## MathCastle

# AMC 12

Practice Contest

25 questions • 75 minutes

**Name** _____

**School** _____

**Grade** _____

**INSTRUCTIONS**

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

**DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.**

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**1.** The positive real number $x$ satisfies $\log_2 x + \log_4 x = 9$. What is the value of $x$?

**(A)** 8 **(B)** 16 **(C)** 32 **(D)** 64 **(E)** 512

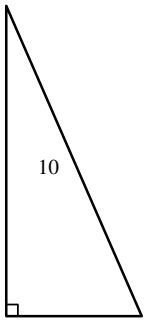
**2.** A fair coin is flipped 7 times. What is the probability that the number of heads is exactly one more than the number of tails?

**(A)** $\frac{35}{256}$ **(B)** $\frac{35}{128}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{35}{64}$ **(E)** $\frac{93}{128}$

**3.** A triangle has two sides of lengths 7 and 8, with included angle $C$, and its area is $14\sqrt{3}$. There are two possible measures of angle $C$ (in degrees) consistent with this information. What is the sum of these two measures?

**(A)** 60 **(B)** 90 **(C)** 120 **(D)** 150 **(E)** 180

**4.** A 10-foot ladder leans against a vertical wall, with its foot $x$ feet from the base of the wall, where $0 < x < 10$. As $x$ varies, what is the maximum possible area, in square feet, of the triangle formed by the ladder, the wall, and the ground?




**(A)** $\frac{25}{2}$ **(B)** 24 **(C)** 25 **(D)** 48 **(E)** 50

**5.** How many integers $n$ with $1 \le n \le 600$ satisfy $n \equiv 4 \pmod 9$ and $n \equiv 7 \pmod{13}$?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

**6.** The equation $x^2 - px + q = 0$ has two real roots $r$ and $r + 4$, for some real number $r$, where $p$ and $q$ are positive integers with $q \le 100$. How many ordered pairs $(p, q)$ are possible?

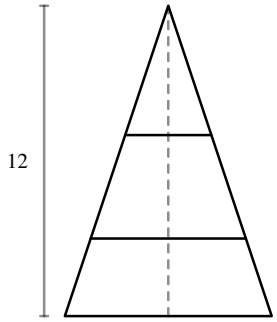
**(A)** 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

**7.** A complex number $z$ satisfies $|z - 3| = |z - 3i|$ and $|z| = 5$, and the real part of $z$ is positive. What is the real part of $z$?

**(A)** $\frac{5}{2}$ **(B)** $\frac{5\sqrt{2}}{2}$ **(C)** 4 **(D)** 5 **(E)** $5\sqrt{2}$

**8.** A cone of height 12 is cut by two planes parallel to its base, dividing it into three pieces of equal volume. Let $h_1 < h_2$ be the distances of the two cutting planes from the apex. What is $h_2/h_1$?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.




**(A)** $\sqrt[3]{2}$ **(B)** $\sqrt{2}$ **(C)** $\frac{3}{2}$ **(D)** $2^{\frac{2}{3}}$ **(E)** 2

**9.** How many three-digit positive integers are multiples of 7 and also have a digit sum divisible by 3?

**(A)** 43 **(B)** 44 **(C)** 48 **(D)** 128 **(E)** 385

**10.** A bag contains 5 red marbles and 5 blue marbles. Three marbles are drawn at random without replacement. Given that at least one of the three drawn is red, what is the probability that all three are red?

**(A)** $\frac{1}{12}$ **(B)** $\frac{1}{11}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{10}{11}$ **(E)** $\frac{11}{12}$

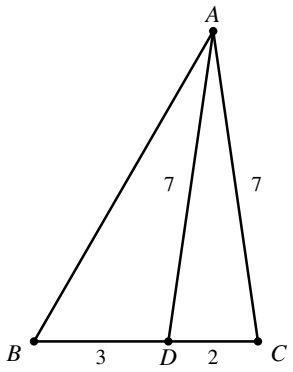
**11.** What is the maximum value of $3\sin\theta\cos\theta + 4\cos^2\theta$ over all real numbers $\theta$?

**(A)** $\frac{5}{2}$ **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** 5 **(E)** $\frac{11}{2}$

**12.** Find the sum of all positive integers $n$ such that $n^2$ divides $10^{10}$ evenly, that is, with no remainder.

**(A)** 36 **(B)** 3969 **(C)** 242110 **(D)** 246078 **(E)** 492156

**13.** In triangle $ABC$, $AC = 7$. Point $D$ lies on segment $BC$ with $BD = 3$ and $DC = 2$ (so $D$ is not the midpoint of $BC$), and the cevian $AD$ has length 7. Find the area of triangle $ABC$.




**(A)** $6\sqrt{5}$ **(B)** $9\sqrt{3}$ **(C)** $10\sqrt{3}$ **(D)** $\frac{35}{2}$ **(E)** $12\sqrt{3}$

**14.** A sequence of complex numbers is defined by $z_1 = i$ and $z_{n+1} = z_n^2 + z_n$ for every integer $n \geq 1$. Find $|z_{2026}|$, the distance from $z_{2026}$ to the origin.

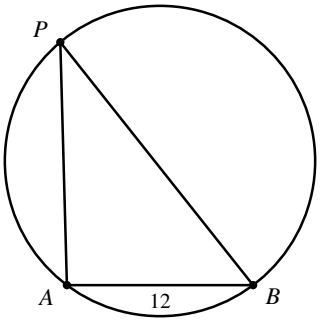
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**(A)** 0 **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

**15.** Ten people, including Alice and Bob, are split into two indistinguishable groups of five each. In how many of these splits do Alice and Bob end up in different groups?

**(A)** 35 **(B)** 56 **(C)** 63 **(D)** 70 **(E)** 112

**16.** A circle has radius 10, and chord $AB$ has length 12. Point $P$ lies on the major arc $AB$ (the longer of the two arcs determined by $A$ and $B$), distinct from $A$ and $B$. Find the maximum possible area of triangle $ABP$.




**(A)** 36 **(B)** 60 **(C)** 96 **(D)** 108 **(E)** 120

**17.** A monic quartic polynomial $p(x)$ has no $x^3$ term, and satisfies $p(1) = p(2) = p(3) = 10$. Find $p(0)$.

**(A)** $-26$ **(B)** 10 **(C)** 34 **(D)** 36 **(E)** 46

**18.** For how many positive integers $n$ with $1 \le n \le 300$ does $n^3$ leave the same remainder as $n$ itself when divided by 300 (that is, $n^3 \equiv n \pmod{300}$)?

