MathCastle

AMC 12

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. An acute angle x (so $0° < x < 90°$) satisfies $\sin x + \cos x = \frac{7}{5}$. What is the value of $\tan x + \cot x$?

(A) $\frac{12}{25}$ **(B)** $\frac{24}{25}$ **(C)** $\frac{25}{24}$ **(D)** $\frac{7}{5}$ **(E)** $\frac{25}{12}$

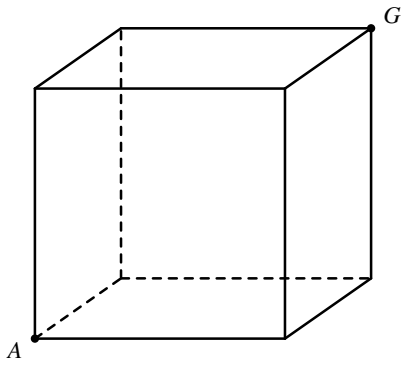
2. A positive real number x has the property that $x^{\log_2 x} = 16x^3$. There are exactly two such numbers x. What is their product?

(A) 8 **(B)** 16 **(C)** $\frac{33}{2}$ **(D)** 18 **(E)** 32

3. Three nonzero real numbers a, b, c, in that order, form a geometric sequence that is not constant. The three numbers a, $2b$, $3c$, in that order, form an arithmetic sequence. Also $a + b + c = 26$. What is c?

(A) 1 **(B)** 2 **(C)** 6 **(D)** $\frac{26}{3}$ **(E)** 18

4. An ant stands at a corner A of a cube and wants to reach the corner G that is farthest from A (the other end of the space diagonal through A). The ant walks only along the edges of the cube, from corner to corner, and it never visits any corner more than once; it stops as soon as it reaches G. How many different routes can the ant take?



(A) 18 **(B)** 24 **(C)** 27 **(D)** 30 **(E)** 36

5. The three side lengths of a right triangle form an arithmetic sequence, and the area of the triangle is 216. What is the length of the altitude drawn from the right angle to the hypotenuse?

(A) $\frac{72}{5}$ **(B)** 15 **(C)** 18 **(D)** 24 **(E)** $\frac{144}{5}$

6. Consider all complex numbers z that satisfy $z^2 = \overline{z}$, where $\overline{z}$ denotes the complex conjugate of z. What is the sum of $|z + 1|$ over all such z?

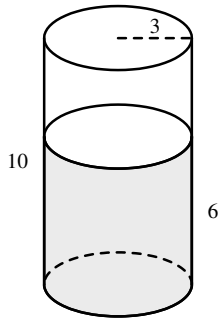
(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

7. A string of 4 letters is formed, each letter chosen from A, B, C, D, with repetition allowed (for example, ABBA and CCCC are such strings). How many of the 256 possible strings have the property that no letter appears in the string exactly once?

(A) 4 **(B)** 22 **(C)** 36 **(D)** 40 **(E)** 76

8. A cylindrical glass (a closed circular base, open at the top) has inner radius 3 and inner height 10. Standing upright, it holds water to a depth of 6. The glass is slowly tilted, without spilling, until the water

surface just reaches the lowest point of the rim, so that one more degree of tilt would spill water. At that moment the water still covers the entire base of the glass. Let θ be the angle between the axis of the glass and the vertical at that moment. What is $\tan\theta$?



(A) $\frac{4}{3}$ **(B)** $\frac{5}{3}$ **(C)** 2 **(D)** $\frac{8}{3}$ **(E)** $\frac{10}{3}$

9. A sequence begins with $a_1 = 1$ and continues by the rule $a_{n+1} = \dfrac{a_n}{1 + n\,a_n}$ for $n = 1, 2, 3, \ldots$. So $a_2 = \frac{1}{2}$, $a_3 = \frac{1/2}{1+2\cdot\frac{1}{2}} = \frac{1}{4}$, and so on. What is the smallest n for which $a_n < \dfrac{1}{1000}$?

