MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A real number x satisfies the equation $x^3 - 3x^2 + 3x = 5$. Using this equation, determine the value of $(x-1)^3$.

(A) -4 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

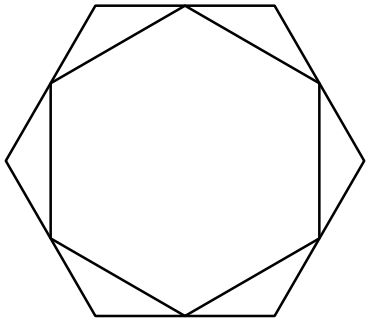
2. A fair coin is flipped repeatedly and the results are recorded in order. The process stops the first time two consecutive flips are both heads or two consecutive flips are both tails (whichever happens first; this can already happen on the first two flips). What is the probability that the process stops with two heads in a row?

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

3. For some angle θ, the numbers $\sin\theta$ and $\cos\theta$ are the two roots of the quadratic equation $2x^2 - x - m = 0$. What is the value of m?

(A) $-\frac{3}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{5}{4}$ **(E)** 2

4. A regular hexagon has area $24\sqrt{3}$. The midpoints of its six sides are joined, in order, to form a smaller regular hexagon. What is the area of the smaller hexagon?



(A) $18\sqrt{3}$ **(B)** $20\sqrt{3}$ **(C)** 36 **(D)** $21\sqrt{3}$ **(E)** $24\sqrt{3}$

5. How many three-digit positive integers (from 100 to 999) have digits that sum to exactly 24?

(A) 6 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 15

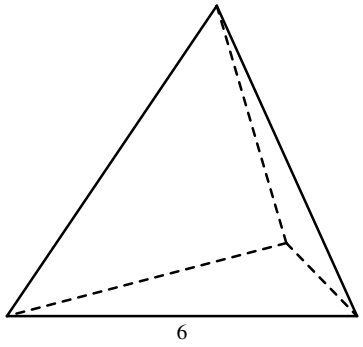
6. A quadratic function $f(x) = x^2 + bx + c$ has integer coefficients with $|b| \le 10$ and $|c| \le 10$. It satisfies $f(f(0)) = 0$ and $f(0) \neq 0$. For how many ordered pairs (b, c) is this possible?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

7. Find the sum of all real numbers x satisfying $4^x + 4^{-x} = 3\,(2^x + 2^{-x}) - 2$.

(A) -4 **(B)** -3 **(C)** -2 **(D)** -1 **(E)** 0

8. A regular tetrahedron has edge length 6. What is the surface area of the smallest sphere that contains it (that is, its circumscribed sphere)?



(A) 54π **(B)** 72π **(C)** 96π **(D)** 108π **(E)** 216π

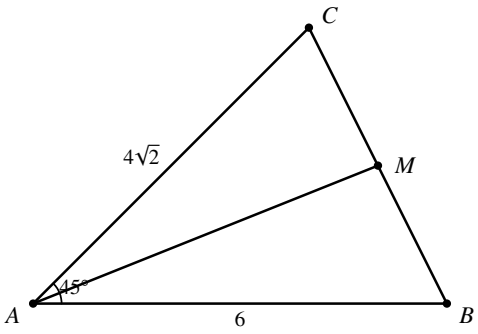
9. Call an integer $n > 1$ *square-factorial* if n^2 divides $n!$. What is the sum of the two smallest square-factorial integers?

(A) 8 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 20

10. Four different numbers are chosen at random from the set $\{1, 2, \ldots, 12\}$, with every set of four numbers equally likely. What is the probability that the four chosen numbers can be arranged, in some order, to form an arithmetic progression (that is, consecutive equal gaps)?

(A) $\frac{1}{55}$ **(B)** $\frac{4}{165}$ **(C)** $\frac{1}{33}$ **(D)** $\frac{2}{55}$ **(E)** $\frac{7}{165}$

11. In triangle ABC, $\angle BAC = 45^\circ$, $AB = 6$, and $AC = 4\sqrt{2}$. Let M be the midpoint of side $\overline{BC}$. What is the length of segment $\overline{AM}$?



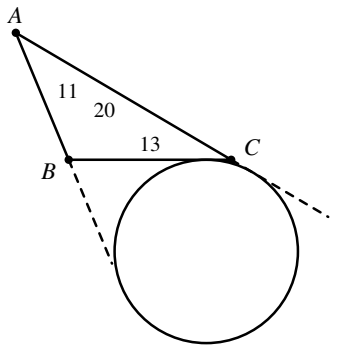
(A) $2\sqrt{5}$ **(B)** 5 **(C)** $\sqrt{29}$ **(D)** $\sqrt{34}$ **(E)** $\sqrt{41}$

12. Let $z = 2 + i$, and let $\overline{z} = 2 - i$ be its complex conjugate. What is the value of $z^5 + \overline{z}^5$?

(A) -152 **(B)** -76 **(C)** -38 **(D)** 38 **(E)** 76

13. Triangle ABC has $AB = 11$, $BC = 13$, and $CA = 20$. The excircle opposite vertex A is the circle outside the triangle that is tangent to side BC, and tangent to the extensions of sides AB and AC beyond B and C respectively. Find the radius of this excircle.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.



(A) 3 **(B)** 4 **(C)** 6 **(D)** $\frac{22}{3}$ **(E)** 33

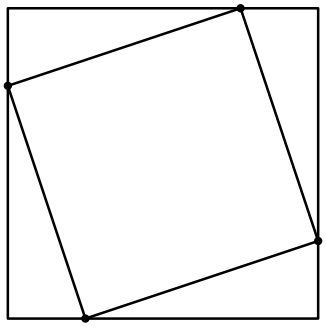
14. For a positive integer n, let $\varphi(n)$ denote Euler's totient function: the number of integers from 1 to n that are relatively prime to n. For how many positive integers n with $1 \le n \le 100$ does $\varphi(3n) = 2\varphi(n)$ hold?

(A) 33 **(B)** 34 **(C)** 66 **(D)** 67 **(E)** 68

15. A monic quadratic polynomial $P(x) = x^2 + bx + c$ (with real b, c) satisfies $P(x)P(-x) = P(x^2)$ for every real x, and $P(1) = 3$. Find $P(5)$.

(A) 16 **(B)** 20 **(C)** 25 **(D)** 30 **(E)** 31

16. Square S_1 has side length 10. To build S_2: on each side of S_1, mark the point that lies $\frac{1}{4}$ of the way from one endpoint to the other, going around the square in the same rotational direction on every side, and connect the four marked points in order; this smaller, tilted square is S_2. Build S_3 from S_2 the same way, then S_4 from S_3, and so on forever. Find the sum of the areas of $S_1, S_2, S_3, \ldots$.