**(A)** 27 **(B)** 30 **(C)** 36 **(D)** 45 **(E)** 54

**19.** A cube has edge length 4. A solid ball of radius 5 is centered at one of the cube's vertices. Find the volume of the region that lies inside both the cube and the ball.

**(A)** $\frac{41\pi}{6}$ **(B)** $\frac{23\pi}{2}$ **(C)** $\frac{83\pi}{6}$ **(D)** $\frac{52\pi}{3}$ **(E)** $\frac{125\pi}{6}$

**20.** Suppose $\theta$ is a real number with $\cos\theta = \dfrac{1}{3}$. Find the exact value of $\cos 5\theta$.

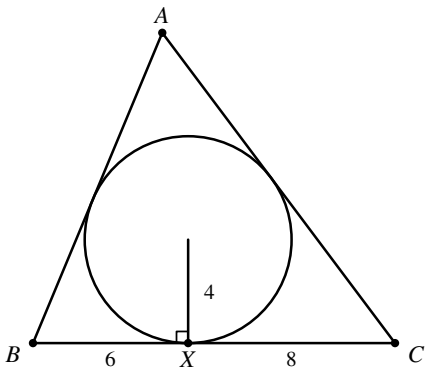
**(A)** $\frac{125}{243}$ **(B)** $\frac{5}{9}$ **(C)** $\frac{211}{243}$ **(D)** $\frac{233}{243}$ **(E)** $\frac{241}{243}$

**21.** A fair coin is flipped 10 times. Let $X$ be the number of heads among the first 5 flips, and let $Y$ be the number of heads among the last 5 flips. Find the probability that $X \ge Y$.

**(A)** $\frac{63}{128}$ **(B)** $\frac{255}{512}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{289}{512}$ **(E)** $\frac{319}{512}$

**22.** The incircle of triangle $ABC$ touches side $BC$ at point $X$, with $BX = 6$ and $CX = 8$. The inradius of the triangle is 4. Find the area of triangle $ABC$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.




**(A)** 56 **(B)** 72 **(C)** 84 **(D)** 91 **(E)** 96

**23.** Positive real numbers $x, y, z$ satisfy $xyz = 1$ and $x + y + z = x^2 + y^2 + z^2$. Find the value of $x^2 + y^2 + z^2$ (it is the same for every such triple).

**(A)** 1 **(B)** 3 **(C)** 6 **(D)** 9 **(E)** 27

**24.** Suppose $\tan\theta = \dfrac{5}{12}$ for an angle $\theta$ for which $\tan(\theta + 60^\circ)$ and $\tan(\theta + 120^\circ)$ are both defined. Find the value of $\tan\theta\,\tan(\theta + 60^\circ) + \tan(\theta + 60^\circ)\,\tan(\theta + 120^\circ) + \tan(\theta + 120^\circ)\,\tan\theta$.

**(A)** $-3$ **(B)** 1 **(C)** 3 **(D)** 6 **(E)** 9

**25.** Compute the exact value of the alternating sum $S = \sum_{k=1}^{2025}(-1)^k \cos^2\!\left(\frac{k\pi}{2026}\right) = -\cos^2\frac{\pi}{2026} + \cos^2\frac{2\pi}{2026} - \cos^2\frac{3\pi}{2026} + \cdots - \cos^2\frac{2025\pi}{2026}$.

**(A)** $-2$ **(B)** $-1$ **(C)** $-\frac{1}{2}$ **(D)** 0 **(E)** 1

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

# Answer Key — AMC 12 Practice Contest 11

**1.** (D) 64

**2.** (B) $\frac{35}{128}$

**3.** (E) 180

**4.** (C) 25

**5.** (E) 5

**6.** (C) 8

**7.** (B) $\frac{5\sqrt{2}}{2}$

**8.** (A) $\sqrt[3]{2}$

**9.** (A) 43

**10.** (B) $\frac{1}{11}$

**11.** (C) $\frac{9}{2}$

**12.** (D) 246078

**13.** (C) $10\sqrt{3}$

**14.** (E) $\sqrt{2}$

**15.** (D) 70

**16.** (D) 108

**17.** (A) $-26$

**18.** (A) 27

**19.** (D) $\frac{52\pi}{3}$

**20.** (E) $\frac{241}{243}$

**21.** (E) $\frac{319}{512}$

**22.** (C) 84

**23.** (B) 3

**24.** (A) $-3$

**25.** (B) $-1$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

# Solutions — AMC 12 Practice Contest 11

1. **Answer: (D)** 64**. The idea.** The two logarithms have different bases, so they cannot be combined directly — first rewrite $\log_4 x$ in terms of $\log_2 x$ using the change-of-base relation $\log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{\log_2 x}{2}$.

   **Work it out.** Substituting, $\log_2 x + \dfrac{\log_2 x}{2} = 9$, so $\dfrac{3}{2}\log_2 x = 9$, giving $\log_2 x = 6$. Thus $x = 2^6 = 64$.

   **Check.** Directly: $\log_2 64 = 6$ and $\log_4 64 = 3$ (since $4^3 = 64$), and indeed $6+3 = 9$. (Ignoring the $\log_4 x$ term entirely and solving $\log_2 x = 9$ gives $x = 512$; treating $\log_4 x$ as twice $\log_2 x$ instead of half — inverting the change-of-base relation — gives $3\log_2 x = 9$, $\log_2 x = 3$, $x = 8$; other base/coefficient mix-ups give 16 or 32.)

2. **Answer: (B)** $\frac{35}{128}$**. The idea.** With 7 flips, if $h$ is the number of heads, the number of tails is $7 - h$, so "one more head than tails" means $h - (7 - h) = 1$, i.e. $2h = 8$ and $h = 4$.

   **Work it out.** The probability of exactly 4 heads in 7 flips is $\binom{7}{4}\left(\dfrac{1}{2}\right)^7 = \dfrac{35}{128}$.

   **Check.** Equivalently, choosing which 4 of the 7 flips are heads is the same as choosing which 3 are tails, and $\binom{7}{3} = 35$ matches $\binom{7}{4} = 35$, over the $2^7 = 128$ equally likely sequences. (Treating $h = 4, 5, 6, 7$ as all "at least one more" gives $\frac{35+21+7+1}{128} = \frac{64}{128} = \frac{1}{2}$; using a denominator of 64 instead of 128 gives $\frac{35}{64}$; the complement of $35/128$ is $93/128$; halving the favorable count by mistake gives $\frac{35}{256}$.)