(A) 46 **(B)** 47 **(C)** 48 **(D)** 50 **(E)** 63

10. How many positive integers n satisfy $\lfloor \log_2 n \rfloor = \lfloor \log_3 n \rfloor + 1$? (Here $\lfloor t \rfloor$ is the greatest integer less than or equal to t.)

(A) 12 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 22

11. What is the largest integer k such that 3^k divides $100! + 101!$? (Here $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

(A) 44 **(B)** 46 **(C)** 47 **(D)** 48 **(E)** 49

12. Three real numbers are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that the largest of the three minus the smallest of the three is less than $\frac{1}{3}$?

(A) $\frac{1}{27}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{5}{27}$ **(D)** $\frac{2}{9}$ **(E)** $\frac{7}{27}$

13. For each whole number of degrees k from 1 to 89, let $a_k = \dfrac{1}{1 + \tan^{2026}(k^\circ)}$, where $\tan^{2026}(k^\circ)$ means $(\tan k^\circ)^{2026}$. What is the value of the sum $a_1 + a_2 + a_3 + \cdots + a_{89}$?

(A) 44 **(B)** $\frac{89}{2}$ **(C)** 45 **(D)** $\frac{91}{2}$ **(E)** 89

14. How many real numbers x with $0 \le x < 2\pi$ satisfy the equation $\sin 5x = \sin x$? (Count each solution x once, even if it can be reached in more than one way.)

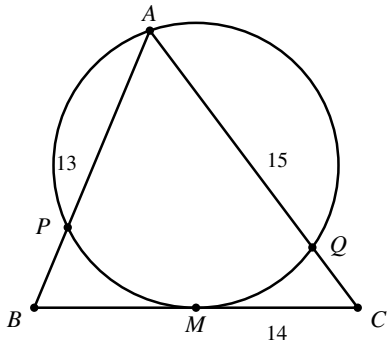
(A) 4 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 10

15. In how many ways can the eight numbers $1, 2, 3, 4, 5, 6, 7, 8$ be split into two groups so that the numbers in one group have the same sum as the numbers in the other? Every number goes into exactly one group,

each group has at least one number, the groups may have different sizes, and the two groups are not labelled (so swapping them does not give a new way).

(A) 4 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 14

16. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$, and M is the midpoint of side $\overline{BC}$. A circle passes through A and is tangent to line BC at M. This circle meets side $\overline{AB}$ again at a point $P \neq A$, and meets side $\overline{AC}$ again at a point $Q \neq A$. What is $BP + CQ$?



(A) $\frac{49}{13}$ **(B)** $\frac{98}{15}$ **(C)** $\frac{203}{30}$ **(D)** 7 **(E)** $\frac{1372}{195}$

17. A sequence is defined by $a_1 = \dfrac{6}{5}$ and $a_{n+1} = a_n^2 - a_n + 1$ for every $n \geq 1$. (So $a_2 = \frac{31}{25}$, and the terms grow without bound.) What is the value of the infinite sum $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \cdots$?

(A) 4 **(B)** $\frac{25}{6}$ **(C)** $\frac{9}{2}$ **(D)** $\frac{24}{5}$ **(E)** 5

18. For how many integers n with $1 \leq n \leq 2026$ is the tens digit of n^3 odd? (For example, $6^3 = 216$ has tens digit 1, which is odd, while $3^3 = 27$ has tens digit 2, which is even.)

(A) 405 **(B)** 505 **(C)** 900 **(D)** 910 **(E)** 1013

19. In triangle ABC, $BC = 14$ and $\tan \angle ABC = 2$. The centroid G of the triangle (where its three medians meet) and its orthocenter H (where its three altitudes meet) are two different points, and the line GH is parallel to $\overline{BC}$. What is the area of triangle ABC?

(A) 49 **(B)** 56 **(C)** 72 **(D)** 84 **(E)** 98

20. A drawer holds 12 socks: 6 pairs, each pair a different colour. In the dark, Dana pulls socks out one at a time, each time choosing uniformly at random from the socks still in the drawer, and she stops as soon as two of the socks she has pulled out have the same colour. What is the expected number of socks she pulls out?