(A) $\frac{400}{3}$ **(B)** 150 **(C)** $\frac{500}{3}$ **(D)** 200 **(E)** $\frac{800}{3}$

17. A right circular cylinder has radius 6. A plane cuts across the lateral surface of the cylinder without touching either circular base, and the resulting cross-section is an ellipse. The two foci of this ellipse are 16 apart. Find the area enclosed by the ellipse.

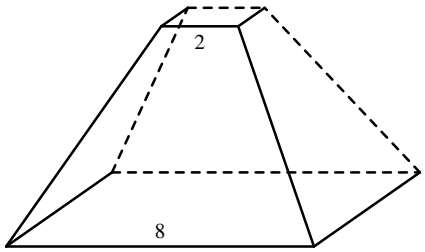
(A) 36π **(B)** 48π **(C)** 60π **(D)** 80π **(E)** 96π

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

18. A fair six-sided die is rolled four times, producing an ordered sequence of results (r_1, r_2, r_3, r_4). What is the probability that *some* three consecutive rolls are strictly increasing – that is, $r_1 < r_2 < r_3$, or $r_2 < r_3 < r_4$ (or both)?

(A) $\frac{25}{144}$ **(B)** $\frac{5}{27}$ **(C)** $\frac{7}{36}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{2}{9}$

19. A frustum of a regular square pyramid has bottom base side length 8 and top base side length 2. The frustum has an inscribed sphere, tangent to both the top and bottom bases and to all four slanted (lateral) faces. Find the radius of this sphere.



(A) 1 **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** 4

20. Find the sum of all real numbers x in the interval $[0, 2\pi)$ that satisfy the equation $2\cos^2(2x) - \cos(2x) - 1 = 0$.

(A) 3π **(B)** $\frac{10\pi}{3}$ **(C)** 4π **(D)** 5π **(E)** 6π

21. Find the smallest positive integer k such that 1001 divides $3^k - 1$ (that is, 3^k leaves remainder 1 when divided by 1001).

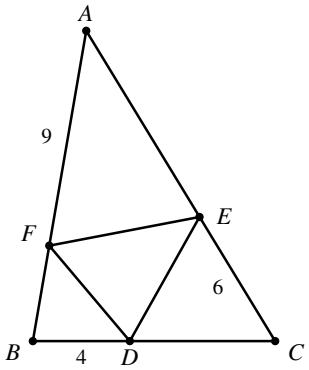
(A) 3 **(B)** 6 **(C)** 10 **(D)** 15 **(E)** 30

22. Real numbers x, y (not assumed positive) satisfy $x + y = xy = x^2 + y^2 - 7$. Find the sum of all possible values of $x + y$.

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

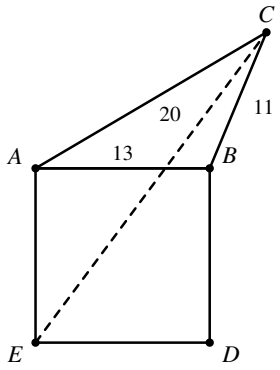
23. In triangle ABC, the incircle is tangent to side BC at D, side CA at E, and side AB at F. The tangent lengths are $BD = 4$, $CE = 6$, and $AF = 9$. Find the ratio $\dfrac{[DEF]}{[ABC]}$ of the area of triangle DEF to the area of triangle ABC.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.



(A) $\frac{72}{325}$ **(B)** $\frac{112}{325}$ **(C)** $\frac{6}{13}$ **(D)** $\frac{207}{325}$ **(E)** $\frac{253}{325}$

24. Triangle ABC has $AB = 13$, $AC = 20$, and $BC = 11$. Square $ABDE$ is drawn outside the triangle on side $\overline{AB}$ (so the square and the triangle lie on opposite sides of line AB), with its four vertices labeled A, B, D, E in order around the square. Find the length of segment $\overline{EC}$.



(A) $\sqrt{305}$ **(B)** $\sqrt{569}$ **(C)** $\sqrt{701}$ **(D)** $7\sqrt{17}$ **(E)** 833

25. A permutation of the numbers 1 through 8 is chosen uniformly at random. Call a position i (for $1 \le i \le 7$) an *ascent* if the number in position i is less than the number in position $i + 1$. Let L be the total number of ascents in the permutation. Find the expected value of L^2.

(A) 12 **(B)** $\frac{49}{4}$ **(C)** 13 **(D)** 14 **(E)** 15

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — AMC 12 Practice Contest 12

1. (B) 4

2. (E) $\frac{1}{2}$

3. (C) $\frac{3}{4}$

4. (A) $18\sqrt{3}$

5. (C) 10

6. (B) 19

7. (E) 0

8. (A) 54π

9. (B) 14

10. (D) $\frac{2}{55}$

11. (C) $\sqrt{29}$

12. (B) -76

13. (D) $\frac{22}{3}$

14. (D) 67

15. (E) 31

16. (E) $\frac{800}{3}$

17. (C) 60π

18. (A) $\frac{25}{144}$

19. (B) 2

20. (D) 5π

21. (E) 30

22. (A) 3

23. (A) $\frac{72}{325}$

24. (D) $7\sqrt{17}$

25. (C) 13

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — AMC 12 Practice Contest 12

1. Answer: (B) 4. The idea. The left side $x^3 - 3x^2 + 3x$ is almost the expansion of $(x-1)^3 = x^3 - 3x^2 + 3x - 1$; it is missing only the constant term -1. Rewriting the equation in terms of $(x-1)^3$ avoids ever solving the cubic for x itself.

Work it out. Since $(x-1)^3 = x^3 - 3x^2 + 3x - 1$, the given equation $x^3 - 3x^2 + 3x = 5$ says $(x-1)^3 + 1 = 5$, so $(x-1)^3 = 4$.

Check. Solving directly, $(x-1)^3 = 4$ gives $x - 1 = \sqrt[3]{4}$, so $x = 1 + \sqrt[3]{4} \approx 2.587$, and substituting back into $x^3 - 3x^2 + 3x$ does equal 5 (it is $(x-1)^3 + 1 = 4 + 1$). Answering with the given value 5 itself forgets the -1 shift; flipping its sign to get a constant $+1$ instead of -1 in the expansion gives the trap 6.

2. Answer: (E) $\frac{1}{2}$. The idea. Swapping the roles of heads and tails throughout the whole process (relabel every H as T and every T as H) turns a run that is "the process ends in HH" into one where "the process ends in TT", and vice versa, without changing any probability, since heads and tails are equally likely at every single flip. So the two outcomes are equally likely, and together they're the only two ways the (certain-to-end) game can finish.

