MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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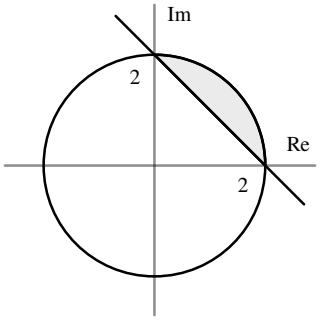
1. In an arithmetic sequence $a_1, a_2, a_3, \ldots$, the sum of the first ten terms, $a_1 + a_2 + \cdots + a_{10}$, is 100, and the sum of the next ten terms, $a_{11} + a_{12} + \cdots + a_{20}$, is 300. What is a_{30}?

(A) 39 **(B)** 55 **(C)** 57 **(D)** 58 **(E)** 59

2. A list of integers contains the number 13. Removing that 13 from the list raises the mean (average) of the list by 1. If instead the number 44 is added to the original list, the mean rises by 2. What is the sum of the numbers in the original list?

(A) 22 **(B)** 198 **(C)** 210 **(D)** 220 **(E)** 242

3. Let S be the set of complex numbers z in the complex plane with $|z| \le 2$ and $\mathrm{Re}(z) + \mathrm{Im}(z) \ge 2$. What is the area of S?



(A) $\pi - 2$ **(B)** $\pi - 1$ **(C)** $2\pi - 4$ **(D)** $2\pi - 2$ **(E)** $3\pi + 2$

4. A three-digit number (from 100 to 999) whose three digits are all different is chosen at random, each such number being equally likely. What is the probability that its digits are either strictly increasing from left to right (like 258) or strictly decreasing from left to right (like 830)?

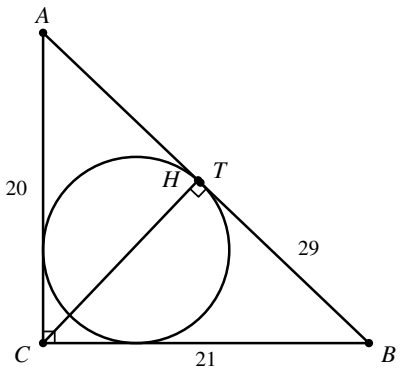
(A) $\frac{17}{75}$ **(B)** $\frac{17}{54}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{10}{27}$ **(E)** $\frac{17}{42}$

5. Let $N = 3^4 \cdot 5^3 \cdot 7^2$. How many positive divisors of N leave a remainder of 1 when divided by 4?

(A) 20 **(B)** 24 **(C)** 28 **(D)** 30 **(E)** 32

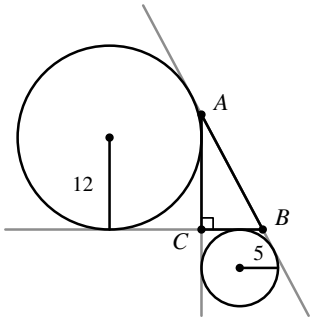
6. Triangle ABC has a right angle at C, with $CA = 20$, $CB = 21$ and $AB = 29$. The inscribed circle of the triangle touches the hypotenuse $\overline{AB}$ at T, and the altitude from C meets $\overline{AB}$ at H. What is the distance TH?

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(A) $\frac{6}{29}$ **(B)** $\frac{12}{29}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** $\frac{35}{29}$

7. An *excircle* of a triangle is a circle outside the triangle that is tangent to one side of the triangle and to the extensions of the other two sides beyond that side. Triangle ABC has a right angle at C. Its excircle tangent to leg $\overline{BC}$ has radius 5, and its excircle tangent to leg $\overline{CA}$ has radius 12. What is the area of triangle ABC?



(A) 30 **(B)** 40 **(C)** 51 **(D)** 60 **(E)** 120

8. Compute the product
$\cos\frac{\pi}{65}\cdot\cos\frac{2\pi}{65}\cdot\cos\frac{4\pi}{65}\cdot\cos\frac{8\pi}{65}\cdot\cos\frac{16\pi}{65}\cdot\cos\frac{32\pi}{65}$,
in which each angle is twice the one before it.

(A) $-\frac{1}{64}$ **(B)** $\frac{1}{128}$ **(C)** $\frac{1}{65}$ **(D)** $\frac{1}{64}$ **(E)** $\frac{1}{32}$

9. A fair coin is flipped repeatedly until heads has come up a total of three times (not necessarily in a row), and then the flipping stops. What is the probability that the total number of flips is even?

(A) $\frac{1}{3}$ **(B)** $\frac{13}{27}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{14}{27}$ **(E)** $\frac{2}{3}$

10. For how many integers n with $1 \le n \le 100$ can n be written as $n = a + b$, where a and b are positive integers and the product ab is divisible by the sum $a + b$? (For example, $n = 12$ works, since $12 = 6 + 6$ and $6 \cdot 6 = 36$ is divisible by 12.)

(A) 9 **(B)** 25 **(C)** 39 **(D)** 42 **(E)** 61

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11. The ten numbers $1, 2, 3, \ldots, 10$ are split into two groups of five numbers each, called group A and group B. For how many of the possible splits is the product of the numbers in group A a multiple of the product of the numbers in group B?

(A) 7 **(B)** 14 **(C)** 16 **(D)** 21 **(E)** 28

12. For a positive integer m, let $s(m)$ be the sum of the digits of m in base ten. For how many integers n with $1 \le n \le 1000$ is $s(2n) = s(n)$? (For example, $n = 9$ works, since $2n = 18$ and $s(18) = 9 = s(9)$.)

(A) 45 **(B)** 90 **(C)** 91 **(D)** 100 **(E)** 111

13. In the coordinate plane let $A = (0, 0)$ and $B = (4, 0)$. Move $\mathcal{R}_A$ rotates the whole plane 60° counterclockwise about A, and move $\mathcal{R}_B$ rotates it 60° counterclockwise about B. A marker starts at $(1, 0)$, and the moves $\mathcal{R}_A, \mathcal{R}_B, \mathcal{R}_A, \mathcal{R}_B, \ldots$ are applied to it alternately, starting with $\mathcal{R}_A$, for 2026 moves in all. How far from A does the marker end up?

(A) 1 **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $\frac{7}{2}$ **(E)** $\sqrt{13}$

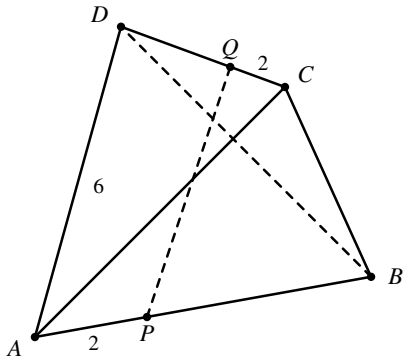
14. Find the smallest value of $\dfrac{x^2 + 7}{\sqrt{x^2 + 3}}$ as x ranges over all real numbers with $|x| \ge 2$.

(A) 4 **(B)** $\frac{7\sqrt{3}}{3}$ **(C)** $\frac{11\sqrt{7}}{7}$ **(D)** $\frac{8\sqrt{3}}{3}$ **(E)** $\frac{11\sqrt{3}}{3}$

15. Sixteen points (i, j) form a 4×4 grid, where i and j each range over $\{0, 1, 2, 3\}$. Two different points are chosen from the grid, with all $\binom{16}{2}$ pairs equally likely. What is the expected value of the square of the distance between the two chosen points?