(A) 4 **(B)** $\frac{1024}{231}$ **(C)** $\frac{9}{2}$ **(D)** $\frac{14}{3}$ **(E)** 5

21. Let $\omega = \cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13}$, and for $k = 1, 2, \ldots, 12$ let $\omega_k = \omega^k$, so that $\omega_1, \ldots, \omega_{12}$ are the 13th roots of unity other than 1. What is the value of the product $\displaystyle\prod_{k=1}^{12} \left(\omega_k^2 - 3\omega_k + 3\right)$?

(A) 531441 **(B)** 1592137 **(C)** 1594322 **(D)** 1594323 **(E)** 1596511

22. In triangle ABC, $\angle ABC = 70^\circ$ and $\angle ACB = 80^\circ$. Point P lies inside the triangle with $\angle PAB = 10^\circ$ and $\angle PBA = 30^\circ$. What is the degree measure of $\angle PCB$?

(A) 20 **(B)** 25 **(C)** 30 **(D)** 35 **(E)** 40

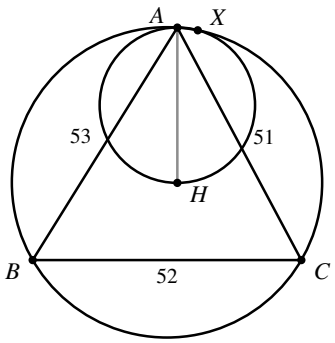
23. For how many positive integers n with $n \le 2026$ is $2^n + 1$ divisible by n? (For example, $n = 3$ works because $2^3 + 1 = 9$ is divisible by 3, and $n = 1$ also works.)

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

24. Compute the sum $\dfrac{1}{\sin^2\left(\frac{\pi}{2026}\right)} + \dfrac{1}{\sin^2\left(\frac{3\pi}{2026}\right)} + \dfrac{1}{\sin^2\left(\frac{5\pi}{2026}\right)} + \cdots + \dfrac{1}{\sin^2\left(\frac{2025\pi}{2026}\right)}$, which has one term $\dfrac{1}{\sin^2\left(\frac{k\pi}{2026}\right)}$ for each odd k from 1 to 2025.

(A) 342056 **(B)** 1026168 **(C)** 1026169 **(D)** 1368225 **(E)** 2052338

25. Triangle ABC has $AB = 53$, $BC = 52$ and $CA = 51$, and H is its orthocenter (the point where its three altitudes meet). The circle with diameter $\overline{AH}$ meets the circumcircle of triangle ABC at A and at one other point X. What is the length AX?



(A) $\frac{451}{226}$ **(B)** $\frac{451}{113}$ **(C)** 4 **(D)** $\frac{902}{113}$ **(E)** $\frac{451}{15}$

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Answer Key — AMC 12 Practice Contest 1

1. (E) $\frac{25}{12}$

2. (A) 8

3. (B) 2

4. (A) 18

5. (A) $\frac{72}{5}$

6. (D) 5

7. (D) 40

8. (A) $\frac{4}{3}$

9. (A) 46

10. (C) 17

11. (E) 49

12. (E) $\frac{7}{27}$

13. (B) $\frac{89}{2}$

14. (C) 8

15. (C) 7

16. (E) $\frac{1372}{195}$

17. (E) 5

18. (D) 910

19. (D) 84

20. (B) $\frac{1024}{231}$

21. (B) 1592137

22. (C) 30

23. (D) 10

24. (C) 1026169

25. (B) $\frac{451}{113}$

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Solutions — AMC 12 Practice Contest 1

1. Answer: (E) $\frac{25}{12}$**. The idea.** We never need x itself. Written in sines and cosines, $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x}$, and squaring the given equation produces exactly $\sin x \cos x$.

Work it out. Squaring, $\sin^2 x + 2\sin x \cos x + \cos^2 x = \frac{49}{25}$, so $1 + 2\sin x \cos x = \frac{49}{25}$ and $\sin x \cos x = \frac{12}{25}$. Therefore $\tan x + \cot x = \frac{25}{12}$.