Work it out. After the first flip, the state is "last flip was H" or "last flip was T", each with probability $\frac{1}{2}$. Let p_H be the probability of eventually ending in HH starting from "last was H", and p_T the same starting from "last was T". From "last was H": the next flip is H (prob $\frac{1}{2}$, game ends in HH) or T (prob $\frac{1}{2}$, state becomes "last was T"). So $p_H = \frac{1}{2} + \frac{1}{2}p_T$. Symmetrically $p_T = \frac{1}{2}p_H$. Substituting, $p_H = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}p_H\right)$, so $\frac{3}{4}p_H = \frac{1}{2}$ and $p_H = \frac{2}{3}$, $p_T = \frac{1}{3}$. The overall probability is $\frac{1}{2}p_H + \frac{1}{2}p_T = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$.

Check. This matches the symmetry argument exactly, as it must. Naively assuming the race is decided by the very first two flips alone (an immediate HH) gives the trap $\frac{1}{4}$; averaging incorrectly as if only one state mattered gives a trap like $\frac{1}{3}$.

3. Answer: (C) $\frac{3}{4}$. The idea. By Vieta's formulas, the sum and product of the roots are determined by the quadratic's coefficients. Since $\sin^2\theta + \cos^2\theta = 1$ always, and this can be written in terms of the sum and product of $\sin\theta$ and $\cos\theta$, the Pythagorean identity pins down m without ever finding θ.

Work it out. By Vieta, $\sin\theta + \cos\theta = \dfrac{1}{2}$ and $\sin\theta\cos\theta = -\dfrac{m}{2}$. Squaring the sum gives $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$, so $\dfrac{1}{4} = 1 + 2\left(-\dfrac{m}{2}\right) = 1 - m$, giving $m = \dfrac{3}{4}$.

Check. With $m = \frac{3}{4}$, the equation $2x^2 - x - \frac{3}{4} = 0$ has roots $\dfrac{1 \pm \sqrt{7}}{4}$, and indeed $\left(\dfrac{1+\sqrt{7}}{4}\right)^2 + \left(\dfrac{1-\sqrt{7}}{4}\right)^2 = \dfrac{(1+\sqrt{7})^2 + (1-\sqrt{7})^2}{16} = \dfrac{2 + 2\cdot 7}{16} = 1$, confirming a real θ exists. Using the product directly as $\frac{m}{2}$ instead of $-\frac{m}{2}$ (a sign slip in Vieta's formula) gives the trap $-\frac{3}{4}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

4. Answer: (A) $18\sqrt{3}$**. The idea.** By the hexagon's six-fold symmetry, the six midpoints form a new regular hexagon centered at the same point. Its "radius" (distance from the center to each midpoint) equals the original hexagon's apothem, the perpendicular distance from the center to a side, and area scales with the square of that radius.

Work it out. For a regular hexagon with circumradius (and side length) R, the apothem is $R\cos 30^\circ = \dfrac{R\sqrt{3}}{2}$. Each midpoint of a side lies at exactly this distance from the center, so the smaller hexagon is similar to the original with scale factor $\dfrac{\text{apothem}}{R} = \dfrac{\sqrt{3}}{2}$, and areas scale by the square of this ratio, $\dfrac{3}{4}$. So the smaller hexagon's area is $24\sqrt{3} \cdot \dfrac{3}{4} = 18\sqrt{3}$.

Check. Direct coordinates confirm this: placing the big hexagon's vertices at angles $0^\circ, 60^\circ, \ldots, 300^\circ$ and computing the midpoints' hexagon area by the shoelace formula gives the same $18\sqrt{3}$. Forgetting to shrink the hexagon at all (simply reporting the original area) gives the trap $24\sqrt{3}$; using the ratio $\frac{7}{8}$ in place of $\frac{3}{4}$ (confusing it with a different similar-looking fraction) gives $21\sqrt{3}$; multiplying by $\frac{\sqrt{3}}{2}$ itself without squaring (mixing up a length ratio with an area ratio) gives 36.

5. Answer: (C) 10**. The idea.** The digit sum 24 is close to the maximum possible digit sum 27 (achieved by 999), so it is easier to count how far each digit falls short of 9 than to count the digits directly.

Work it out. Write the number as $\overline{abc}$ with $a \in \{1, \ldots, 9\}$, $b, c \in \{0, \ldots, 9\}$, and $a + b + c = 24$. Let $a' = 9 - a$, $b' = 9 - b$, $c' = 9 - c$; each of a', b', c' is a digit from 0 to 9 (and $a' \le 8$, though that bound never binds below), and $a' + b' + c' = 27 - 24 = 3$. The number of ways to write 3 as an ordered sum of three digits 0 through 9 is just the number of nonnegative integer solutions of $a' + b' + c' = 3$, which is $\binom{3+2}{2} = \binom{5}{2} = 10$ (none of them need a digit bigger than 3, so no upper bound is ever an issue).

Check. Listing a few: (a, b, c) with $a+b+c = 24$ include $(9, 9, 6)$ and its rearrangements, $(9, 8, 7)$ and its rearrangements, and $(8, 8, 8)$ – that's 3+6+1 = 10 numbers total (699, 969, 996, 789, 798, 879, 897, 978, 987 matching the complement count exactly. Using $\binom{4}{2} = 6$ (miscounting the stars-and-bars formula) or $\binom{6}{2} = 15$ (an off-by-one in the other direction) are the traps.

6. Answer: (B) 19**. The idea.** Since $f(0) = c$, the condition $f(f(0)) = 0$ really says $f(c) = 0$. Plugging c into the quadratic and factoring out the (nonzero) value c turns this into a simple linear relation between b and c.

Work it out. $f(c) = c^2 + bc + c = c(c + b + 1) = 0$. Since $f(0) = c \neq 0$, the other factor must vanish: $c + b + 1 = 0$, i.e., $b = -c - 1$. Now impose the bounds: $|c| \le 10$ gives $c \in \{-10, \ldots, 10\}$, and $|b| = |-c - 1| \le 10$ means $-10 \le c + 1 \le 10$, i.e., $-11 \le c \le 9$. Combining both ranges and excluding $c = 0$: c ranges over $\{-10, \ldots, 9\} \setminus \{0\}$, which has $20 - 1 = 19$ values, each giving exactly one valid b.

Check. At the extremes: $c = -10$ gives $b = 9$ (both within bounds), and $c = 9$ gives $b = -10$ (also within bounds), while $c = 10$ would need $b = -11$, out of range – correctly excluded. Forgetting to exclude $c = 0$ (which would force $b = -1$ and $f(0) = 0$, violating the problem's condition) gives the trap 20.