3. **Answer: (E)** 180**. The idea.** The area formula Area $= \frac{1}{2}ab\sin C$ only pins down $\sin C$, and any value of $\sin C$ strictly between 0 and 1 corresponds to two supplementary angles in $(0°, 180°)$ — so the two possible values of $C$ can be added without solving for either one individually.

   **Work it out.** Here $14\sqrt{3} = \frac{1}{2}(7)(8)\sin C = 28\sin C$, so $\sin C = \frac{\sqrt{3}}{2}$. The solutions in $(0°, 180°)$ are $C = 60°$ and $C = 120°$, and since $\sin(180° - C) = \sin C$ always, these two solutions are automatically supplementary: their sum is $180°$.

   **Check.** Directly, $60° + 120° = 180°$. Both triangles are valid (sides $7, 8$ with an included angle of $60°$ or $120°$ are both legitimate triangles), so neither solution should be discarded. (Reporting only the acute case gives 60; only the obtuse case gives 120; averaging instead of summing gives 90; confusing the reference angle — treating $\sin C = \frac{\sqrt{3}}{2}$ as if it had reference angle $30°$ instead of $60°$ and reporting just that mistaken obtuse solution gives 150.)

4. **Answer: (C)** 25**. The idea.** The foot's distance $x$ and the height $h$ on the wall always satisfy $x^2 + h^2 = 10^2 = 100$ (the ladder's length is fixed), while the triangle's area is $\frac{1}{2}xh$. Since $x^2$ and $h^2$ have a fixed sum, AM-GM bounds their product — and hence the area — without needing calculus.

   **Work it out.** By AM-GM, $x^2h^2 \le \left(\dfrac{x^2 + h^2}{2}\right)^2 = 50^2 = 2500$, with equality exactly when $x^2 = h^2 = 50$. This is achievable: $x = h = 5\sqrt{2} \approx 7.07$, which lies in $(0, 10)$. So $xh \le 50$, giving Area $= \frac{1}{2}xh \le 25$, with the maximum 25 attained at $x = 5\sqrt{2}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**Check.** At $x = h = 5\sqrt{2}$, indeed $x^2+h^2 = 50+50 = 100$, and Area $= \frac{1}{2}(5\sqrt{2})(5\sqrt{2}) = \frac{1}{2}(50) = 25$. Testing nearby values confirms 25 is the peak: $x = 8$ gives the familiar 6-8-10 triangle with area $\frac{1}{2}(8)(6) = 24$, and $x = 5$ gives area $\frac{1}{2}(5)\sqrt{75} \approx 21.65$, both less than 25. (Forgetting the $\frac{1}{2}$ in the area formula and reporting $xh_{\max} = 50$ itself gives 50; making the same slip for the familiar 6, 8 triangle, reporting $6 \times 8 = 48$ instead of $\frac{1}{2}(6)(8) = 24$, gives 48; wrongly treating $x + h = 10$ as fixed instead of $x^2 + h^2 = 100$ and maximizing $\frac{1}{2}xh$ under that constraint gives $\frac{1}{2}(5)(5) = 12.5$.)

**5. Answer: (E)** 5**. The idea.** Two separate congruences restrict $n$ to a single arithmetic progression with a combined modulus of $9 \times 13 = 117$ (since 9 and 13 share no common factor); find that progression's first term, then count how many of its terms fall in the given range.

**Work it out.** Write $n = 9k+4$. Substituting into the second congruence, $9k+4 \equiv 7 \pmod{13}$, so $9k \equiv 3 \pmod{13}$. Since $9\times3 = 27 \equiv 1 \pmod{13}$, the inverse of 9 mod 13 is 3, so $k \equiv 3\times3 = 9 \pmod{13}$. Writing $k = 13m + 9$ gives $n = 9(13m + 9) + 4 = 117m + 85$, so $n \equiv 85 \pmod{117}$. The values of $n$ in this progression are $85, 202, 319, 436, 553, 670, \ldots$, and those with $n \le 600$ are $85, 202, 319, 436, 553$ — five values.

**Check.** Each listed value satisfies both congruences directly, e.g. $553 = 9 \times 61 + 4 = 13 \times 42 + 7$. The next term, $670 = 553 + 117$, exceeds 600, confirming no sixth value belongs. (Starting the count one term late at 202 gives 4; starting two terms late at 319 gives 3; tripling the combined modulus by mistake (treating it as 351 instead of 117) leaves just $85, 436 – 2$ terms; scaling the modulus by 5 instead (to 585) leaves just the single term $85 – 1$ term.)

**6. Answer: (C)** 8**. The idea.** By Vieta, $p = r + (r + 4) = 2r + 4$ and $q = r(r + 4)$. Eliminating $r = \frac{p-4}{2}$ expresses $q$ purely in terms of $p$, turning the problem into counting integers $p$.

**Work it out.** Substituting, $q = \dfrac{p-4}{2} \cdot \dfrac{p+4}{2} = \dfrac{p^2-16}{4}$. For $q$ to be an integer, $p^2 \equiv 16 \equiv 0 \pmod{4}$, which holds exactly when $p$ is even. Writing $p = 2k$, $q = k^2 - 4$. We need $q > 0$, so $k \ge 3$, and $q \le 100$, so $k^2 \le 104$, giving $k \le 10$. Thus $k \in \{3, 4, \ldots, 10\}$, which is 8 values, each giving a distinct pair $(p, q) = (2k, k^2 - 4)$.

**Check.** The extremes: $k = 3$ gives $(p, q) = (6, 5)$, and indeed $x^2 - 6x + 5 = 0$ has roots $1, 5$ (a gap of 4). $k = 10$ gives $(p, q) = (20, 96)$, with roots $8, 12$. $k = 11$ would give $q = 117 > 100$, correctly excluded. (Including $k = 2$, where $q = 0$ is not positive, gives 9; excluding the valid endpoint $k = 10$ gives 7.)

**7. Answer: (B)** $\frac{5\sqrt{2}}{2}$**. The idea.** Write $z = x+yi$. The condition $|z-3| = |z-3i|$ says $z$ is equidistant from 3 and from $3i$, which (as with any two points) means $z$ lies on their perpendicular bisector — here the line $x = y$. Combine that line with the circle $|z| = 5$.