(A) $\frac{5}{2}$ **(B)** $\frac{8}{3}$ **(C)** 4 **(D)** 5 **(E)** $\frac{16}{3}$

16. Regular tetrahedron $ABCD$ has every edge of length 6. Point P lies on edge $\overline{AB}$ with $AP = 2$, and point Q lies on the opposite edge $\overline{CD}$ with $CQ = 2$. What is the distance PQ?



(A) $2\sqrt{5}$ **(B)** $\sqrt{22}$ **(C)** $2\sqrt{6}$ **(D)** $3\sqrt{3}$ **(E)** $2\sqrt{7}$

17. A permutation $(p_1, p_2, \ldots, p_7)$ of the numbers $1, 2, \ldots, 7$ is chosen uniformly at random. What is the probability that $p_i \ge i - 2$ for every $i = 1, 2, \ldots, 7$?

(A) $\frac{4}{315}$ **(B)** $\frac{9}{280}$ **(C)** $\frac{27}{560}$ **(D)** $\frac{9}{140}$ **(E)** $\frac{27}{280}$

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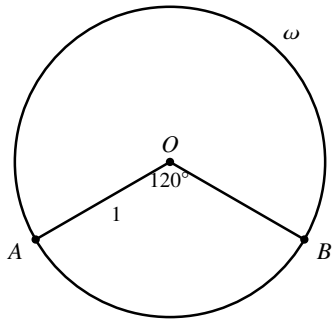
18. Each of the angles $10°$, $50°$ and $70°$ has an irrational sine, yet the sum of the sixth powers of these three sines is a rational number. What is the exact value of $\sin^6 10° + \sin^6 50° + \sin^6 70°$?

(A) $\frac{27}{32}$ **(B)** $\frac{57}{64}$ **(C)** $\frac{15}{16}$ **(D)** $\frac{63}{64}$ **(E)** $\frac{9}{8}$

19. For a positive integer n, let $f(n)$ be the number of integers x with $0 \le x \le n-1$ such that $x^2 - 1$ is divisible by n. For every $n \ge 3$ the two values $x = 1$ and $x = n - 1$ work, so $f(n) \ge 2$; for example $f(8) = 4$, because $1, 3, 5, 7$ all work. For how many integers n with $3 \le n \le 100$ is $f(n) = 2$?

(A) 39 **(B)** 44 **(C)** 47 **(D)** 48 **(E)** 49

20. Points A and B lie on a circle ω with centre O and radius 1, and $\angle AOB = 120°$. Consider the points P inside or on ω, other than A and B, for which $\angle APB \ge 90°$ (points of segment $\overline{AB}$ count, with $\angle APB = 180°$). What is the area of the region formed by these points?



(A) $\frac{3\pi}{8}$ **(B)** $-\frac{\sqrt{3}}{2} + \frac{17\pi}{24}$ **(C)** $-\frac{\sqrt{3}}{4} + \frac{17\pi}{24}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{\sqrt{3}}{4} + \frac{17\pi}{24}$

21. Three different complex numbers are chosen at random from the 12 solutions of $z^{12} = 1$, with all $\binom{12}{3}$ choices equally likely. What is the probability that the sum of the three chosen numbers has absolute value exactly 1?

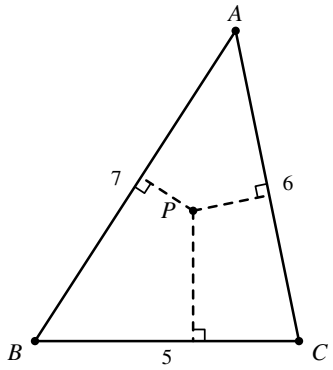
(A) $\frac{3}{11}$ **(B)** $\frac{3}{10}$ **(C)** $\frac{18}{55}$ **(D)** $\frac{4}{11}$ **(E)** $\frac{5}{11}$

22. The number 2029 is prime. How many ordered pairs (x, y) of integers with $0 \le x \le 2028$ and $0 \le y \le 2028$ have the property that $x^2 + y^2 - 1$ is divisible by 2029?

(A) 2028 **(B)** 2029 **(C)** 2030 **(D)** 2031 **(E)** 2032

23. Triangle ABC has side lengths $BC = 5$, $CA = 6$ and $AB = 7$. A point P is chosen uniformly at random from the interior of the triangle, and let d_a, d_b, d_c be the distances from P to the lines BC, CA, AB respectively. What is the probability that d_a, d_b, d_c are the side lengths of a nondegenerate triangle?

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(A) $\frac{35}{286}$ **(B)** $\frac{35}{143}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{108}{143}$

24. Call a positive integer a *semiprime* if it is the product of exactly two primes, not necessarily different; for example 4, 6, 9 and 10 are semiprimes, but 8 and 12 are not. Let S be the set of all semiprimes from 1 to 60. Call a subset of S *square* if the product of its elements is a perfect square (the empty set, whose product is 1, is square). What is the sum of the sizes of all square subsets of S, that is, the total number of elements, counted over all square subsets?

(A) 19456 **(B)** 30720 **(C)** 34816 **(D)** 38912 **(E)** 43008

25. Let $\mathcal{S}$ be the set of all ordered triples (x, y, z) of real numbers that satisfy $y = \dfrac{3x - x^3}{1 - 3x^2}$, $z = \dfrac{3y - y^3}{1 - 3y^2}$, $x = \dfrac{3z - z^3}{1 - 3z^2}$ (so in particular none of the three denominators is 0). The set $\mathcal{S}$ is finite. What is the sum of $x^2 + y^2 + z^2$ over all triples (x, y, z) in $\mathcal{S}$?

(A) 200 **(B)** 575 **(C)** 600 **(D)** 625 **(E)** 675

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Answer Key — AMC 12 Practice Contest 10

1. (E) 59

2. (D) 220

3. (A) $\pi - 2$

4. (B) $\frac{17}{54}$

5. (E) 32

6. (A) $\frac{6}{29}$

7. (D) 60

8. (D) $\frac{1}{64}$

9. (B) $\frac{13}{27}$

10. (C) 39

11. (B) 14

12. (C) 91

13. (E) $\sqrt{13}$

14. (C) $\frac{11\sqrt{7}}{7}$

15. (E) $\frac{16}{3}$

16. (A) $2\sqrt{5}$

17. (E) $\frac{27}{280}$

18. (B) $\frac{57}{64}$

19. (D) 48

20. (C) $-\frac{\sqrt{3}}{4} + \frac{17\pi}{24}$

21. (A) $\frac{3}{11}$

22. (A) 2028

23. (B) $\frac{35}{143}$

24. (D) 38912

25. (C) 600

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Solutions — AMC 12 Practice Contest 10

1. Answer: (E) 59**. The idea.** Compare the two blocks term by term instead of writing two sum formulas. Each term of the second block sits exactly ten places after a term of the first block, so it is $10d$ larger.

Work it out. Pair a_{11} with a_1, a_{12} with a_2, and so on up to a_{20} with a_{10}. Each pair differs by $10d$, and there are ten pairs, so the second sum exceeds the first by $100d$. Thus $100d = 300-100 = 200$ and $d = 2$.

Now the first block: $a_1 + \cdots + a_{10} = 10a_1 + (0 + 1 + \cdots + 9)d = 10a_1 + 45 \cdot 2 = 100$, so $a_1 = 1$. Then $a_{30} = a_1 + 29d = 1 + 58 = 59$.

Check. The sequence is $1, 3, 5, \ldots$, the odd numbers. The first ten odd numbers sum to $10^2 = 100$ and the first twenty sum to $20^2 = 400$, so the second block is 300. The 30th odd number is 59. (Stopping at $a_{20} = 39$ gives 39; using $a_{20} = 39$ and adding only $9d$ gives 57; computing $29d = 58$ but forgetting to add a_1 gives 58.)