7. Answer: (E) 0**. The idea.** Let $t = 2^x + 2^{-x}$. Since $4^x + 4^{-x} = (2^x + 2^{-x})^2 - 2 = t^2 - 2$, the whole equation becomes a quadratic in t, and $t \ge 2$ always (AM-GM). Once t is known, a second

substitution $u = 2^x$ reduces the problem to a quadratic whose two roots multiply to 1 – and that is exactly what controls the sum of the x-values.

Work it out. The equation becomes $t^2 - 2 = 3t - 2$, i.e., $t^2 - 3t = 0$, so $t = 0$ or $t = 3$. Since $t = 2^x + 2^{-x} \geq 2$ for all real x, only $t = 3$ is possible. Let $u = 2^x > 0$; then $u + \frac{1}{u} = 3$, i.e., $u^2 - 3u + 1 = 0$, with roots $u_1, u_2 = \dfrac{3 \pm \sqrt{5}}{2}$, both positive. By Vieta, $u_1 u_2 = 1$. The corresponding x-values are $x_1 = \log_2 u_1$ and $x_2 = \log_2 u_2$, and $x_1 + x_2 = \log_2(u_1 u_2) = \log_2 1 = 0$.

Check. Numerically, $u_1 \approx 2.618$ and $u_2 \approx 0.382$, so $x_1 \approx 1.389$ and $x_2 \approx -1.389$: they really are negatives of each other, since $u_2 = 1/u_1$ makes $x_2 = -x_1$ automatically. Clearing denominators directly and solving the resulting quartic in u gives the same two positive roots (and two non-real ones), confirming there are exactly two real solutions. Mistaking $t = 0$ as valid (ignoring $t \geq 2$) would introduce extra, non-existent solutions.

8. Answer: (A) 54π**. The idea.** A regular tetrahedron can be formed by four alternating vertices of a cube (every edge of the tetrahedron is then a face diagonal of the cube). Those four vertices automatically lie on the cube's circumscribed sphere, which is therefore the tetrahedron's circumsphere too – the smallest sphere containing it, since a regular tetrahedron's circumcenter lies inside it.

Work it out. If the cube has edge a, its face diagonal is $a\sqrt{2}$, which must equal the tetrahedron's edge: $a\sqrt{2} = 6$, so $a = 3\sqrt{2}$. The cube's space diagonal is $a\sqrt{3} = 3\sqrt{6}$, and the circumradius of the cube (half the space diagonal) is $R = \dfrac{3\sqrt{6}}{2}$. So $R^2 = \dfrac{9 \cdot 6}{4} = \dfrac{27}{2}$, and the surface area is $4\pi R^2 = 4\pi \cdot \dfrac{27}{2} = 54\pi$.

Check. Placing the cube's corners at $(0,0,0)$, $(a,a,0)$, $(a,0,a)$, $(0,a,a)$ (an alternating set) and solving directly for the point equidistant from all four confirms the center is $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$ (the cube's center) with $R^2 = 3\left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} = \frac{3 \cdot 18}{4} = \frac{27}{2}$, the same value. Using the cube's edge $a = 6$ directly (forgetting to solve $a\sqrt{2} = 6$) gives the trap 108π.

9. Answer: (B) 14**. The idea.** For small n, $n!$ simply does not have enough factors to supply n^2; the best way in is to check each n directly, prime by prime, rather than look for a general formula.

Work it out. Check $n = 2, 3, 4, 5$: $n! = 2, 6, 24, 120$, and none is divisible by $4, 9, 16, 25$ respectively. At $n = 6$: $6! = 720$ and $6^2 = 36$; indeed $720/36 = 20$, an integer, so $n = 6$ works. At $n = 7$: $7! = 5040$ and $7^2 = 49$; $5040/49$ is not an integer (since only one factor of 7 appears in $7!$). At $n = 8$: $8! = 40320$ and $8^2 = 64$; $40320/64 = 630$, an integer, so $n = 8$ works. So the two smallest square-factorial integers are 6 and 8, and their sum is 14.

Check. The key reason $n = 6$ and $n = 8$ succeed while $n = 7$ fails is the exponent of the relevant prime in $n!$: $6 = 2 \cdot 3$ needs one more 2 and one more 3 beyond what's in 6 itself, and $5! = 120 = 2^3 \cdot 3 \cdot 5$ supplies both; $8 = 2^3$ needs $v_2(8!) \geq 6$, and indeed $v_2(8!) = 7$. But 7 is prime, so $7^2 \mid 7!$ would require two multiples of 7 at or below 7, and there is only one. Answering with just the first success, 8 (forgetting 6 is smaller), or including 9 (which also works, since $v_3(9!) = 4$, but is the *third* smallest, not one of the two smallest) are the traps.

10. Answer: (D) $\frac{2}{55}$**. The idea.** A set of four numbers can be arranged into an arithmetic progression exactly when, sorted into increasing order, consecutive differences are all equal. Such a set is determined by its smallest term a and common difference $d \geq 1$, so count by casework on d.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Work it out. For a 4-term progression $a, a+d, a+2d, a+3d$ to fit in $\{1, \ldots, 12\}$, we need $a \geq 1$ and $a+3d \leq 12$. For $d = 1$: a ranges from 1 to 9, giving 9 sets. For $d = 2$: $a+6 \leq 12$, so a from 1 to 6: 6 sets. For $d = 3$: $a+9 \leq 12$, so a from 1 to 3: 3 sets. For $d \geq 4$: $a+3d \geq 1+12 = 13 > 12$, impossible. So there are $9+6+3 = 18$ favorable sets, out of $\binom{12}{4} = 495$ total. The probability is $\dfrac{18}{495} = \dfrac{2}{55}$.

Check. A direct computer enumeration of all 495 four-element subsets of $\{1, \ldots, 12\}$, testing whether the sorted gaps are equal, finds exactly 18 such subsets, confirming the casework. Forgetting the $d = 3$ case gives the trap $\frac{15}{495} = \frac{1}{33}$; counting only $d = 1$ progressions gives $\frac{9}{495} = \frac{1}{55}$.

11. Answer: (C) $\sqrt{29}$. The idea. With one angle and its two adjacent sides known, the law of cosines gives the third side BC. Then the median-length formula $m_a^2 = \dfrac{2AB^2 + 2AC^2 - BC^2}{4}$ (from doubling the median to form a parallelogram) finishes the problem without any coordinates.

Work it out. By the law of cosines, $BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cos 45^\circ = 36 + 32 - 2(6)(4\sqrt{2}) \cdot \dfrac{\sqrt{2}}{2} = 68 - 48 = 20$. Then $AM^2 = \dfrac{2(36) + 2(32) - 20}{4} = \dfrac{72 + 64 - 20}{4} = \dfrac{116}{4} = 29$, so $AM = \sqrt{29}$.