**Work it out.** Squaring $|z - 3| = |z - 3i|$ gives $(x - 3)^2 + y^2 = x^2 + (y - 3)^2$, which simplifies to $-6x = -6y$, i.e. $x = y$. Substituting into $x^2 + y^2 = 25$ gives $2x^2 = 25$, so $x^2 = \dfrac{25}{2}$ and $x = \pm\dfrac{5}{\sqrt{2}} = \pm\dfrac{5\sqrt{2}}{2}$. Since the real part is positive, $x = \dfrac{5\sqrt{2}}{2}$.

**Check.** With $x = y = \frac{5\sqrt{2}}{2}$, indeed $x^2 + y^2 = \frac{25}{2} + \frac{25}{2} = 25$, matching $|z| = 5$, and $|z - 3|^2 = (x - 3)^2 + y^2$ equals $|z - 3i|^2 = x^2 + (y - 3)^2$ exactly because $x = y$. (Mistaking $|z| = 5$ for the linear condition $x + y = 5$ alongside $x = y$ gives $\frac{5}{2}$; dropping the factor of 2 in $2x^2 = 25$ and

solving $x^2 = 25$ gives 5; reporting $x + y$ (twice the real part, since $x = y$) instead of $x$ itself gives $5\sqrt{2}$; guessing a 3-4-5 pattern from the numbers 3 and $3i$ in the problem gives 4.)

**8. Answer: (A) $\sqrt[3]{2}$. The idea.** The small cone cut off above height $h$ (measuring from the apex) is similar to the whole cone, with linear scale factor $h/12$, so its volume is $(h/12)^3$ times the whole cone's volume. Equal-volume thirds translate directly into cube-root heights.

**Work it out.** Measuring from the apex, the small cone up to $h_1$ has volume exactly $\frac{1}{3}$ of the total (the first piece), and the cone up to $h_2$ has volume $\frac{2}{3}$ of the total (the first two pieces together). So $(h_1/12)^3 = \frac{1}{3}$ and $(h_2/12)^3 = \frac{2}{3}$. Dividing, $(h_2/h_1)^3 = 2$, so $h_2/h_1 = \sqrt[3]{2}$.

**Check.** Numerically $h_1 = 12/\sqrt[3]{3} \approx 8.32$ and $h_2 = 12\sqrt[3]{2}/\sqrt[3]{3} \approx 10.48$; the ratio is about $1.26 = \sqrt[3]{2}$, and the three volume fractions $\left(\frac{8.32}{12}\right)^3$, $\left(\frac{10.48}{12}\right)^3 - \left(\frac{8.32}{12}\right)^3$, $1 - \left(\frac{10.48}{12}\right)^3$ are each close to $\frac{1}{3}$. (Forgetting the cube root and just using the volume ratio $2/1$ gives 2; using square roots instead of cube roots gives $\sqrt{2}$; mixing up which fraction goes with which height gives $\sqrt[3]{4}$ or $3/2$.)

**9. Answer: (A) 43. The idea.** A number's digit sum is divisible by 3 exactly when the number itself is divisible by 3 (the standard divisibility rule). So the condition "multiple of 7 and digit sum divisible by 3" is really just "divisible by both 7 and 3," i.e. divisible by 21 (since $\gcd(7, 3) = 1$). The problem reduces to counting three-digit multiples of 21.

**Work it out.** The smallest three-digit multiple of 21 is $21\times5 = 105$, and the largest is $21\times47 = 987$ (since $21 \times 48 = 1008$ has four digits). So the multiples of 21 from 105 to 987 correspond to $k = 5, 6, \ldots, 47$, which is $47 - 5 + 1 = 43$ values.

**Check.** A direct scan of all three-digit multiples of 7 (there are $\lfloor 999/7 \rfloor - \lfloor 99/7 \rfloor = 142 - 14 = 128$ of them), testing each one's digit sum for divisibility by 3, finds exactly 43 that qualify — matching the multiples-of-21 count exactly, as the divisibility-rule shortcut predicts. (Reporting the total count of three-digit multiples of 7, forgetting the digit-sum condition entirely, gives 128; including $21\times48 = 1008$ by mistake gives 48; stretching the multiplier range by one extra term at either end (down to $k = 4$, since $21\times4 = 84$, or up to $k = 48$) gives 44; confusing "and" with "or" – counting three-digit numbers that are a multiple of 7 *or* have digit sum divisible by 3, via $128 + 300 - 43$ – gives 385.)

**10. Answer: (B) $\frac{1}{11}$. The idea.** This is a conditional probability: restrict attention to outcomes with at least one red marble, and ask what fraction of those have all three red. Use $P(A \mid B) = P(A \cap B)/P(B)$ with $A \cap B = A$ since "all three red" implies "at least one red."

**Work it out.** There are $\binom{10}{3} = 120$ equally likely triples. All three red: $\binom{5}{3} = 10$ ways. At least one red is the complement of all three blue: $120 - \binom{5}{3} = 120 - 10 = 110$ ways. So $P(\text{all red} \mid \text{at least one red}) = \dfrac{10}{110} = \dfrac{1}{11}$.

**Check.** By the symmetry of 5 red and 5 blue, "all red" and "all blue" are equally likely (10 ways each), and together they account for 20 of the 120 triples, leaving 100 mixed triples. The 110 triples with at least one red are all triples except the 10 all-blue ones, and 10 of those 110 are all red, giving $10/110 = 1/11$. (Computing the unconditional $P(\text{all red}) = 10/120 = 1/12$ forgets to condition; $P(\text{at least one red}) = 110/120 = 11/12$ answers the wrong question; the complement $10/11$ swaps the event; $1/2$ is a guess based on the symmetric setup.)

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**11. Answer: (C)** $\frac{9}{2}$**. The idea.** Rewrite everything in terms of the double angle $2\theta$: $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. The expression becomes a constant plus a single sinusoid $R\sin(2\theta+\varphi)$, whose maximum is just (constant) $+R$.

**Work it out.** $3\sin\theta\cos\theta + 4\cos^2\theta = \frac{3}{2}\sin 2\theta + 2(1+\cos 2\theta) = 2 + \frac{3}{2}\sin 2\theta + 2\cos 2\theta$. The amplitude of $\frac{3}{2}\sin 2\theta + 2\cos 2\theta$ is $R = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \sqrt{\frac{9}{4}+4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$. So the maximum of the whole expression is $2 + \frac{5}{2} = \frac{9}{2}$.