Check. The numbers $\sin x$ and $\cos x$ have sum $\frac{7}{5}$ and product $\frac{12}{25}$, so they are $\frac{3}{5}$ and $\frac{4}{5}$ in some order. If $\tan x = \frac{3}{4}$ then $\tan x + \cot x = \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$, and the other order gives the same sum. The traps are stopping at $\sin x \cos x = \frac{12}{25}$ or at $\sin 2x = \frac{24}{25}$.

2. Answer: (A) 8**. The idea.** An unknown in the exponent and in the base at once is tamed by taking a logarithm: with $t = \log_2 x$ the equation becomes an ordinary quadratic in t, and the product of the x's corresponds to the *sum* of the t's.

Work it out. Both sides are positive, so take $\log_2$ of each: $\log_2 x \cdot \log_2 x = \log_2 16 + 3\log_2 x$, that is, $t^2 = 4 + 3t$, or $t^2 - 3t - 4 = 0$. This factors as $(t-4)(t+1) = 0$, so $t = 4$ or $t = -1$, giving $x = 16$ or $x = \frac{1}{2}$. Their product is $2^{4+(-1)} = 2^3 = 8$.

Check. For $x = 16$: $16^4 = 65536$ and $16 \cdot 16^3 = 16^4$. For $x = \frac{1}{2}$: $(\frac{1}{2})^{-1} = 2$ and $16 \cdot \frac{1}{8} = 2$. Both work. By Vieta the sum of the two t's is 3 without solving, so the product of the x's is $2^3 = 8$. (Keeping only the root $x = 16$ gives 16; adding the roots instead of multiplying gives $\frac{33}{2}$; turning $t = -1$ into $x = 2$ instead of $\frac{1}{2}$ gives a product of 32, or a sum of 18.)

3. Answer: (B) 2**. The idea.** Write everything in terms of a and the common ratio r: then $b = ar$, $c = ar^2$, and the arithmetic condition (middle term is the average of its neighbours) becomes an equation for r alone, because $a \neq 0$ divides out.

Work it out. The arithmetic condition says $2 \cdot 2b = a + 3c$, so $4ar = a + 3ar^2$. Dividing by a: $3r^2 - 4r + 1 = 0$, which factors as $(3r-1)(r-1) = 0$. The value $r = 1$ would make the geometric sequence constant, which is not allowed, so $r = \frac{1}{3}$. Then $a + b + c = a\left(1 + \frac{1}{3} + \frac{1}{9}\right) = \frac{13a}{9} = 26$, so $a = 18$, $b = 6$ and $c = 2$.

Check. $18, 6, 2$ is geometric with ratio $\frac{1}{3}$, and $a, 2b, 3c = 18, 12, 6$ is arithmetic with difference -6; the sum is 26. The forbidden root $r = 1$ would give $a = b = c = \frac{26}{3}$, and it is a tempting wrong answer; reporting $a = 18$ or $b = 6$ instead of c are the other slips.

4. Answer: (A) 18**. The idea.** The first two steps are forced up to symmetry, and after them the rest of the route has very few options. So count the routes for one choice of the first two steps and multiply by the number of such choices.

Work it out. Label corners by coordinates, $A = 000$ and $G = 111$; an edge changes exactly one coordinate. From A there are 3 choices of first step, say to 100. From 100 the ant cannot return to A, so it goes to 110 or 101: 2 choices, say 110. That makes $3 \cdot 2 = 6$ symmetric starts, and each has the same number of completions. From 110 the neighbours are 100 (visited), $111 = G$ and 010. Going to G finishes a route of length 3. Going to 010: its neighbours 000 and 110 are visited, so the ant must go to 011. From 011 it can go to G (a route of length 5) or to 001; from 001 the only unvisited neighbour is 101, and from 101 the only unvisited neighbour is G (a route of length 7 through all eight corners). So each start has 3 completions, and the total is $6 \cdot 3 = 18$.