Check. Coordinates confirm this: put $A = (0,0)$, $B = (6,0)$, and $C = \left(4\sqrt{2}\cos 45^\circ, 4\sqrt{2}\sin 45^\circ\right) = (4,4)$. Then $M = (5,2)$ and $AM = \sqrt{25+4} = \sqrt{29}$, matching exactly (and indeed $BC = \sqrt{(6-4)^2 + 4^2} = \sqrt{20}$, confirming the law-of-cosines step too). Swapping which side plays the role of a in the median formula (using BC and AB in place of AB and AC) gives the trap $\sqrt{20}$, which is really just BC itself.

12. Answer: (B) -76. The idea. Since z and $\overline{z}$ are the two roots of the quadratic $t^2 - 4t + 5 = 0$ (sum 4, product 5, matching $z + \overline{z} = 4$ and $z\overline{z} = |z|^2 = 5$), the power sums $s_n = z^n + \overline{z}^n$ satisfy the linear recurrence $s_n = 4s_{n-1} - 5s_{n-2}$. This avoids expanding z^5 by repeated multiplication.

Work it out. Start with $s_0 = 2$ and $s_1 = z + \overline{z} = 4$. Then $s_2 = 4(4) - 5(2) = 6$, $s_3 = 4(6) - 5(4) = 4$, $s_4 = 4(4) - 5(6) = -14$, and $s_5 = 4(-14) - 5(4) = -56 - 20 = -76$.

Check. Direct multiplication confirms it: $z^2 = 3 + 4i$, $z^3 = (3+4i)(2+i) = 2 + 11i$, $z^4 = (2+11i)(2+i) = -7 + 24i$, $z^5 = (-7+24i)(2+i) = -38 + 41i$. Since $\overline{z}^5 = \overline{z^5} = -38 - 41i$, the sum is $-38 + 41i + (-38 - 41i) = -76$, matching the recurrence exactly. Dropping the conjugate term entirely (just doubling the real part incorrectly, or reporting $2\,\mathrm{Re}(z^5)/2 = -38$) gives the trap -38.

13. Answer: (D) $\frac{22}{3}$. The idea. For the excircle opposite A, with center I_A and radius r_A, splitting triangle ABC using I_A gives $[ABC] = [ABI_A] + [ACI_A] - [BCI_A]$ (the triangle BCI_A is subtracted because I_A lies on the far side of BC from A). Each of these three triangles has height r_A to the respective side AB, AC, or BC, so $[ABC] = \frac{1}{2} r_A(c + b - a) = r_A(s - a)$, where s is the semiperimeter and $a = BC$. So $r_A = \frac{[ABC]}{s-a}$.

Work it out. With $a = 13, b = 20, c = 11$: $s = \frac{13+20+11}{2} = 22$, so $s - a = 9$. Heron's formula gives $[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{22 \cdot 9 \cdot 2 \cdot 11} = \sqrt{4356} = 66$. So $r_A = \frac{66}{9} = \frac{22}{3}$.

Check. Placing $B = (0,0)$, $C = (13,0)$ and solving for A from $AB = 11, CA = 20$ gives $A = \left(-\frac{55}{13}, \frac{132}{13}\right)$. The excenter opposite A has the barycentric formula $I_A = \frac{-a \cdot A + b \cdot B + c \cdot C}{-a+b+c}$, which

evaluates to $I_A = \left(11, -\frac{22}{3}\right)$; its distance to line BC (the x-axis) is $\frac{22}{3}$, confirming the answer by an entirely different route. Using the *inradius* formula $r = \frac{[ABC]}{s}$ instead of the exradius formula gives the trap $\frac{66}{22} = 3$; using $s - b = 2$ or $s - c = 11$ in place of $s - a = 9$ (the exradii opposite the other two vertices) gives the traps 33 and 6.

14. Answer: (D) 67**. The idea.** Euler's totient function is multiplicative, but $3n$ and n are not generally coprime, so the behavior of $\varphi(3n)$ splits into two cases depending on whether 3 already divides n.

Work it out. *Case 1:* $3 \nmid n$. Then $\gcd(3, n) = 1$, so $\varphi(3n) = \varphi(3)\varphi(n) = 2\varphi(n)$ – the desired equation holds automatically for every such n. *Case 2:* $3 \mid n$. Write $n = 3^k m$ with $k \geq 1$ and $3 \nmid m$. Then $\varphi(n) = \varphi(3^k)\varphi(m) = 2 \cdot 3^{k-1}\varphi(m)$, while $\varphi(3n) = \varphi(3^{k+1}m) = 2 \cdot 3^k \varphi(m) = 3\varphi(n)$. Since $\varphi(n) > 0$, this gives $\varphi(3n) = 3\varphi(n) \neq 2\varphi(n)$ – the equation always fails here. So the condition holds exactly for n not divisible by 3. Among 1 through 100, there are $\left\lfloor \frac{100}{3} \right\rfloor = 33$ multiples of 3, leaving $100 - 33 = 67$ values of n that work.

Check. Computing $\varphi(3n)$ and $\varphi(n)$ directly for every n from 1 to 100 and comparing confirms exactly 67 matches, all of them exactly the multiples of 3 excluded – consistent with the case analysis. Reporting the count of multiples of 3 instead of its complement gives the trap 33; an off-by-one slip in $\lfloor 100/3 \rfloor$ gives 34 or the traps 66/68.

15. Answer: (E) 31**. The idea.** Expand both sides of $P(x)P(-x) = P(x^2)$ as polynomials in x and match coefficients; this pins down b and c up to a short list of cases, and the extra condition $P(1) = 3$ picks out the right one.

Work it out. With $P(x) = x^2+bx+c$, $P(x)P(-x) = (x^2+bx+c)(x^2-bx+c) = x^4+(2c-b^2)x^2+c^2$, while $P(x^2) = x^4 + bx^2 + c$. Matching coefficients of x^2 and the constant term: $2c - b^2 = b$ and $c^2 = c$. The second equation gives $c = 0$ or $c = 1$. If $c = 0$: $-b^2 = b \Rightarrow b = 0$ or $b = -1$, giving $P(1) = 1 + b + c \in \{1, 0\}$. If $c = 1$: $2 - b^2 = b \Rightarrow b^2 + b - 2 = 0 \Rightarrow b = 1$ or $b = -2$, giving $P(1) \in \{3, 0\}$. Only $b = 1$, $c = 1$ gives $P(1) = 3$, so $P(x) = x^2+x+1$, and $P(5) = 25+5+1 = 31$.