2. Answer: (D) 220**. The idea.** Call the original list's length n and its mean m, so its sum is nm. Each change gives one equation, and in each the nm terms cancel, leaving a simple linear relation between n and m.

Work it out. Removing 13: the new list has $n - 1$ numbers with mean $m + 1$, so

$nm - 13 = (n - 1)(m + 1) = nm + n - m - 1$,

which gives $m = n + 12$.

Adding 44: the new list has $n + 1$ numbers with mean $m + 2$, so

$nm + 44 = (n + 1)(m + 2) = nm + 2n + m + 2$,

which gives $2n + m = 42$. Substituting $m = n + 12$: $3n + 12 = 42$, so $n = 10$ and $m = 22$. The sum is $nm = 220$.

Check. Without the 13: $\frac{220-13}{9} = \frac{207}{9} = 23 = 22 + 1$. With the 44: $\frac{264}{11} = 24 = 22 + 2$. Both work. The mean itself is 22, and 198 or 242 come from using 9 or 11 numbers instead of 10.

3. Answer: (A) $\pi - 2$**. The idea.** Writing $z = x + yi$, the set is the part of the disk $x^2 + y^2 \le 4$ on the far side of the line $x + y = 2$. That line passes through $(2, 0)$ and $(0, 2)$, both on the circle, so it cuts off a circular segment over a quarter of the circle.

Work it out. The chord from $(2, 0)$ to $(0, 2)$ subtends a right angle at the origin. The region on the side away from the origin (where $x + y \ge 2$) is the quarter disk minus the right triangle with vertices $(0, 0)$, $(2, 0)$, $(0, 2)$:

$\frac{1}{4}\pi \cdot 2^2 - \frac{1}{2} \cdot 2 \cdot 2 = \pi - 2$.

Check. Integrate: $\int_0^2 \left(\sqrt{4 - x^2} - (2 - x)\right) dx = \pi - 2$, since the first part is a quarter of the disk's area and the second is 2. The rest of the disk, $4\pi - (\pi - 2) = 3\pi + 2$, is the side containing the origin, where $x + y \le 2$.

4. Answer: (B) $\frac{17}{54}$**. The idea.** An increasing or decreasing number is determined by its set of digits: choose three different digits, and there is exactly one way to arrange them in increasing order and one in decreasing order. The only asymmetry is the digit 0, which can never lead.

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Work it out. The sample space: the first digit has 9 choices (1 to 9), the second 9 (anything else, including 0), the third 8, for $9 \cdot 9 \cdot 8 = 648$ numbers.

Increasing: the smallest digit comes first, so 0 cannot be used at all (it would lead). Choose 3 of the digits 1 to 9: $\binom{9}{3} = 84$.

Decreasing: 0 can only be last, which is allowed, so choose any 3 of the ten digits: $\binom{10}{3} = 120$.

No number is both, so there are $84 + 120 = 204$ favourable numbers, and the probability is $\frac{204}{648} = \frac{17}{54}$.

Check. Without the leading-zero rule, every 3-set of digits gives $3! = 6$ arrangements of which 2 are monotone, a probability of $\frac{1}{3}$; the true answer $\frac{17}{54} \approx 0.315$ is a bit lower because the excluded numbers (with 0 in front) never included the increasing ones. Using $\binom{10}{3}$ for both directions gives $\frac{240}{648} = \frac{10}{27}$; counting only $9 \cdot 8 \cdot 7 = 504$ numbers (forgetting that 0 may appear after the first digit) gives $\frac{204}{504} = \frac{17}{42}$; dividing by all 900 three-digit numbers gives $\frac{17}{75}$.

5. Answer: (E) 32**. The idea.** Every divisor is $3^a 5^b 7^c$, and modulo 4 each prime is ± 1: $5 \equiv 1$, while $3 \equiv 7 \equiv -1$. So the remainder depends only on the parity of $a + c$.

Work it out. A divisor $3^a 5^b 7^c$ with $0 \le a \le 4$, $0 \le b \le 3$, $0 \le c \le 2$ is congruent to $(-1)^a \cdot 1^b \cdot (-1)^c = (-1)^{a+c} \pmod 4$. It is $\equiv 1$ exactly when $a + c$ is even.

a even (0, 2, 4: 3 ways) with c even (0, 2: 2 ways) gives 6; a odd (1, 3: 2 ways) with c odd (1: 1 way) gives 2. That is 8 choices of (a, c), and b is free (4 ways): $8 \cdot 4 = 32$.

Check. N has $5 \cdot 4 \cdot 3 = 60$ divisors, all odd, so the other $60 - 32 = 28$ are $\equiv 3 \pmod 4$: those have $a + c$ odd, $(3 \cdot 1 + 2 \cdot 2) \cdot 4 = 28$. The total checks. Guessing "half" gives 30, which cannot be right because the exponent ranges are uneven.

6. Answer: (A) $\frac{6}{29}$**. The idea.** Both points sit on $\overline{AB}$ at distances from A that have clean formulas: tangent lengths from a vertex for T, and the projection of a leg onto the hypotenuse for H. Subtract.

Work it out. The two tangent segments from each vertex to the incircle are equal. With semiperimeter $s = \frac{20+21+29}{2} = 35$, the tangent length from A is $s - BC = 35 - 21 = 14$, so $AT = 14$.

For H: triangles AHC and ACB are similar (they share angle A and each has a right angle), so $\frac{AH}{AC} = \frac{AC}{AB}$, giving $AH = \frac{AC^2}{AB} = \frac{400}{29}$.

Both are measured from A along $\overline{AB}$, so $TH = 14 - \frac{400}{29} = \frac{406-400}{29} = \frac{6}{29}$.

Check. Measure from B instead: $BT = s - CA = 15$ and $BH = \frac{CB^2}{AB} = \frac{441}{29}$, so $TH = \frac{441}{29} - 15 = \frac{6}{29}$. Same, and $AT + BT = 29$, $AH + BH = \frac{841}{29} = 29$ as they should. Mixing up which leg goes with which vertex gives $\frac{441}{29} - 14 = \frac{35}{29}$; using the midpoint of AB in place of T gives $\frac{1}{2}$.

7. Answer: (D) 60**. The idea.** An excircle's radius is the area divided by a "tangent length": the excircle tangent to side a has radius $r_a = \frac{K}{s-a}$, where K is the area and s the semiperimeter. In a right triangle the area itself is a product of two tangent lengths, $K = (s - a)(s - b)$, so each leg's exradius is the *other* tangent length, and the area is just the product of the two given radii.

Work it out. Let $a = BC$, $b = CA$, $c = AB$, so $a^2 + b^2 = c^2$. The excircle tangent to $\overline{BC}$ has centre I_A at distance r_a from all three side lines, and triangle ABC is what remains of quadrilateral ABI_AC after removing triangle I_ABC: $K = [I_AAB] + [I_AAC] - [I_ABC] = \frac{r_a}{2}(c + b - a) = r_a(s - a)$, so $r_a = \frac{K}{s-a}$, and likewise $r_b = \frac{K}{s-b}$.

Next, $(s-a)(s-b) = \frac{(c-a+b)(c+a-b)}{4} = \frac{c^2-(a-b)^2}{4} = \frac{c^2-a^2-b^2+2ab}{4} = \frac{ab}{2} = K$.