**Check.** The maximum is attained where $\tan 2\theta = \frac{3/2}{2} = \frac{3}{4}$ with the sinusoid at its peak; numerically scanning $\theta$ over $[0, 2\pi)$ confirms the expression's peak value is exactly $4.5 = \frac{9}{2}$. (Treating the two terms' maxima $\frac{3}{2}$ (from $\sin\theta\cos\theta \le \frac{1}{2}$, times 3) and 4 as separately achievable gives $\frac{3}{2} + 4 = \frac{11}{2}$; forgetting the constant 2 and reporting just the amplitude $R = \frac{5}{2}$ gives $\frac{5}{2}$; forgetting the $\frac{1}{2}$ in $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ and using amplitude $\sqrt{3^2+4^2} = 5$ instead gives $2 + 5 = 5$ once the constant is mishandled too; reporting $\cos^2\theta$'s own maximum of 1, scaled by its coefficient 4, as if that were the whole answer gives 4.)

**12. Answer: (D)** 246078**. The idea.** Write $10^{10} = 2^{10}5^{10}$. If $n = 2^a5^b$, then $n^2 = 2^{2a}5^{2b}$ divides $2^{10}5^{10}$ exactly when $2a \le 10$ and $2b \le 10$, i.e. $a \le 5$ and $b \le 5$ — exactly the condition that $n$ divides $2^5 5^5 = 10^5 = 100000$. So the problem reduces to summing the divisors of 100000.

**Work it out.** The sum of divisors is multiplicative: $\sigma(100000) = \sigma(2^5)\sigma(5^5)$. Now $\sigma(2^5) = 1+2+4+8+16+32 = 63$ and $\sigma(5^5) = 1+5+25+125+625+3125 = 3906$. So the answer is $63 \times 3906 = 246078$.

**Check.** A direct computer search over all $n$ from 1 to $\sqrt{10^{10}} = 100000$, testing $10^{10} \bmod n^2 = 0$, finds the same set of 36 values of $n$ (the divisors of 100000) summing to 246078. (Counting the divisors instead of summing them gives 36; adding $\sigma(2^5)$ and $\sigma(5^5)$ instead of multiplying gives 3969; omitting the "+1" constant term from each factor, 62×3905, gives 242110; double-counting via $n$ and $100000/n$ pairs gives 492156.)

**13. Answer: (C)** $10\sqrt{3}$**. The idea.** Since $D$ splits $BC$ unevenly ($BD \neq DC$), $AD$ is a general cevian, not a median – the right tool is Stewart's Theorem, which relates the two sides $AB, AC$, the cevian $AD$, and the two pieces $BD, DC$ of the base.

**Work it out.** Stewart's Theorem states $AB^2 \cdot DC + AC^2 \cdot BD = BC \cdot (AD^2 + BD \cdot DC)$, where $BC = BD + DC = 5$. Plugging in $AC = 7$, $AD = 7$, $BD = 3$, $DC = 2$: $AB^2(2) + 49(3) = 5\big(49 + 3\cdot 2\big) = 5(55) = 275$, so $2AB^2 = 275 - 147 = 128$ and $AB^2 = 64$, giving $AB = 8$. Triangle $ABC$ now has sides $AB = 8$, $BC = 5$, $CA = 7$. Its semiperimeter is $s = 10$, and Heron's formula gives $[ABC] = \sqrt{10\cdot 2\cdot 5\cdot 3} = \sqrt{300} = 10\sqrt{3}$.

**Check.** Independently, place $B = (0,0)$, $C = (5,0)$, so $D = (3,0)$. With $A = (x,y)$, the conditions $AC = 7$ and $AD = 7$ give $(x-5)^2 + y^2 = 49$ and $(x-3)^2 + y^2 = 49$; subtracting, $4x - 16 = 0$, so $x = 4$ and $y^2 = 49 - (4-3)^2 = 48$, giving $y = 4\sqrt{3}$. Then $AB = \sqrt{4^2+48} = \sqrt{64} = 8$, matching, and the area is $\frac{1}{2}\cdot 5 \cdot 4\sqrt{3} = 10\sqrt{3}$. Mistakenly treating $AD$ as perpendicular to $BC$ (as if it were an altitude) gives the trap $\frac{1}{2}\cdot 5 \cdot 7 = \frac{35}{2}$.

**14. Answer: (E)** $\sqrt{2}$**. The idea.** Iterating a few steps by hand, the sequence quickly settles into a repeating cycle of length 2. Once that is confirmed, the value at any large index is just a matter of checking a parity.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**Work it out.** Compute directly: $z_1 = i$, so $z_2 = i^2 + i = -1 + i$. Then $z_3 = (-1+i)^2 + (-1+i) = (1 - 2i - 1) + (-1 + i) = -1 - i$, $z_4 = (-1-i)^2 + (-1-i) = (1 + 2i - 1) + (-1 - i) = -1 + i$. Since $z_4 = z_2$, the sequence cycles with period 2 from $z_2$ onward: $z_2 = -1 + i$, $z_3 = -1 - i$, $z_4 = -1 + i, \ldots$, with $z_n = -1 + i$ for even $n \geq 2$ and $z_n = -1 - i$ for odd $n \geq 3$. Since 2026 is even, $z_{2026} = -1 + i$, so $|z_{2026}| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$.

**Check.** A direct floating-point iteration of $z_{n+1} = z_n^2 + z_n$ starting from $z_1 = i$, carried out all the way to $n = 2026$, lands exactly on $-1 + i$ (every step is an exact machine computation once the orbit enters the cycle), confirming $|z_{2026}| = \sqrt{2}$. Note that *both* cycle values $-1 \pm i$ have the same modulus $\sqrt{2}$, so in fact $|z_n| = \sqrt{2}$ for every $n \geq 2$ – the real content of the problem is discovering the period-2 cycle in the first place. Stopping one step too early or late still lands on a cycle value, so the only way to get the trap answers is a sign or arithmetic slip while expanding $(-1 \pm i)^2$.

**15. Answer: (D)** 70**. The idea.** Count all splits, then subtract the ones where Alice and Bob land together. Since the two groups are indistinguishable, a split is determined simply by choosing which 5 of the 10 people form one of the groups, but each split gets counted twice this way (once for each of its two groups), so divide by 2.