Check. Directly verify the identity for $P(x) = x^2 + x + 1$: $P(x)P(-x) = (x^2 + x + 1)(x^2 - x + 1) = x^4 + x^2 + 1$, and $P(x^2) = x^4 + x^2 + 1$ – they match exactly, for every x, confirming this is a genuine solution (and $P(1) = 1+1+1 = 3$ as required). The other three branches give $P(5) \in \{16, 20, 25\}$ but all fail $P(1) = 3$, so none is a valid trap answer by accident – they appear only as distractors here.

16. Answer: (E) $\frac{800}{3}$**. The idea.** At each corner of S_n, the marked point cuts off a small right triangle with legs $\frac{1}{4}$ and $\frac{3}{4}$ of S_n's side s; the side of S_{n+1} is the hypotenuse of that triangle. So every square's area shrinks by the same constant factor when you pass to the next one, and the full list of areas is a geometric series.

Work it out. If S_n has side s, then S_{n+1}'s side length squared is $\left(\frac{s}{4}\right)^2 + \left(\frac{3s}{4}\right)^2 = \frac{1+9}{16}s^2 = \frac{5}{8}s^2$. So $[S_{n+1}] = \frac{5}{8}[S_n]$ for every n, and the areas $[S_1], [S_2], [S_3], \ldots$ form a geometric series with first term $[S_1] = 10^2 = 100$ and common ratio $\frac{5}{8}$. Its sum is $\dfrac{100}{1 - \frac{5}{8}} = \dfrac{100}{\frac{3}{8}} = \dfrac{800}{3}$.

Check. Place S_1 at $(0, 0), (10, 0), (10, 10), (0, 10)$. The points a quarter of the way around (in the same rotational direction) are $(2.5, 0), (10, 2.5), (7.5, 10), (0, 7.5)$; the shoelace formula on these four points gives area 62.5, and indeed $62.5 = \frac{5}{8} \cdot 100$ – confirming the ratio directly

from coordinates rather than assuming the Pythagorean argument generalizes. Using the familiar midpoint-square shrink factor $\frac{1}{2}$ instead of $\frac{5}{8}$ gives the trap 200; summing only $S_2, S_3, \ldots$ and forgetting to include S_1 gives $\frac{500}{3}$; shrinking the *side* (not the area) by $\frac{1}{4}$ each time gives $\frac{400}{3}$; and mistaking the $\frac{1}{4}$ division ratio itself for the area-shrink ratio (skipping the Pythagorean step) gives 150.

17. Answer: (C) 60π**. The idea.** Picture the cylinder's axis as the z-axis and the cutting plane tilted at some angle θ to the base. In the direction perpendicular to the tilt, the cross-section is unstretched, so the ellipse's semi-minor axis equals the cylinder's radius, $b = 6$. In the tilt direction, lengths are stretched by a factor of $\frac{1}{\cos\theta}$, giving semi-major axis $a = \frac{6}{\cos\theta}$. The distance between the foci, $2c$, then pins down θ (and hence a) via $c = \sqrt{a^2 - b^2}$.

Work it out. Writing $t = \tan\theta$, $a = 6\sqrt{1+t^2}$ (since $\frac{1}{\cos\theta} = \sqrt{1+\tan^2\theta}$), so $c^2 = a^2 - b^2 = 36(1+t^2) - 36 = 36t^2 \implies c = 6t$. Given $2c = 16$, $c = 8$, so $t = \frac{4}{3}$ and $a = 6\sqrt{1+\frac{16}{9}} = 6\cdot\frac{5}{3} = 10$. The area is $\pi ab = \pi(10)(6) = 60\pi$.

Check. Parametrize the cylinder's surface by angle φ: a point is $(6\cos\varphi,\ 6\sin\varphi,\ 6t\cos\varphi)$, which indeed lies in the tilted plane $z = tx$. Measuring this point's coordinates *within* that plane (using an orthonormal basis for the plane) gives exactly $u = 6\sqrt{1+t^2}\cos\varphi$ and $v = 6\sin\varphi$ – an ellipse of semi-axes $6\sqrt{1+t^2}$ and 6, confirming the shortcut used above without just asserting it. Using the diameter 12 in place of the radius 6 for b gives the trap 96π; mistaking $c = 8$ itself for the semi-major axis gives $\pi\cdot 8\cdot 6 = 48\pi$.

18. Answer: (A) $\frac{25}{144}$**. The idea.** Let A be the event $r_1 < r_2 < r_3$ and B be the event $r_2 < r_3 < r_4$. We want $P(A \cup B)$, which by inclusion-exclusion is $P(A) + P(B) - P(A \cap B)$ – the overlap $A \cap B$ is exactly the event that all four rolls are strictly increasing, and it must be subtracted so outcomes satisfying both aren't double-counted.

Work it out. For A: choose any 3 of the 6 values for r_1, r_2, r_3 (they must be distinct, and there's only one increasing order for a chosen set), giving $\binom{6}{3} = 20$ ways, times 6 free choices for r_4: $|A| = 120$. By the same count, $|B| = 120$. For $A \cap B$ (all four strictly increasing), choose 4 of the 6 values: $\binom{6}{4} = 15$ ways, with only one valid order: $|A\cap B| = 15$. So $|A\cup B| = 120+120-15 = 225$, out of $6^4 = 1296$ total sequences, giving probability $\frac{225}{1296} = \frac{25}{144}$.

Check. A direct computer enumeration of all 1296 ordered quadruples, flagging each one that has $r_1 < r_2 < r_3$ or $r_2 < r_3 < r_4$, counts exactly 225 of them – matching without using inclusion-exclusion at all. Forgetting to subtract the overlap $|A \cap B| = 15$ gives the trap $\frac{240}{1296} = \frac{5}{27}$.

19. Answer: (B) 2**. The idea.** Look at the cross-section cut by the vertical plane through the axis and perpendicular to one pair of opposite lateral faces: it is an isosceles trapezoid with parallel sides 8 and 2 (the two base edges it crosses), and the inscribed sphere shows up in this cross-section as the trapezoid's own inscribed circle. A circle can be inscribed in a trapezoid exactly when the sum of the two parallel sides equals the sum of the two legs (it's a tangential quadrilateral); that condition pins down the leg length, and then the height follows from the Pythagorean theorem. Finally, since the circle is tangent to both parallel sides, its diameter equals the trapezoid's height.