Hence $r_a = \frac{(s-a)(s-b)}{s-a} = s-b$ and $r_b = s-a$, and

$K = (s-a)(s-b) = r_b \cdot r_a = 12 \cdot 5 = 60$.

Check. Find the triangle. The hypotenuse is $c = (s-a)+(s-b) = 12+5 = 17$, and $a+b = 2s-c$ with $ab = 2K = 120$. From $a = s-12$, $b = s-5$ and $a^2+b^2 = 289$: $s^2-17s-60 = 0$, so $s = 20$, giving legs $a = 8$, $b = 15$: the 8–15–17 triangle, of area $\frac{8\cdot 15}{2} = 60$. Directly: $r_a = \frac{60}{20-8} = 5$ and $r_b = \frac{60}{20-15} = 12$, as given. Taking 5 and 12 to be the legs gives 30; forgetting the $\frac{1}{2}$ in $\frac{ab}{2}$ gives 120; the inradius is $s-c = 3$, and $3 \cdot 17 = 51$ uses the hypotenuse where the semiperimeter belongs ($3 \cdot 20 = 60$); 40 is the perimeter.

8. Answer: (D) $\frac{1}{64}$. The idea. The angles double, so multiplying by $\sin\frac{\pi}{65}$ lets $2\sin\theta\cos\theta = \sin 2\theta$ collapse the product one factor at a time.

Work it out. Let P be the product and $\theta = \frac{\pi}{65}$. Then

$64\sin\theta \cdot P = 32 \cdot (2\sin\theta\cos\theta)\cos 2\theta \cdots \cos 32\theta = 32\sin 2\theta\cos 2\theta \cdots \cos 32\theta = \cdots = \sin 64\theta$.

Six uses of the double-angle formula consume the six factors of 2. Now $64\theta = \frac{64\pi}{65} = \pi - \frac{\pi}{65}$, so $\sin 64\theta = \sin\theta$. Hence $64\sin\theta \cdot P = \sin\theta$, and since $\sin\theta \neq 0$, $P = \frac{1}{64}$.

Check. Every angle is below $\frac{\pi}{2}$ (the largest is $\frac{32\pi}{65}$), so every factor is positive and $P > 0$, ruling out $-\frac{1}{64}$. The same trick with denominator 7: $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \frac{\sin(8\pi/7)}{8\sin(\pi/7)} = -\frac{1}{8}$, where the sign flips because $8\theta = \pi + \theta$; here $64\theta = \pi - \theta$ keeps it positive.

9. Answer: (B) $\frac{13}{27}$. The idea. Cut the flips into three blocks, each a run of tails followed by one head (for example, T T H | H | T H). The blocks are independent and each has the same distribution, so only the parity of each block's length matters.

Work it out. A block has length $k+1$ when it starts with k tails, which happens with probability $\left(\frac{1}{2}\right)^{k+1}$. Its length is odd when k is even:

$P(\text{odd block}) = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{1/2}{1-1/4} = \frac{2}{3}$, $\qquad P(\text{even block}) = \frac{1}{3}$.

The total length is even exactly when an even number of the three blocks have odd length: none of them, or exactly two. So

$P(\text{even total}) = \left(\frac{1}{3}\right)^3 + \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{1}{27} + \frac{12}{27} = \frac{13}{27}$.

Check. The stopping flip is flip n with probability $\binom{n-1}{2}/2^n$ (two heads among the first $n-1$ flips, then a head). The first even terms are $n = 4$: $\frac{3}{16}$, $n = 6$: $\frac{10}{64}$, $n = 8$: $\frac{21}{256}$, $n = 10$: $\frac{36}{1024}$, already summing to about 0.46, and the rest bring it up to $\frac{13}{27} \approx 0.481$. Guessing $\frac{1}{2}$ by symmetry is wrong because short games favour the odd value $n = 3$; $\frac{14}{27}$ is the probability of an odd total.

10. Answer: (C) 39. The idea. Replace b by $n-a$. Then $ab = a(n-a) = an - a^2$, so $n \mid ab$ exactly when $n \mid a^2$. The question becomes: for which n is there an a with $0 < a < n$ whose square is a multiple of n? That happens exactly when n is divisible by a perfect square larger than 1.

Work it out. If $n = m^2k$ with $m \geq 2$, take $a = mk$, which lies strictly between 0 and n; then $a^2 = m^2k^2 = n \cdot k$ is a multiple of n, so $n = mk + (n-mk)$ works. If instead n is a product of

distinct primes (squarefree), then $n \mid a^2$ forces each prime of n to divide a, so $n \mid a$, impossible for $0 < a < n$. (The case $n = 1$ fails because it cannot be split at all.)

So we count $n \le 100$ divisible by the square of a prime: by 4: 25; by 9: 11; by 25: 4; by 49: 2. Numbers counted twice are the multiples of 36 (36, 72) and of 100 (100); no number up to 100 is a multiple of three of these squares, and $4 \cdot 49$, $9 \cdot 25$ already exceed 100. Total $25 + 11 + 4 + 2 - 2 - 1 = 39$.

Check. $n = 18 = 6 + 12$: $72 = 4 \cdot 18$. $n = 50 = 10 + 40$: $400 = 8 \cdot 50$. And $n = 6$ fails: the products $1 \cdot 5, 2 \cdot 4, 3 \cdot 3$ are $5, 8, 9$, none a multiple of 6. The 61 squarefree numbers are the ones that fail ($39 + 61 = 100$). Forgetting the overlaps gives 42; only allowing $a = b$ (multiples of 4) gives 25; only perfect squares gives 9.

11. Answer: (B) 14**. The idea.** The two products multiply to 10!. If P_A and P_B are the products, then $\frac{P_A}{P_B} = \frac{10!}{P_B^2}$, so the condition only concerns group B: P_B^2 must divide $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, i.e. P_B must divide $2^4 \cdot 3^2 \cdot 5 = 720$.

Work it out. So group B is a set of five numbers whose product divides 720: it avoids 7, contains at most one of 5 and 10, its numbers carry at most four factors of 2 in total ($2, 6, 10$ carry one each, 4 two, 8 three), and at most two factors of 3 (3 and 6 carry one, 9 carries two, so 9 cannot go with 3 or 6).

If $8 \in B$, only one more factor of 2 is allowed, so $4 \notin B$ and at most one of $2, 6, 10$ is in B; the rest come from $\{1, 3, 5, 9\}$, which cannot supply four numbers (3 and 9 clash). With 2: $\{1, 3, 5\}$ or $\{1, 5, 9\}$; with 6: $\{1, 3, 5\}$; with 10: we would need $\{1, 3, 9\}$, which clashes. That is 3 sets.

If $8 \notin B$ and $9 \in B$: then $3, 6 \notin B$, and the other four come from $\{1, 2, 4, 5, 10\}$ without both 5 and 10: $\{1, 2, 4, 5\}$ or $\{1, 2, 4, 10\}$. That is 2 sets.

If $8, 9 \notin B$: choose 5 of $\{1, 2, 3, 4, 5, 6, 10\}$, $\binom{7}{5} = 21$ ways. Remove the $\binom{5}{3} = 10$ that contain both 5 and 10, and the ones containing all of $2, 4, 6, 10$ (five factors of 2) that are not already removed: $\{2, 4, 6, 10, 1\}$ and $\{2, 4, 6, 10, 3\}$. That leaves $21 - 10 - 2 = 9$.

In total $3 + 2 + 9 = 14$ choices of B, and each gives one split.