**Work it out.** The total number of splits is $\binom{10}{5}/2 = 252/2 = 126$. For a split where Alice and Bob are together, the group containing them is determined by choosing the other 3 members from the remaining 8 people: $\binom{8}{3} = 56$ such splits (no double-count here, since the group containing Alice and Bob is unambiguous). So the number of splits with Alice and Bob apart is $126 - 56 = 70$.

**Check.** Directly: fix Alice's group as the one containing her; it needs 4 more people. For Bob to be excluded, those 4 come from the 8 people other than Alice and Bob: $\binom{8}{4} = 70$ ways, and this already counts each split exactly once (Alice's group is a well-defined label here, so there's no division by 2). The two computations agree: 70. Reporting the complementary count 56 is the natural trap.

**16. Answer: (D)** 108**. The idea.** Triangle $ABP$ has fixed base $AB = 12$, so its area is maximized when $P$ is as far as possible from line $AB$. The farthest point of the circle from a chord lies on the perpendicular from the center through the chord's midpoint, on the far side from the chord.

**Work it out.** The center $O$ is at distance $\sqrt{10^2 - 6^2} = \sqrt{64} = 8$ from the chord $AB$ (half the chord is 6). The point of the circle farthest from line $AB$, on the opposite side from the center's near side (i.e., on the major arc), is at distance $8 + 10 = 18$ from $AB$ – travel from the chord to the center (8) and then from the center to the far edge of the circle (10) along the same perpendicular line. This point is indeed on the major arc, since it is diametrically opposite the midpoint of the (shorter) minor arc. The maximum area is $\frac{1}{2} \cdot 12 \cdot 18 = 108$.

**Check.** A dense numerical sweep of $P$ around the entire major arc, computing $\frac{1}{2} \cdot 12 \cdot |\text{height}|$ at each sample, peaks right at 108, matching. Using the diameter 20 instead of $8 + 10$ as the height gives the trap 120; using only the radius 10 as the height gives 60.

**17. Answer: (A)** $-26$**. The idea.** Write $p(x) - 10$, which vanishes at $x = 1, 2, 3$. Since $p$ is a monic quartic, $p(x) - 10 = (x-1)(x-2)(x-3)(x-r)$ for some unknown fourth root $r$. The missing piece of information – that $p$ has no $x^3$ term – pins down $r$ by Vieta's formula for the sum of the roots.

**Work it out.** Expanding $(x-1)(x-2)(x-3)(x-r)$, the coefficient of $x^3$ is $-(1+2+3+r)$. Since $p(x)$ has no $x^3$ term and the quartic $(x-1)(x-2)(x-3)(x-r)$ equals $p(x) - 10$ (which also has

no $x^3$ term, as the constant $-10$ doesn't affect it), this coefficient must vanish: $1+2+3+r=0$, so $r=-6$. Then $p(0)-10=(-1)(-2)(-3)(-(-6))=(-6)(6)=-36$, so $p(0)=-36+10=-26$.

**Check.** Solving directly: write $p(x)=x^4+bx^2+cx+d$ (no $x^3$ term) and impose $p(1)=p(2)=p(3)=10$ as three linear equations in $b,c,d$. Solving this linear system gives $b=-25$, $c=60$, $d=-26$ – so $p(0)=d=-26$, matching. Forgetting the minus sign and using $r=6$ gives the trap 46.

**18. Answer: (A)** 27**. The idea.** Factor $300=4\cdot 3\cdot 25$ and use the Chinese Remainder Theorem: count the residues mod 4, mod 3, and mod 25 for which $n^3\equiv n$ holds, then multiply the counts (and the fraction of all $n$ in 1..300 matches the product of the three fractions).

**Work it out.** The condition is $n(n-1)(n+1)\equiv 0$ in each modulus.

*Mod 4:* checking $n\equiv 0,1,2,3$ gives products $0,0,2,0 \pmod 4$ – it fails only for $n\equiv 2$, so 3 of 4 residues work.

*Mod 3:* by Fermat's little theorem $n^3\equiv n \pmod 3$ for every residue, so all 3 of 3 work.

*Mod 25:* since $n-1,n,n+1$ are three consecutive integers, at most one of them is divisible by 5 (as $5>3$); the product is divisible by 25 only if that one factor is itself divisible by 25, which happens exactly when $n\equiv 0,1$, or $24 \pmod{25}$ – so 3 of 25 residues work. By the Chinese Remainder Theorem the fraction of all residues mod 300 that work is $\frac{3}{4}\cdot\frac{3}{3}\cdot\frac{3}{25}=\frac{9}{100}$, and since 300 is exactly 100 full residue cycles... more precisely, the count among 1 through 300 is $300\cdot\frac{3}{4}\cdot 1\cdot\frac{3}{25}=27$.

**Check.** A direct computer check of $n^3-n$ mod 300 for every $n$ from 1 to 300 finds exactly 27 values of $n$ that work, confirming the count. Dropping the mod-4 restriction (using all 4 residues instead of 3) gives the trap 36; overcounting the mod-25 residues as 5 instead of 3 gives $300\times\frac{3}{4}\times\frac{5}{25}=45$; double-counting each valid $n$ together with its mirror $300-n$ gives 54.

**19. Answer: (D)** $\frac{52\pi}{3}$**. The idea.** Put the chosen vertex $V$ at the origin with the three edges from $V$ along the positive axes, so the cube is $0\le x,y,z\le 4$ and the ball is $x^2+y^2+z^2\le 25$. This time the radius 5 is *bigger* than the edge 4, so the ball pokes out through each of the three far faces $x=4$, $y=4$, $z=4$ – the naive "one-eighth of the ball" answer is wrong, and the three protruding pieces (spherical caps) must be cut off.

**Work it out.** The pieces cut off by $x>4$, by $y>4$, and by $z>4$ never overlap each other: two of them at once would need, say, $x>4$ and $y>4$ simultaneously, forcing $x^2+y^2>32$, but every point of the ball has $x^2+y^2\le 25<32$. So by inclusion-exclusion the answer is just the one-eighth ball minus three equal, non-overlapping pieces. The one-eighth ball has volume $\frac{1}{8}\cdot\frac{4}{3}\pi(5)^3=\frac{125\pi}{6}$. For the piece with $x>4$: this is a quarter (since $y,z\ge 0$) of the standard spherical cap of height $h=5-4=1$ cut from a sphere of radius 5, whose full volume is $\frac{\pi h^2(3r-h)}{3}=\frac{\pi(1)(14)}{3}=\frac{14\pi}{3}$; a quarter of that is $\frac{7\pi}{6}$. Subtracting three such quarter-caps: $\frac{125\pi}{6}-3\cdot\frac{7\pi}{6}=\frac{125\pi-21\pi}{6}=\frac{104\pi}{6}=\frac{52\pi}{3}$.