Work it out. The tangential condition gives (leg)+(leg)= 8 + 2 = 10, so each leg is 5. The two parallel sides have half-lengths (apothems) 4 and 1, which differ by 3 horizontally, so if h is the frustum's height, $h^2+3^2 = 5^2$, giving $h = 4$. The sphere's diameter is h, so its radius is $r = \frac{h}{2} = 2$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Check. Place the bottom square's center at the origin with the base in the plane $z = 0$ and the top square's center at $(0, 0, h)$. By symmetry the sphere's center is $(0, 0, r)$ with $2r = h$. Building the plane of one lateral face explicitly from the points $(4, 0, 0)$ and $(1, 0, h)$ (it doesn't depend on y) and setting the distance from $(0, 0, r)$ to that plane equal to r gives $h = 4$ directly, with no use of the tangential-trapezoid shortcut – confirming $r = 2$ by an independent route. Reporting the top apothem itself gives the trap 1; using the leg length 5 as the height (skipping the Pythagorean step) and halving it gives $\frac{5}{2}$; using the apothem difference 3 as the radius gives 3; and reporting the height $h = 4$ itself instead of halving it gives 4.

20. Answer: (D) 5π**. The idea.** Treat $\cos(2x)$ as a single unknown and factor the resulting quadratic; each resulting value of $\cos(2x)$ then has to be solved for x over the full range $[0, 2\pi)$, remembering that $2x$ itself sweeps out $[0, 4\pi)$ – two full periods – as x ranges over $[0, 2\pi)$, so each cosine equation can contribute more solutions than it would over a single period.

Work it out. Let $c = \cos(2x)$. Then $2c^2 - c - 1 = 0$ factors as $(2c + 1)(c - 1) = 0$, so $c = -\frac{1}{2}$ or $c = 1$. For $2x \in [0, 4\pi)$: $\cos(2x) = 1$ at $2x = 0, 2\pi$, i.e. $x = 0, \pi$; and $\cos(2x) = -\frac{1}{2}$ at $2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$, i.e. $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. In all, the six solutions in $[0, 2\pi)$ are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, and their sum is $(0 + 1 + 2 + 3 + 4 + 5) \cdot \frac{\pi}{3} = 15 \cdot \frac{\pi}{3} = 5\pi$.

Check. Solving $\cos(2x) = 1$ and $\cos(2x) = -\frac{1}{2}$ directly for x over the real line and intersecting with $[0, 2\pi)$ (rather than hand-tracking the doubled period) gives the same six values, confirming none were missed or duplicated. Only counting the four solutions from $\cos(2x) = -\frac{1}{2}$ and forgetting $x = 0, \pi$ gives the trap 4π.

21. Answer: (E) 30**. The idea.** Factor $1001 = 7 \cdot 11 \cdot 13$. The condition $1001 \mid 3^k - 1$ means $3^k \equiv 1$ modulo each of 7, 11, and 13 simultaneously, so k must be a multiple of the multiplicative order of 3 modulo each prime; the smallest such k is the *lcm* of those three orders (not their product).

Work it out. Modulo 7: $3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5, 3^6 = 1$, so the order is 6. Modulo 11: $3^1 = 3, 3^2 = 9, 3^3 = 27 \equiv 5, 3^4 \equiv 15 \equiv 4, 3^5 \equiv 12 \equiv 1$, so the order is 5. Modulo 13: $3^1 = 3, 3^2 = 9, 3^3 = 27 \equiv 1$, so the order is 3. The smallest k making $3^k \equiv 1$ in all three moduli at once is $\operatorname{lcm}(6, 5, 3) = 30$.

Check. Direct computation confirms $3^{30} - 1$ is divisible by 1001: indeed $3^{30} = (3^6)^5 \equiv 1^5 = 1 \pmod 7$, $3^{30} = (3^5)^6 \equiv 1^6 = 1 \pmod{11}$, and $3^{30} = (3^3)^{10} \equiv 1^{10} = 1 \pmod{13}$, and no smaller positive k works since k must be a multiple of 6, 5, and 3 all at once, and 30 is the least common multiple. Multiplying the three orders instead of taking their lcm gives the trap 90; using only the order mod 11 and mod 13 (forgetting the factor of 7) gives the trap 15.

22. Answer: (A) 3**. The idea.** Let $s = x + y$. Since s also equals xy (given), and $x^2 + y^2 = s^2 - 2xy = s^2 - 2s$, the third equation $xy = x^2 + y^2 - 7$ becomes a single equation in s alone.

Work it out. Substituting, $s = (s^2 - 2s) - 7$, i.e. $s^2 - 3s - 7 = 0$. By Vieta's formula, the sum of the two roots of this quadratic is $\frac{-(-3)}{1} = 3$ – without even needing to find the roots individually (they are $\frac{3 \pm \sqrt{37}}{2}$). Both roots must still be checked for validity: x, y are themselves the two roots of $t^2 - st + s = 0$ (since $x + y = s$ and $xy = s$), which are real exactly when $s^2 - 4s \geq 0$, i.e. $s \leq 0$ or $s \geq 4$; both $\frac{3 - \sqrt{37}}{2} \approx -1.54$ and $\frac{3 + \sqrt{37}}{2} \approx 4.54$ satisfy this (the first is negative, the second exceeds 4), so both values of s genuinely come from a real pair (x, y). The sum of all possible values of $x + y$ is 3.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Check. For $s = \frac{3+\sqrt{37}}{2} \approx 4.54$, solving $t^2 - st + s = 0$ numerically gives two positive real roots $x \approx 3.05$, $y \approx 1.49$ (their sum and product both check out to ≈ 4.54). For $s = \frac{3-\sqrt{37}}{2} \approx -1.54$, the same equation gives real roots $x \approx 0.69$, $y \approx -2.23$ – genuinely real, though not both positive, which is exactly why the problem allows any real x, y rather than requiring them positive. Discarding the negative branch (as one would if x, y had to be positive) and reporting only the other root's value of s (rounded) is the source of the nearby traps.

23. Answer: (A) $\frac{72}{325}$. The idea. Since tangents from the same point to a circle are equal, $BD = BF = 4$, $CD = CE = 6$, and $AE = AF = 9$ (writing $x = 4$, $y = 6$, $z = 9$). This pins down the side lengths $a = BC = x+y = 10$, $b = CA = y+z = 15$, $c = AB = z+x = 13$. Triangle DEF is what's left of ABC after cutting off the three small corner triangles BDF, CDE, AEF at the vertices, and each corner triangle's area compares to $[ABC]$ through a shared angle: $\frac{[BDF]}{[ABC]} = \frac{BD \cdot BF}{BA \cdot BC} = \frac{x^2}{ac}$ (both triangles have the angle at B, so their areas are in the ratio of the products of the two sides enclosing it), and likewise for the other two corners.