Check. The largest allowed product is 720 itself, e.g. $B = \{2, 3, 4, 5, 6\}$ and $A = \{1, 7, 8, 9, 10\}$ with $P_A = 5040 = 7 \cdot 720$; the smallest is $B = \{1, 2, 3, 4, 5\}$ with $P_B = 120$ and $P_A = 30240 = 252 \cdot 120$. The groups are named, so a split and its swap are different; only one of them can work unless $P_A = P_B$, impossible since 10! is not a perfect square. Halving to 7 treats the groups as unnamed.

12. Answer: (C) 91**. The idea.** Double n digit by digit. Each digit d becomes $2d$, and a carry happens exactly at the digits $d \ge 5$. Each carry replaces a 10 in one place by a 1 in the next, losing 9 from the digit sum. So $s(2n) = 2s(n) - 9c$, where c is the number of digits of n that are at least 5.

Work it out. Whether a place sends a carry depends only on its own digit: $2d \le 8$ (plus a possible incoming 1 gives at most 9) when $d \le 4$, and $2d \ge 10$ when $d \ge 5$. The condition $s(2n) = s(n)$ becomes $s(n) = 9c$: the digit sum must be 9 times the number of "big" digits (5 to 9).

$n = 1000$ fails ($s = 1$, $c = 0$), so write n as three digits with leading zeros, $1 \le n \le 999$.

$c = 0$: digit sum 0, only $n = 0$; none.

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$c = 1$: one big digit b, two small digits (each 0 to 4) summing to $9 - b$. For $b = 5, 6, 7, 8, 9$ the small pair has sum $4, 3, 2, 1, 0$, with $5, 4, 3, 2, 1$ ordered choices: 15, times 3 positions for the big digit: 45.

$c = 2$: two big digits and one small digit t with total 18, so the big digits sum to $18 - t \in \{14, \ldots, 18\}$, giving $5 + 4 + 3 + 2 + 1 = 15$ ordered big pairs, times 3 positions for the small digit: 45.

$c = 3$: three big digits summing to 27: only 999. That is 1.

Total $45 + 45 + 1 = 91$.

Check. Since $s(m) \equiv m \pmod 9$, the equation $s(2n) = s(n)$ forces $2n \equiv n$, so $9 \mid n$; the answer is at most the 111 multiples of 9 up to 1000, which is why 111 is tempting. Not every multiple of 9 works: 144 has digit sum 9 but no big digit, and $2 \cdot 144 = 288$ has digit sum 18. Examples that do work, matching the rule: $45 \to 90$ (one big digit, sum 9) and $468 \to 936$ (two big digits, sum 18). Forgetting 999 gives 90; counting only the case $c = 1$ gives 45.

13. Answer: (E) $\sqrt{13}$. The idea. Treat points as complex numbers. A pair of moves $\mathcal{R}_B \circ \mathcal{R}_A$ is again a rotation, by $60° + 60° = 120°$, so doing the pair three times returns every point to where it started. That cuts 2026 moves down to 4.

Work it out. Let $\zeta = \cos 60° + i \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then $\mathcal{R}_A(z) = \zeta z$ and $\mathcal{R}_B(z) = 4 + \zeta(z - 4)$, so one pair of moves is $T(z) = \mathcal{R}_B(\mathcal{R}_A(z)) = \zeta^2 z + 4(1 - \zeta)$. Applying it three times, $T^3(z) = \zeta^6 z + 4(1 - \zeta)(1 + \zeta^2 + \zeta^4) = z$, because $\zeta^6 = 1$ and $1 + \zeta^2 + \zeta^4 = 0$ (three cube roots of unity). The 2026 moves are 1013 pairs, and $1013 = 3 \cdot 337 + 2$, so the marker ends at $T^2(1)$.

$T(1) = \zeta^2 + 4 - 4\zeta = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 2 - 2\sqrt{3}\,i = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$.

$T^2(1) = \zeta^2\left(\frac{3}{2} - \frac{3\sqrt{3}}{2}i\right) + 2 - 2\sqrt{3}\,i = \left(\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right) + 2 - 2\sqrt{3}\,i = \frac{7}{2} - \frac{\sqrt{3}}{2}i$. Its distance from A is $\sqrt{\frac{49}{4} + \frac{3}{4}} = \sqrt{13}$.

Check. Follow the marker for six moves: $(1, 0) \to \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \to \left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right) \to (3, 0) \to \left(\frac{7}{2}, -\frac{\sqrt{3}}{2}\right) \to \left(\frac{5}{2}, \frac{3\sqrt{3}}{2}\right) \to (1, 0)$, back at the start after six moves. Since $2026 = 6 \cdot 337 + 4$, the answer is the fourth point, $\left(\frac{7}{2}, -\frac{\sqrt{3}}{2}\right)$, at distance $\sqrt{13}$. Stopping one move early gives the distractor 3, and reading off only the x-coordinate gives $\frac{7}{2}$.

14. Answer: (C) $\frac{11\sqrt{7}}{7}$. The idea. With $t = \sqrt{x^2 + 3}$ the expression becomes $\frac{t^2+4}{t} = t + \frac{4}{t}$. AM-GM says $t + \frac{4}{t} \geq 4$ with equality at $t = 2$, but the condition $|x| \geq 2$ forces $t \geq \sqrt{7} > 2$, so that equality can never happen. Instead use that $t + \frac{4}{t}$ is increasing once $t \geq 2$.

Work it out. $|x| \geq 2$ means $x^2 \geq 4$, so $t = \sqrt{x^2 + 3} \geq \sqrt{7}$, and every $t \geq \sqrt{7}$ occurs. For $2 \leq s < t$, $\left(t + \frac{4}{t}\right) - \left(s + \frac{4}{s}\right) = (t - s)\left(1 - \frac{4}{st}\right) > 0$, since $st > 4$. So $t + \frac{4}{t}$ increases on $[\sqrt{7}, \infty)$ and its minimum there is at $t = \sqrt{7}$ (that is, $x = \pm 2$): $\sqrt{7} + \frac{4}{\sqrt{7}} = \frac{11}{\sqrt{7}} = \frac{11\sqrt{7}}{7} \approx 4.157$.

Check. At $x = 2$ directly: $\frac{11}{\sqrt{7}}$. At $x = 3$: $\frac{16}{\sqrt{12}} \approx 4.62$, larger, as it should be. The trap 4 is the unrestricted AM-GM bound (reached at $x = \pm 1$, which is not allowed), and $\frac{7}{\sqrt{3}}$ is the value at $x = 0$.

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15. Answer: (E) $\frac{16}{3}$**. The idea.** The squared distance is $(\Delta x)^2 + (\Delta y)^2$, and squares of differences add up nicely if we first allow the two points to coincide (then the coordinates are independent). A coinciding pair contributes 0, so we only have to correct the number of pairs at the end.

Work it out. Over ordered pairs of coordinates $i, i' \in \{0, 1, 2, 3\}$, $\sum(i-i')^2 = 2(3 \cdot 1^2 + 2 \cdot 2^2 + 1 \cdot 3^2) = 2(3 + 8 + 9) = 40$. Over all $16^2 = 256$ ordered pairs of grid points (repeats allowed), each pair of x-coordinates comes with 16 choices of the y-coordinates, so $\sum(\Delta x)^2 = 16 \cdot 40 = 640$, and likewise $\sum(\Delta y)^2 = 640$. The total of squared distances is 1280.

The 16 pairs with the same point twice contribute 0, so the $16 \cdot 15 = 240$ ordered pairs of different points also total 1280. The expected value is $\frac{1280}{240} = \frac{16}{3}$.

