}$ comes from $DE = DB \cdot DC \cdot \frac{AD}{\ldots}$, inverting the power relation.

24. Answer: (E) $\frac{3}{7}$. The idea. A plane divides the cube into equal volumes exactly when it passes through the centre of the cube. (If it does, the reflection through the centre swaps the two pieces, so they have equal volume; if it does not, the plane and its reflection through the centre cut off a slab of positive volume between them, so one side is strictly bigger.) So count the vertex triples whose plane contains the centre – and the centre is the midpoint of each of the four space diagonals, so a plane through two opposite vertices automatically contains it.

Work it out. A plane through the centre is mapped to itself by the reflection through the centre, so it contains the vertex opposite each of its vertices. It therefore contains a pair of opposite vertices, i.e. a space diagonal (three vertices and their three opposites would be six vertices on one plane, and no plane holds more than four vertices of a cube). So the good triples are: choose a space

diagonal (4 ways) and any third vertex (6 ways): 24 triples, no double counting (a triple cannot contain two different space diagonals). Probability $\dfrac{24}{56} = \dfrac{3}{7}$.

Check. Testing all 56 triples by computer (does the plane through them contain $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$?) finds exactly 24. The 24 good planes are the 6 "diagonal rectangles" of the cube, each counted by its 4 vertex triples. The traps: $\frac{3}{14}$ counts each rectangle only twice; $\frac{2}{7} = \frac{16}{56}$ and $\frac{5}{14} = \frac{20}{56}$ miscount the third vertex; $\frac{1}{7} = \frac{8}{56}$ counts only the triples with two space diagonals' worth of symmetry, i.e. one per octant.

25. Answer: (C) 2. The idea. Test whether the map $x \mapsto x^2 - 2$ sends roots to roots. By Vieta, $r + s + t = -1$, $rs + st + tr = -2$, $rst = 1$. The numbers $r^2 - 2$, $s^2 - 2$, $t^2 - 2$ have sum $(r^2 + s^2 + t^2) - 6 = \big((-1)^2 - 2(-2)\big) - 6 = -1$; pairwise-product sum $\sum r^2 s^2 - 4\sum r^2 + 12 = \big((-2)^2 - 2\cdot 1\cdot(-1)\big) - 4\cdot 5 + 12 = -2$; and product $\prod(r^2-2) = \prod(r-\sqrt{2})(r+\sqrt{2}) = P(\sqrt{2})\,P(-\sqrt{2})$ where $P(x) = x^3 + x^2 - 2x - 1$, which is $(2\sqrt{2} + 2 - 2\sqrt{2} - 1)(-2\sqrt{2} + 2 + 2\sqrt{2} - 1) = 1$. Same three symmetric functions, so $\{r^2 - 2, s^2 - 2, t^2 - 2\} = \{r, s, t\}$: squaring-minus-two *permutes* the roots.

Work it out. Which root is $r^2 - 2$? Not r itself: $r^2 - r - 2 = (r - 2)(r + 1)$ and neither 2 nor -1 is a root ($P(2) = 7$, $P(-1) = 1$). Locate the roots by sign changes: $P(1) = -1 < 0 < P(\frac{3}{2}) = \frac{13}{8}$ gives $1 < r < \frac{3}{2}$; $P(-1) = 1 > 0 > P(0) = -1$ gives $-1 < s < 0$; $P(-2) = -1 < 0 < P(-1)$ gives $-2 < t < -1$. Then $r^2 - 2 \in (-1, \frac{1}{4})$, which rules out $t < -1$. So $r^2 - 2 = s$ and $r^2 - s = 2$.

Check. The roots are $2\cos\frac{2\pi}{7} \approx 1.247$, $2\cos\frac{4\pi}{7} \approx -0.445$, $2\cos\frac{6\pi}{7} \approx -1.802$ (substituting $x = 2\cos\theta$ turns the equation into $\cos 3\theta + \cos 2\theta + \cos\theta + \frac{1}{2} = 0$, which holds for $\theta = \frac{2\pi k}{7}$), and $r^2 - 2 = 2\cos\frac{4\pi}{7} = s$ is just the double-angle formula. Numerically $1.247^2 - (-0.445) = 2.000$. ✓ The traps: 1 and 3 come from $r^2 - t$ or $r^2 + s$ guesses; $\frac{3}{2}$ and $\frac{5}{2}$ from estimating $r^2 \approx 1.5$ and mis-signing s.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.