Work it out. With $x = 4, y = 6, z = 9$ and $a = 10, b = 15, c = 13$: $\frac{[BDF]}{[ABC]} = \frac{x^2}{ac} = \frac{16}{130} = \frac{8}{65}$, $\frac{[CDE]}{[ABC]} = \frac{y^2}{ab} = \frac{36}{150} = \frac{6}{25}$, $\frac{[AEF]}{[ABC]} = \frac{z^2}{bc} = \frac{81}{195} = \frac{27}{65}$. Over a common denominator of 325: $\frac{8}{65} = \frac{40}{325}$, $\frac{6}{25} = \frac{78}{325}$, $\frac{27}{65} = \frac{135}{325}$, summing to $\frac{253}{325}$. So $\frac{[DEF]}{[ABC]} = 1 - \frac{253}{325} = \frac{72}{325}$.

Check. Place $B = (0,0)$, $C = (10,0)$, solve for A from $AB = 13, CA = 15$ to get $A = \left(\frac{26}{5}, \frac{4\sqrt{114}}{5}\right)$, then build D, E, F directly on the three sides from the given tangent lengths and compute $[DEF]$ and $[ABC]$ by the shoelace formula. This gives $[ABC] = 6\sqrt{114}$ and $[DEF] = \frac{432\sqrt{114}}{325}$, whose ratio is $\frac{72}{325}$ exactly – matching, with no use of the shared-angle corner-triangle argument at all (and confirming that the irrational area of ABC cancels cleanly out of the ratio). Forgetting to cut off just one of the three corners gives three nearby traps: forgetting AEF (the $\frac{z^2}{bc}$ term) gives $\frac{207}{325}$, forgetting BDF (the $\frac{x^2}{ac}$ term) gives $\frac{112}{325}$, and forgetting CDE (the $\frac{y^2}{ab}$ term) gives $\frac{6}{13}$; and reporting the total cut away, $\frac{253}{325}$, instead of what's left gives another trap.

24. Answer: (D) $7\sqrt{17}$. The idea. Rotating the plane 90° about A, in the direction that carries B out to where the square puts it, sends B exactly to E (since $AE = AB$ and $\angle BAE = 90°$ by construction of the square). So $\angle EAC = \angle EAB + \angle BAC = 90° + \angle BAC$. Applying the Law of Cosines to triangle EAC turns this awkward angle back into something about triangle ABC itself: since $\cos(90° + \theta) = -\sin\theta$, the term $-2 \cdot AE \cdot AC \cos(\angle EAC)$ becomes $+2 \cdot AB \cdot AC \sin(\angle BAC)$, which is exactly $4[ABC]$ (twice the usual area formula $\frac{1}{2} AB \cdot AC \sin(\angle BAC)$). So the rotation converts the whole problem into $EC^2 = AB^2 + AC^2 + 4[ABC]$.

Work it out. First find $[ABC]$ by Heron's formula: with $AB = 13, AC = 20, BC = 11$, the semiperimeter is $s = \frac{13+20+11}{2} = 22$, so $[ABC] = \sqrt{22 \cdot 9 \cdot 2 \cdot 11} = \sqrt{4356} = 66$. Then $EC^2 = 13^2 + 20^2 + 4(66) = 169 + 400 + 264 = 833 = 49 \cdot 17$, so $EC = \sqrt{833} = 7\sqrt{17}$.

Check. Place $A = (0,0)$ and $B = (13,0)$; solving $AC = 20$, $BC = 11$ for the vertex on the side where C belongs gives $C = \left(\frac{224}{13}, \frac{132}{13}\right)$ (and indeed $\frac{1}{2} \cdot 13 \cdot \frac{132}{13} = 66$, matching the area above). Rotating B by 90° about A in the matching sense (away from C) gives $E = (0, -13)$, so square $ABDE$ sits below the x-axis. The distance formula then gives $EC = \sqrt{\left(\frac{224}{13}\right)^2 + \left(\frac{132}{13} + 13\right)^2} = \sqrt{833} = 7\sqrt{17}$ directly, confirming the answer with no use of the rotation-angle shortcut at all. Rotating B the wrong way (placing the square on the *same* side as C, overlapping the triangle)

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

gives the trap $\sqrt{305}$; forgetting the square entirely and just combining $AB^2 + AC^2$ gives $\sqrt{569}$; halving the area contribution (using $2[ABC]$ in place of $4[ABC]$, i.e. forgetting the extra factor of 2 that comes from doubling the sine) gives $EC^2 = 169 + 400 + 132 = 701$, i.e. $\sqrt{701}$; and reporting EC^2 itself instead of taking the square root gives 833.

25. Answer: (C) 13**. The idea.** Write $L = A_1 + A_2 + \cdots + A_7$, where A_i is 1 if position i is an ascent and 0 otherwise. By symmetry (swapping the values at positions $i, i+1$ is a bijection on permutations that flips A_i), $E[A_i] = \frac{1}{2}$ for each i. Expanding $L^2 = \sum_i A_i^2 + \sum_{i \neq j} A_i A_j$ and using linearity, $E[L^2]$ reduces to computing $E[A_i A_j]$ for pairs of positions – and this depends on whether the two adjacent-pairs $(i, i+1)$ and $(j, j+1)$ overlap in a shared position or not.

Work it out. Since $A_i^2 = A_i$, the diagonal terms contribute $\sum_{i=1}^{7} E[A_i] = 7 \cdot \frac{1}{2} = \frac{7}{2}$. For $|i - j| \geq 2$ (the two pairs of positions are entirely disjoint), the relative order on one pair is independent of the relative order on the other (for a uniformly random permutation, the relative ranks on any two disjoint sets of positions are independent), so $E[A_i A_j] = E[A_i]E[A_j] = \frac{1}{4}$. For $j = i + 1$ (the pairs share position $i + 1$), $A_i A_j = 1$ exactly when the three values at positions $i, i+1, i+2$ are in increasing order – one of their $3! = 6$ equally likely relative orders – so $E[A_i A_{i+1}] = \frac{1}{6}$. Among the $7 \times 6 = 42$ ordered pairs (i, j) with $i \neq j$, there are $2 \times 6 = 12$ adjacent ones (for $i = 1, \ldots, 6$, both $(i, i+1)$ and $(i+1, i)$), leaving $42 - 12 = 30$ non-adjacent ones. So $E[L^2] = \frac{7}{2} + 12 \cdot \frac{1}{6} + 30 \cdot \frac{1}{4} = \frac{7}{2} + 2 + \frac{15}{2} = 13$.

Check. A direct computer enumeration over all $8! = 40320$ permutations of $\{1, \ldots, 8\}$, counting ascents and averaging their squares, gives exactly 13 – matching, with no use of the indicator-covariance argument at all. Mistakenly treating every pair (i, j) as independent (even the adjacent ones, using $\frac{1}{4}$ throughout) gives $E[L^2] \approx \frac{7}{2} + 42 \cdot \frac{1}{4} = 14$; computing only $E[L]^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$ instead of $E[L^2]$ (ignoring the variance entirely) gives another trap.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.