Check. A single coordinate is uniform on $\{0, 1, 2, 3\}$ with variance $\frac{5}{4}$, so for independent points $E[(\Delta x)^2] = 2 \cdot \frac{5}{4} = \frac{5}{2}$ and $E[d^2] = 5$ when repeats are allowed (the distractor; stopping after one coordinate gives $\frac{5}{2}$). Removing the $\frac{1}{16}$ chance of a repeat rescales: $5 \cdot \frac{16}{15} = \frac{16}{3}$.

16. Answer: (A) $2\sqrt{5}$**. The idea.** A regular tetrahedron sits inside a cube: four alternate corners of a cube are pairwise joined by face diagonals, all equal. With the cube in coordinates, P and Q are simple weighted averages of corners and the distance is one square root.

Work it out. Take a cube of side $s = 3\sqrt{2}$ (so a face diagonal is $s\sqrt{2} = 6$) and $A = (0, 0, 0)$, $B = (s, s, 0)$, $C = (s, 0, s)$, $D = (0, s, s)$. Every two of these differ in exactly two coordinates by s, so all six edges equal 6. Then $P = A + \frac{1}{3}(B - A) = \left(\frac{s}{3}, \frac{s}{3}, 0\right)$, $Q = C + \frac{1}{3}(D - C) = \left(\frac{2s}{3}, \frac{s}{3}, s\right)$, so $Q - P = \left(\frac{s}{3}, 0, s\right)$ and $PQ^2 = \frac{s^2}{9} + s^2 = \frac{10}{9}s^2 = \frac{10}{9} \cdot 18 = 20$, $PQ = 2\sqrt{5}$.

Check. With vectors $\vec{b} = \vec{AB}, \vec{c} = \vec{AC}, \vec{d} = \vec{AD}$ (each of length 6, pairwise dot product 18 because the angles are 60°): $\vec{PQ} = \frac{2}{3}\vec{c} + \frac{1}{3}\vec{d} - \frac{1}{3}\vec{b}$, so $PQ^2 = 36\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) + 2 \cdot 18\left(\frac{2}{9} - \frac{2}{9} - \frac{1}{9}\right) = 24 - 4 = 20$.

For comparison, the two midpoints are $3\sqrt{2}$ apart, the shortest possible distance between the opposite edges, so PQ must be at least that; getting the sign of the last dot-product term wrong gives $24 + 4 = 28$ and $PQ = 2\sqrt{7}$.

17. Answer: (E) $\frac{27}{280}$**. The idea.** Read the condition from the value's side: the value v may sit only in a position $i \le v + 2$. Place the values in increasing order $1, 2, 3, \ldots$; each new value's allowed positions contain all the positions used so far, so the number of choices at each step is easy to count.

Work it out. Value v is allowed in positions $1, 2, \ldots, \min(v + 2, 7)$. When we come to place v, the values $1, \ldots, v - 1$ already occupy $v - 1$ positions, all inside value v's allowed range (their ranges are smaller). So v has $\min(v + 2, 7) - (v - 1)$ choices: $v = 1, \ldots, 5$: 3 choices each; $v = 6$: $7 - 5 = 2$; $v = 7$: 1. The number of good permutations is $3^5 \cdot 2 = 486$, and the probability is $\frac{486}{5040} = \frac{27}{280}$.

Check. For $n = 3$ the condition $p_i \ge i - 2$ only says $p_3 \ge 1$, so all 6 permutations work; the formula gives $3 \cdot 2 \cdot 1 = 6$. For $n = 4$ it gives $3 \cdot 3 \cdot 2 \cdot 1 = 18$: of the 24 permutations exactly those with $p_4 = 1$ fail, which are 6. The distractor $\frac{4}{315}$ is the answer to the stricter condition $p_i \ge i - 1$ (2^6 permutations).

18. Answer: (B) $\frac{57}{64}$**. The idea.** Write $\sin^6\theta$ as a combination of $\cos 2\theta$, $\cos 4\theta$, $\cos 6\theta$. Since $\sin 50° = \sin 130°$, the three angles can be taken as $10°, 70°, 130°$, which step by $60°$; then most cosine terms cancel in threes, but the $\cos 6\theta$ terms do not.

Work it out. From $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\sin^6\theta = \frac{(1-c)^3}{8}$ with $c = \cos 2\theta$, and using $c^2 = \frac{1+\cos 4\theta}{2}$, $c^3 = \frac{3\cos 2\theta+\cos 6\theta}{4}$: $\sin^6\theta = \frac{10-15\cos 2\theta+6\cos 4\theta-\cos 6\theta}{32}$. For $\theta = 10°, 70°, 130°$: the angles $2\theta = 20°, 140°, 260°$ are $120°$ apart, so their cosines sum to 0; the angles $4\theta = 40°, 280°, 520°$ are $240°$ apart (again three equally spaced directions), so their cosines sum to 0; but $6\theta = 60°, 420°, 780°$ all point the same way, so each $\cos 6\theta = \frac{1}{2}$. Hence the sum is $\frac{3\cdot 10-3\cdot\frac{1}{2}}{32} = \frac{57/2}{32} = \frac{57}{64}$.

Check. Numerically, $\sin 10° \approx 0.17365$, $\sin 50° \approx 0.76604$, $\sin 70° \approx 0.93969$; their sixth powers are about 0.00003, 0.20200, 0.68858, summing to $0.89061 \approx \frac{57}{64} = 0.890625$. Forgetting the $\cos 6\theta$ term gives the distractor $\frac{30}{32} = \frac{15}{16}$.

19. Answer: (D) 48**. The idea.** Split n into prime powers. By the Chinese Remainder Theorem, choosing x modulo n is the same as choosing it independently modulo each prime-power factor, so f is multiplicative: $f(n)$ is the product of the values $f(p^k)$ over the prime powers p^k exactly dividing n. Then find f on prime powers; odd primes and the prime 2 behave differently.

Work it out. *Odd prime powers.* If p is odd and $p^k \mid (x-1)(x+1)$, then p cannot divide both factors (they differ by 2), so all of p^k divides one of them: $x \equiv 1$ or $x \equiv -1 \pmod{p^k}$. These are different since $p^k \geq 3$, so $f(p^k) = 2$.

Powers of 2. $f(1) = 1$, $f(2) = 1$ (only $x = 1$), $f(4) = 2$ ($x = 1, 3$). For 2^e with $e \geq 3$: x must be odd, so $x-1$ and $x+1$ are consecutive even numbers and one of them is $\equiv 2 \pmod 4$; the other must then carry 2^{e-1}. That gives $x \equiv \pm 1 \pmod{2^{e-1}}$, which is four residues modulo 2^e: $1,\ 2^{e-1}-1,\ 2^{e-1}+1,\ 2^e-1$. So $f(2^e) = 4$.

Putting it together. If $n = 2^e m$ with m odd having j different prime factors, then $f(n) = f(2^e)\cdot 2^j$. This equals 2 exactly when either $j = 1$ and $e \in \{0, 1\}$ (that is, $n = p^k$ or $n = 2p^k$ with p an odd prime), or $j = 0$ and $e = 2$ (that is, $n = 4$).

Counting up to 100. Odd prime powers: the 24 odd primes below 100, plus $9, 27, 81, 25, 49$, for 29. Twice an odd prime power: $p^k \leq 50$, namely the 14 odd primes up to 47 plus $9, 25, 27, 49$, for 18. Plus $n = 4$. Total $29 + 18 + 1 = 48$.