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Check. Sorting by length: 6 shortest routes (the 3! orders in which to change the coordinates), 6 of length 5 and 6 of length 7, total 18. A route can never have even length, since each step changes the parity of the coordinate sum and A, G have opposite parities, so no lengths were missed. Guessing 3 choices at each of three stages gives $3^3 = 27$, and counting each detour of length 5 or 7 twice (once in each direction around the face) gives $6 + 12 + 12 = 30$.

5. Answer: (A) $\frac{72}{5}$**. The idea.** An arithmetic sequence has only two unknowns, a middle term and a common difference, and the Pythagorean theorem fixes their ratio. So every right triangle with sides in arithmetic progression has the same shape; the area then fixes the size, and the altitude to the hypotenuse is twice the area divided by the hypotenuse.

Work it out. Call the sides $a - d$, a, $a + d$ with $d > 0$; the hypotenuse is the longest, $a + d$. Then $(a - d)^2 + a^2 = (a + d)^2$ simplifies to $a^2 = 4ad$, so $a = 4d$ and the sides are $3d$, $4d$, $5d$. The area is $\frac{1}{2} \cdot 3d \cdot 4d = 6d^2 = 216$, so $d^2 = 36$ and $d = 6$: the sides are 18, 24, 30. Computing the area with the hypotenuse as base, $\frac{1}{2} \cdot 30 \cdot h = 216$, so $h = \frac{432}{30} = \frac{72}{5}$.

Check. $18^2 + 24^2 = 324 + 576 = 900 = 30^2$, and $18, 24, 30$ has common difference 6. Also $h = \frac{18 \cdot 24}{30} = \frac{72}{5} = 14.4$, which is shorter than both legs, as an altitude inside a right triangle must be. Confusing the altitude with the median to the hypotenuse gives half of 30, which is 15; reporting a leg gives 18 or 24; doubling the area twice, $\frac{4 \cdot 216}{30}$, gives $\frac{144}{5}$.

6. Answer: (D) 5**. The idea.** Take absolute values first. Since $|\overline{z}| = |z|$, the equation forces $|z|^2 = |z|$, so $|z| = 0$ or $|z| = 1$. On the unit circle the conjugate is the reciprocal, $\overline{z} = \frac{1}{z}$, and the equation turns into $z^3 = 1$.

Work it out. If $|z| = 0$ then $z = 0$, which works. If $|z| = 1$ then $z\overline{z} = 1$, so $z^2 = \frac{1}{z}$ and $z^3 = 1$. So z is 1, ω or ω^2, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Each of these has modulus 1, so all of them are solutions. The distances are: $|0 + 1| = 1$ and $|1 + 1| = 2$. Also $1 + \omega + \omega^2 = 0$ gives $\omega + 1 = -\omega^2$, so $|\omega + 1| = 1$, and in the same way $|\omega^2 + 1| = |-\omega| = 1$. The sum is $1 + 2 + 1 + 1 = 5$.

Check. Directly, $\omega + 1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ has modulus $\sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. As a check on ω itself, $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \overline{\omega}$. Writing $z = x + yi$ gives $x^2 - y^2 = x$ and $2xy = -y$. Either $y = 0$, so $x \in \{0, 1\}$, or $x = -\frac{1}{2}$, so $y^2 = \frac{3}{4}$: four solutions, as found. Forgetting $z = 0$ gives 4. Forgetting $z = 1$ gives 3.

7. Answer: (D) 40**. The idea.** Do not count letter by letter; sort the strings by their pattern of repetition. The multiset of letter counts is a partition of 4, and the condition says no part equals 1. That leaves only two patterns, each easy to count.

Work it out. The partitions of 4 are 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1; only 4 and 2 + 2 have no part equal to 1. Pattern 4 (one letter four times): 4 strings. Pattern 2 + 2: choose the two letters that appear, $\binom{4}{2} = 6$ ways, then arrange two copies of each, $\frac{4!}{2!\,2!} = 6$ ways, for 36 strings. The total is $4 + 36 = 40$.