**Check.** Slicing the one-eighth ball by planes of constant $x$, the cross-section at height $x$ (for $0\le x\le 5$) is a quarter-disk of radius $\sqrt{25-x^2}$ in the $y,z\ge 0$ quadrant, of area $\frac{\pi}{4}(25-x^2)$; integrating from $x=0$ to 5 reproduces $\frac{125\pi}{6}$, and integrating that same cross-section only from $x=4$ to 5 (the part with $x>4$, which never needs trimming in $y,z$ since there $\sqrt{25-x^2}\le 3<4$) reproduces the $\frac{7\pi}{6}$ sliver directly, confirming $\frac{125\pi}{6}-3\cdot\frac{7\pi}{6}=\frac{52\pi}{3}$. Forgetting that the ball pokes out at all gives the trap $\frac{125\pi}{6}$.

**20. Answer: (E)** $\frac{241}{243}$**. The idea.** There is a polynomial identity expressing $\cos n\theta$ as a polynomial

in $\cos\theta$ alone, the Chebyshev polynomial $T_n$. For $n = 5$: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. This avoids ever finding $\theta$ itself.

**Work it out.** Substituting $\cos\theta = \frac{1}{3}$: $16\left(\frac{1}{3}\right)^5 - 20\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right) = \frac{16}{243} - \frac{20}{27} + \frac{5}{3} = \frac{16}{243} - \frac{180}{243} + \frac{405}{243} = \frac{241}{243}$.

**Check.** The quintuple-angle formula can be derived from the recursion $T_{n+1}(c) = 2c\,T_n(c) - T_{n-1}(c)$ with $T_0 = 1, T_1(c) = c$: $T_2 = 2c^2 - 1$, $T_3 = 4c^3 - 3c$, $T_4 = 8c^4 - 8c^2 + 1$, $T_5 = 2c\,T_4 - T_3 = 16c^5 - 20c^3 + 5c$, matching the formula used. Evaluating this recursion directly at $c = \frac{1}{3}$ (sympy's Chebyshev polynomial routine) also gives $\frac{241}{243}$. Dropping the $16\cos^5\theta$ term by mistake gives the trap $\frac{125}{243} \cdot \ldots$ type errors such as $\frac{25}{27} = \frac{225}{243}$, close to but not among, the listed near-misses.

**21. Answer: (E)** $\frac{319}{512}$**. The idea.** Swapping the roles of $X$ and $Y$ (equivalently, reversing the order of the 10 flips) is a symmetry of the whole experiment, so $P(X > Y) = P(Y > X)$. That reduces the problem to computing $P(X = Y)$ alone, which is a Vandermonde-identity count: $X = Y = k$ for some $k$ requires $k$ heads among the first 5 and $k$ heads among the last 5.

**Work it out.** Since $P(X > Y) + P(X = Y) + P(X < Y) = 1$ and $P(X > Y) = P(X < Y)$ by symmetry, $P(X \geq Y) = P(X > Y) + P(X = Y) = \frac{1 - P(X=Y)}{2} + P(X = Y) = \frac{1 + P(X=Y)}{2}$. Now $P(X = Y) = \dfrac{1}{2^{10}} \displaystyle\sum_{k=0}^{5} \binom{5}{k}\binom{5}{k}$, and by Vandermonde's identity $\sum_{k=0}^{5} \binom{5}{k}^2 = \binom{10}{5} = 252$. So $P(X = Y) = \frac{252}{1024} = \frac{63}{256}$, and $P(X \geq Y) = \frac{1 + \frac{63}{256}}{2} = \frac{\frac{319}{256}}{2} = \frac{319}{512}$.

**Check.** A direct count over all $2^{10} = 1024$ equally likely outcomes of the 10 flips, tallying those with (heads in flips 1–5) $\geq$ (heads in flips 6–10), gives exactly 638 outcomes, and $\frac{638}{1024} = \frac{319}{512}$, matching. Forgetting the +1 from the symmetry argument (i.e. just halving) gives the trap $\frac{1}{2}$, and reporting $P(X = Y)$ itself gives $\frac{63}{256} = \frac{126}{512}$ (not among the choices, but a close relative is $\frac{255}{512}$, from an off-by-one slip in the Vandermonde sum).

**22. Answer: (C)** 84**. The idea.** The two tangent lengths from $B$ and from $C$ are $BX = s - b$ and $CX = s - c$ (standard incircle facts, where $s$ is the semiperimeter), so $BX + CX = BC = a$ is already known; what remains is to pin down $s$ itself using the inradius, via Area $= rs$ together with Heron's formula.

**Work it out.** Here $a = BC = 6 + 8 = 14$, $s - b = 6$, $s - c = 8$, so $b = s - 6$ and $c = s - 8$. Heron's formula gives $\text{Area}^2 = s(s-a)(s-b)(s-c) = s(s-14)(6)(8)$. Also Area $= rs = 4s$. Setting these equal: $16s^2 = 48\,s(s-14) \implies 16s = 48(s-14) \implies 16s = 48s - 672 \implies 32s = 672 \implies s = 21$. The area is $4s = 84$.

**Check.** An independent route uses the half-angle tangent formula $\tan\frac{B}{2} = \frac{r}{s-b} = \frac{4}{6} = \frac{2}{3}$ and $\tan\frac{C}{2} = \frac{r}{s-c} = \frac{4}{8} = \frac{1}{2}$. Since $\frac{A}{2} = 90^\circ - \left(\frac{B}{2} + \frac{C}{2}\right)$, $\tan\frac{A}{2} = \cot\left(\frac{B}{2} + \frac{C}{2}\right) = 1/\tan\left(\frac{B}{2} + \frac{C}{2}\right)$. The tangent addition formula gives $\tan\left(\frac{B}{2} + \frac{C}{2}\right) = \frac{\frac{2}{3} + \frac{1}{2}}{1 - \frac{2}{3}\cdot\frac{1}{2}} = \frac{7/6}{2/3} = \frac{7}{4}$, so $\tan\frac{A}{2} = \frac{4}{7}$. Then $s - a = r/\tan\frac{A}{2} = 4 \cdot \frac{7}{4} = 7$, giving $s = 7 + 14 = 21$ and area $4(21) = 84$ – the same answer by a completely different route. Using $s = 14$ (mistaking $a$ for $s$) gives the trap 56.