Check. $f(15) = 4$: $1, 4, 11, 14$ (here $4^2 - 1 = 15$). $f(18) = 2$: only 1 and 17, since x must be odd and $\equiv \pm 1 \pmod 9$. $f(12) = 4$: $1, 5, 7, 11$. Forgetting $n = 4$ gives 47; counting $n = 8$ as well gives 49; leaving out the doubled prime powers $18, 50, 54, 98$ gives 44; and counting only primes, twice primes and 4 gives 39.

20. Answer: (C) $-\frac{\sqrt{3}}{4} + \frac{17\pi}{24}$**. The idea.** $\angle APB \geq 90°$ exactly when P lies in the closed disk D with diameter $\overline{AB}$ (Thales: right angle on the circle, obtuse inside, acute outside). So we need the area of the overlap of D with the disk bounded by ω. Split it along line AB.

Work it out. The chord has length $AB = 2\sin 60° = \sqrt{3}$, so D has centre M (the midpoint of AB) and radius $\frac{\sqrt{3}}{2}$, and $OM = \cos 60° = \frac{1}{2}$. The two circles meet only at A and B (two circles share at most two points), so each of the four arcs cut off by line AB lies entirely inside or entirely outside the other disk; one test point per arc decides.

On O's side of AB: the point of ∂D farthest towards O is at distance $\frac{\sqrt{3}}{2} - \frac{1}{2} < 1$ from O, so that half of D lies inside ω. Contribution: $\frac{1}{2}\pi\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3\pi}{8}$.

On the far side: the point of ω farthest from AB is at distance $1 - \frac{1}{2} = \frac{1}{2} < \frac{\sqrt{3}}{2}$ from M, so the whole minor segment of ω cut off by AB lies inside D. Its area is sector minus triangle: $\frac{120}{360}\pi - \frac{1}{2}\sin 120° = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$.

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Total: $\frac{3\pi}{8} + \frac{\pi}{3} - \frac{\sqrt{3}}{4} = \frac{17\pi}{24} - \frac{\sqrt{3}}{4} \approx 1.792$.

Check. The general lens-area formula for circles of radii 1 and $\frac{\sqrt{3}}{2}$ with centres $\frac{1}{2}$ apart gives the same value. It is less than the whole of D ($\frac{3\pi}{4} \approx 2.36$), since part of D sticks out of ω beyond AB, and more than the half disk $\frac{3\pi}{8} \approx 1.18$.

21. Answer: (A) $\frac{3}{11}$. The idea. Choosing two opposite points u and $-u$ plus any third point w obviously works: the sum is w. The surprise is that nothing else works. If $|u| = |v| = |w| = 1$ and $|u + v + w| = 1$, then two of the three numbers are opposite. Use $\bar{u} = \frac{1}{u}$ on the unit circle to turn the condition $|u + v + w| = 1$ into a polynomial identity that factors.

Work it out. *The key fact.* Let $s = u + v + w$ with $|u| = |v| = |w| = 1$. Then $\bar{s} = \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = \frac{uv+vw+wu}{uvw}$, so $|s| = 1$, i.e. $s\bar{s} = 1$, means $(u + v + w)(uv + vw + wu) = uvw$. Expanding shows $(u + v + w)(uv + vw + wu) - uvw = (u + v)(v + w)(w + u)$, so $|s| = 1$ exactly when $(u + v)(v + w)(w + u) = 0$, that is, when two of the numbers are opposite.

Counting. The 12 solutions of $z^{12} = 1$ form 6 opposite pairs $\{\zeta, -\zeta\}$. A good choice is an opposite pair plus one of the other 10 numbers: $6 \cdot 10 = 60$ choices. None is counted twice, because three numbers cannot contain two different opposite pairs. So the probability is $\frac{60}{\binom{12}{3}} = \frac{60}{220} = \frac{3}{11}$.

Check. The identity: both sides of $(u + v)(v + w)(w + u) = (u + v + w)(uv + vw + wu) - uvw$ expand to $\sum u^2v + 2uvw$, where the sum runs over the six terms $u^2v, u^2w, v^2u, v^2w, w^2u, w^2v$. Spot checks: $\{1, i, -1\}$ has sum i, absolute value 1, and contains an opposite pair; $\{1, \zeta, \zeta^{11}\}$ with $\zeta = \cos 30^\circ + i \sin 30^\circ$ has sum $1 + 2\cos 30^\circ = 1 + \sqrt{3}$; $\{1, \zeta^4, \zeta^8\}$ has sum 0. Counting a pair with any third point, $6 \cdot 12$, gives the distractor $\frac{72}{220} = \frac{18}{55}$.

22. Answer: (A) 2028**. The idea.** Work in arithmetic modulo $p = 2029$ and count points on the "circle" $x^2 + y^2 = 1$ the way one finds rational points on the unit circle: every line through the known point $Q = (-1, 0)$ meets the circle in at most one more point, and every other point lies on exactly one such line. The only subtlety is the slopes t with $1 + t^2 = 0$, which exist because $p \equiv 1 \pmod 4$.

Work it out. *Vertical line* $x = -1$: then $y^2 = 0$, so it meets the circle only at Q.

Line $y = t(x + 1)$ *with slope* t (p choices): substituting, $x^2 - 1 + t^2(x + 1)^2 = 0$, i.e. $(x + 1)\big[(1 + t^2)x - (1 - t^2)\big] = 0$. Besides $x = -1$ (the point Q), we need $(1 + t^2)x = 1 - t^2$. If $1 + t^2 \neq 0$, this gives exactly one point $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$, and it is not Q (that would need $1 - t^2 = -1 - t^2$, i.e. $2 = 0$). If $1 + t^2 = 0$, the equation reads $0 = 2$: no new point.

Every point other than Q has $x \neq -1$ and lies on exactly one of these lines (slope $t = \frac{y}{x+1}$), so the points other than Q correspond one-to-one to the slopes with $t^2 \neq -1$. Since $2029 = 4 \cdot 507 + 1$, -1 is a square modulo 2029 and $t^2 = -1$ has exactly 2 solutions. So there are $2029 - 2 = 2027$ points besides Q, and 2028 in all.

Check. For $p = 5$ (also $\equiv 1 \pmod 4$) the formula predicts $p - 1 = 4$, and indeed the solutions of $x^2 + y^2 \equiv 1 \pmod 5$ are $(0, \pm 1)$ and $(\pm 1, 0)$ only. For a prime $\equiv 3 \pmod 4$ no slope is lost and the count would be $p + 1$ (the distractor 2030).

23. Answer: (B) $\frac{35}{143}$. The idea. Inside the triangle each distance d_a, d_b, d_c is a linear function of the position of P (a signed distance to a line). So each condition such as $d_a < d_b + d_c$ cuts the triangle by a straight line, and the good region is a smaller triangle. Its corners are easy to locate:

on side AB we have $d_c = 0$, and there the condition $d_a = d_b + d_c$ becomes $d_a = d_b$, which on AB happens exactly at the foot of the angle bisector from C. So the good region is the triangle whose vertices are the feet of the three angle bisectors.

Work it out. Let the bisectors from A, B, C meet the opposite sides at D, E, F. The line $d_a = d_b + d_c$ meets side AB at F (where $d_c = 0$, $d_a = d_b$) and side CA at E (where $d_b = 0$, $d_a = d_c$), so it is line EF. At the incentre all three distances are equal and every inequality holds, while near A we have $d_a > 0 = d_b = d_c$; so $d_a < d_b + d_c$ holds exactly on the BC side of line EF. Likewise for the other two conditions. The good region is triangle DEF, and the probability is $\frac{[DEF]}{[ABC]}$, found by removing three corner triangles.

By the angle bisector theorem, $\frac{AF}{AB} = \frac{CA}{CA+BC} = \frac{6}{11}$ and $\frac{AE}{AC} = \frac{AB}{AB+BC} = \frac{7}{12}$, so $\frac{[AEF]}{[ABC]} = \frac{6}{11} \cdot \frac{7}{12} = \frac{7}{22}$. Similarly $\frac{[BFD]}{[ABC]} = \frac{BF}{BA} \cdot \frac{BD}{BC} = \frac{5}{11} \cdot \frac{7}{13} = \frac{35}{143}$ and $\frac{[CDE]}{[ABC]} = \frac{CD}{CB} \cdot \frac{CE}{CA} = \frac{6}{13} \cdot \frac{5}{12} = \frac{5}{26}$.

So the probability is $1 - \frac{7}{22} - \frac{35}{143} - \frac{5}{26} = 1 - \frac{91+70+55}{286} = 1 - \frac{216}{286} = \frac{70}{286} = \frac{35}{143}$.

Check. In general the same computation gives $\frac{2abc}{(a+b)(b+c)(c+a)}$, here $\frac{2 \cdot 210}{11 \cdot 13 \cdot 12} = \frac{35}{143}$. For an equilateral triangle it gives $\frac{1}{4}$, the medial triangle, which is the trap of assuming the boundary lines always pass through midpoints. A point test: the incentre (all three distances equal the inradius) is good, while every point of a side is bad (one distance is 0, so the other two would have to be equal and the triangle degenerates); the region touches the sides only at the single points D, E, F, where exactly that happens. The complement $\frac{108}{143}$ is the total area of the three corners.

24. Answer: (D) 38912**. The idea.** Two layers. First count the square subsets: a prime square p^2 never affects squareness, and a product pq of two different primes is an *edge* joining p and q in a graph on the primes; a product is a square exactly when every prime touches an even number of chosen edges, and such edge sets are counted by a spanning-tree argument. Second, a symmetric-difference trick: if some square subset T_0 contains an element e, then $T \mapsto T \triangle T_0$ (toggle every element of T_0) pairs the square subsets containing e with those not containing e. So each element lies in exactly half of all square subsets, or in none of them.

Work it out. *The count.* The semiprimes up to 60 are 4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, 34, 35, 38, 39, 46, 49, 51, (21 of them). The four squares 4, 9, 25, 49 are free, a factor 2^4. The other 17 are edges on the 10 primes up to 29: 2-3, 2-5, 2-7, 2-11, 2-13, 2-17, 2-19, 2-23, 2-29, 3-5, 3-7, 3-11, 3-13, 3-17, 3-19, 5-7, 5-11. Every prime is joined to 2; take those 9 edges as a spanning tree. The other 8 edges may be chosen freely, and then the tree edges are forced: an edge 2-p is chosen exactly when p already touches an odd number of chosen non-tree edges, and then the vertex 2 is automatically even, because the degrees in any edge set add up to an even number. So there are 2^8 even edge sets and $2^8 \cdot 2^4 = 2^{12} = 4096$ square subsets.

Which elements occur. The symmetric difference of two square subsets is square (its product is the product of the two products divided by the square of the product of the common elements), so the pairing above works. Each square p^2 occurs (as $\{p^2\}$). Each edge lying on a cycle occurs: $\{6, 10, 15\}$ handles 2-3, 2-5, 3-5, and for $p = 7, 11, 13, 17, 19$ the triangle $\{6, 2p, 3p\}$ handles 2-p and 3-p, while $\{10, 14, 35\}$ and $\{10, 22, 55\}$ handle 5-7 and 5-11. But 23 divides only 46 and 29 divides only 58 among the elements of S, so 46 and 58 are in no square subset (the prime would appear exactly once).

So $21 - 2 = 19$ elements each lie in $\frac{4096}{2} = 2048$ square subsets, and the total is $19 \cdot 2048 = 38912$.

Check. Small version with the semiprimes up to 15: 4, 6, 9, 10, 14, 15. The square subsets are the

8 sets made from any of $\{4\}$, $\{9\}$ together with either nothing or $\{6, 10, 15\}$; 14 (the only multiple of 7) never occurs. Their sizes total $2 \cdot 4 + 3 \cdot 4 = 20$, and the rule gives $5 \cdot \frac{8}{2} = 20$. Forgetting that 46 and 58 never occur gives $21 \cdot 2048 = 43008$; counting only the 17 two-prime elements, with 46 and 58 but without the squares, gives $17 \cdot 2048 = 34816$; forgetting the four squares gives $15 \cdot 2048 = 30720$; using 2^{11} square subsets gives 19456.

25. Answer: (C) 600**. The idea.** The map $t \mapsto \frac{3t-t^3}{1-3t^2}$ is the triple-angle formula: if $t = \tan\theta$ then it gives $\tan 3\theta$. So write $x = \tan\theta$; then $y = \tan 3\theta$, $z = \tan 9\theta$, and the last equation says $\tan 27\theta = \tan\theta$. This lists all solutions explicitly, and the sum of squares becomes a classical sum of $\tan^2$ over equally spaced angles, which we evaluate with the identity $\sum_{j=1}^{n-1} \csc^2 \frac{j\pi}{n} = \frac{n^2-1}{3}$.

Work it out. *Solutions.* Every real x is $\tan\theta$ for exactly one $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The first denominator is nonzero iff $\cos 3\theta \neq 0$, and then $y = \tan 3\theta$; similarly $z = \tan 9\theta$ and the third equation reads $\tan\theta = \tan 27\theta$, i.e. 26θ is a multiple of π: $\theta = \frac{k\pi}{26}$ with $-12 \le k \le 12$. A denominator vanishes only when $3^j k \equiv 13 \pmod{26}$ for some j, and since 3 is invertible modulo 26 with $3 \cdot 13 \equiv 13$, that happens only for $k \equiv 13$, which is not in our range. So $\mathcal{S}$ has exactly 25 triples $(\tan\theta, \tan 3\theta, \tan 9\theta)$, $\theta = \frac{k\pi}{26}$, k running over the residues mod 26 other than 13.

The sum. As k runs over the residues $\neq 13$, so do $3k$ and $9k$ (multiplication by 3 permutes them and fixes 13). Hence each of $\sum x^2, \sum y^2, \sum z^2$ equals $T = \sum_{k\neq 13} \tan^2 \frac{k\pi}{26}$, and the answer is $3T$. Now $\tan^2 = \sec^2 - 1$ and $\sec \frac{k\pi}{26} = \csc \frac{(13-k)\pi}{26}$; as k runs over the residues $\neq 13$, $j = 13 - k$ runs over the nonzero residues, and $\csc^2 \frac{j\pi}{26}$ depends only on j mod 26. So $T = \sum_{j=1}^{25} \csc^2 \frac{j\pi}{26} - 25 = \frac{26^2-1}{3} - 25 = 225 - 25 = 200$, and the requested sum is $3 \cdot 200 = 600$.

Check. The $\csc^2$ identity on a small case: $n = 4$ gives $\csc^2 45^\circ + \csc^2 90^\circ + \csc^2 135^\circ = 2+1+2 = 5 = \frac{16-1}{3}$. The analogous count with $\tan 3\theta = \tan\theta$ alone ($n = 2$ instead of 26) gives only $\theta = 0$, i.e. the single fixed point $x = 0$, and the formula $\frac{2^2-1}{3} - 1 = 0$ agrees. Forgetting to subtract the 25 ones gives 675; summing only x^2 gives 200.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.