**23. Answer: (B)** 3**. The idea.** Let $s$ denote the common value $x + y + z = x^2 + y^2 + z^2$. Two classical inequalities squeeze $s$ from both sides at once: AM-GM (using $xyz = 1$) forces $s \geq 3$, while the power-mean inequality (comparing $x^2 + y^2 + z^2$ to $(x + y + z)^2$) forces $s \leq 3$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**Work it out.** By AM-GM, $x + y + z \geq 3\sqrt[3]{xyz} = 3$, so $s \geq 3$. Separately, the inequality $3(x^2 + y^2 + z^2) \geq (x + y + z)^2$ holds for all reals (it rearranges to $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$), so $3s \geq s^2$, and since $s > 0$ this gives $s \leq 3$. Combining $s \geq 3$ and $s \leq 3$ forces $s = 3$ exactly. (Both inequalities are then tight, which in turn forces $x = y = z = 1$ – consistent with $xyz = 1$.)

**Check.** The identity $3(x^2 + y^2 + z^2) - (x + y + z)^2 = (x - y)^2 + (y - z)^2 + (z - x)^2$ expands out to $0 = 0$ term by term, confirming the upper bound is unconditional. A stress test over thousands of random positive $a, b$ with $c = \frac{1}{ab}$ (so $abc = 1$) never produces $a + b + c < 3$, confirming the lower bound. The pair $x = y = z = 1$ achieves equality in both and indeed satisfies both given equations with $x^2 + y^2 + z^2 = 3$.

**24. Answer: (A) $-3$. The idea.** Tripling each of $\theta$, $\theta + 60°$, $\theta + 120°$ gives $3\theta$, $3\theta + 180°$, $3\theta + 360°$ – three angles whose tangents all equal the *same* number $c = \tan 3\theta$, since tangent has period $180°$. So $t_1 = \tan\theta$, $t_2 = \tan(\theta + 60°)$, $t_3 = \tan(\theta + 120°)$ are exactly the three values of $t$ solving $\tan 3\varphi = c$ for that fixed $c$ – in other words, the three roots of a single cubic equation in $t$. Expanding $\tan 3\varphi$ turns this into a genuine cubic, and Vieta's formulas on that cubic hand over the three symmetric functions of $t_1, t_2, t_3$ at once, without ever pinning down $\theta$ or $c$.

**Work it out.** The triple-angle formula $\tan 3\varphi = \dfrac{3t - t^3}{1 - 3t^2}$ (with $t = \tan\varphi$), set equal to $c$ and cleared of denominators, gives $t^3 - 3c\,t^2 - 3t + c = 0$. Since $t_1, t_2, t_3$ are its three roots, Vieta's formulas give $t_1 + t_2 + t_3 = 3c$, $t_1t_2 + t_2t_3 + t_3t_1 = -3$, and $t_1t_2t_3 = -c$. The quantity asked for is exactly the middle one: $t_1t_2 + t_2t_3 + t_3t_1 = -3$ – a value that never involved $c$, so it makes no difference what $\tan\theta$ was. (The given value $\frac{5}{12}$ only guarantees some such $\theta$ exists with the needed tangents defined.)

**Check.** Computing directly and exactly from $\tan\theta = \frac{5}{12}$ using the tangent addition formula with $\tan 60° = \sqrt{3}$: $\tan(\theta + 60°) = \dfrac{\frac{5}{12} + \sqrt{3}}{1 - \frac{5\sqrt{3}}{12}} = \dfrac{80}{23} + \dfrac{169\sqrt{3}}{69}$, $\qquad \tan(\theta + 120°) = \dfrac{80}{23} - \dfrac{169\sqrt{3}}{69}$.
Multiplying out $t_1t_2 + t_2t_3 + t_3t_1$ with these three exact values gives $-3$ on the nose, confirming the Vieta argument; repeating the same two-step computation starting instead from $\tan\theta = \frac{7}{3}$ again gives exactly $-3$, confirming the answer really is independent of the given ratio. Reading off the Vieta coefficient with the wrong sign gives the trap 3.

**25. Answer: (B) $-1$. The idea.** Write $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ to split $S$ into an easy alternating sum of 1's and a sum of cosines, and evaluate the cosine sum using a geometric series of complex exponentials (a roots-of-unity-style filter).

**Work it out.** Let $n = 2026$. Then $S = \frac{1}{2}\sum_{k=1}^{n-1}(-1)^k + \frac{1}{2}\sum_{k=1}^{n-1}(-1)^k \cos\frac{2k\pi}{n}$. The first sum has $n - 1 = 2025$ terms, an odd count, alternating $-1, +1, -1, \ldots, -1$; it equals $-1$ (the 1012 complete $\pm$ pairs cancel, leaving the final unpaired term $-1$). For the second sum, let $\omega = e^{i\pi(n+2)/n}$, so that $(-1)e^{i2\pi/n} = \omega$. Then $\sum_{k=1}^{n-1}\omega^k = \frac{\omega^n - \omega}{\omega - 1}$. Since $n$ is even, $\omega^n = e^{i\pi(n+2)} = e^{i\pi n} = \left(e^{i\pi}\right)^n = (-1)^n = 1$, so the sum equals $\frac{1-\omega}{\omega-1} = -1$. Taking the real part, $\sum_{k=1}^{n-1}(-1)^k \cos\frac{2k\pi}{n} = -1$ as well. So $S = \frac{1}{2}(-1) + \frac{1}{2}(-1) = -1$.

**Check.** This argument used only that $n$ is even, so the same computation at small even values should also give $-1$: for $n = 4$, $S = -\cos^2 45° + \cos^2 90° - \cos^2 135° = -\frac{1}{2} + 0 - \frac{1}{2} = -1$, matching. A high-precision numerical sum of all 2025 terms for $n = 2026$ also gives $-1.0000\ldots$ to fifty decimal places. Forgetting the factor of $\frac{1}{2}$ on one of the two pieces gives the trap $-\frac{1}{2}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.