Check. Count all five patterns: 4 (pattern 4) + $4 \cdot 3 \cdot 4 = 48$ (3 + 1) + 36 (2 + 2) + $4 \cdot 3 \cdot 12 = 144$ (2 + 1 + 1: the doubled letter, the pair of singles, and $\frac{4!}{2!} = 12$ orders) + 24 (1 + 1 + 1 + 1) = $256 = 4^4$. Choosing the two letters in order double counts to get 76, halving the arrangements gives 22, and forgetting the one-letter strings gives 36.

8. Answer: (A) $\frac{4}{3}$**. The idea.** Measure water heights along the wall of the glass. When the water surface is a tilted plane that stays above the base, the water is a cylinder cut off by a slanted plane,

and its volume is (base area) × (the average of the heights at two diametrically opposite points). The two opposite heights are symmetric about the centre height, so the height at the centre stays 6 no matter how the glass is tilted.

Work it out. Let the water reach height 10 on the low side of the rim (the spilling point) and height h on the diametrically opposite side of the wall. By the volume rule, $\pi \cdot 3^2 \cdot \frac{10+h}{2} = \pi \cdot 3^2 \cdot 6$, so $h = 2$. (Since $h > 0$, the water covers the base, as the problem says.) Across a diameter of length 6, the water surface rises $10 - 2 = 8$ relative to the base. The angle between the water surface and the base equals the angle between the vertical and the axis, since the surface is horizontal and the base is perpendicular to the axis. Hence $\tan\theta = \frac{8}{6} = \frac{4}{3}$.

Why the volume rule holds. In coordinates on the base, the surface height over the point (x, y) is linear, $6 + kx$. Integrating over the disc, the kx part cancels (the disc is symmetric about $x = 0$), leaving $6 \cdot \pi \cdot 9$.

Check. Comparing only one side, $\frac{10-6}{3} = \frac{4}{3}$ as well: the centre height 6 and the rim height 10 differ by 4 over a radius 3. Dividing the full rise 8 by the radius instead of the diameter gives $\frac{8}{3}$, the classic slip; dividing the depth or the rim height by the radius gives $\frac{6}{3} = 2$ or $\frac{10}{3}$; forgetting that the water stays at height 2 on the far side (taking 0) gives $\frac{10}{6} = \frac{5}{3}$.

9. Answer: (A) 46**. The idea.** The rule is awkward for a_n but becomes simple for the reciprocal: flipping both sides turns a fraction-of-a-fraction into a plain addition, and the reciprocals are partial sums of $1, 2, 3, \ldots$.

Work it out. Let $b_n = \frac{1}{a_n}$. Then $b_{n+1} = \frac{1+na_n}{a_n} = \frac{1}{a_n} + n = b_n + n$. Starting from $b_1 = 1$: $b_n = 1 + (1 + 2 + \cdots + (n-1)) = 1 + \frac{n(n-1)}{2}$. We need $b_n > 1000$, that is, $\frac{n(n-1)}{2} > 999$, or $n(n-1) > 1998$. Now $45 \cdot 44 = 1980$ is too small, and $46 \cdot 45 = 2070$ works. So $a_{45} = \frac{1}{991}$, still above $\frac{1}{1000}$, and $a_{46} = \frac{1}{1036}$ is the first term below it. The answer is 46.

Check. The formula gives $b_2 = 2$, $b_3 = 4$, matching $a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{4}$ from the statement, and $b_4 = 7$: indeed $a_4 = \frac{1/4}{1+3/4} = \frac{1}{7}$. The estimate $\frac{n^2}{2} \approx 1000$ gives $n \approx 44.7$, so the answer is near 45; the exact count shows $a_{45} = \frac{1}{991}$ just misses, and a_{46} is the first term below. Shifting the index by one when writing b_n gives 47.

10. Answer: (C) 17**. The idea.** Fix the common integer part. If $\lfloor \log_3 n \rfloor = k$, then n lies in the block $[3^k, 3^{k+1})$. The condition also says $\lfloor \log_2 n \rfloor = k + 1$, so n lies in $[2^{k+1}, 2^{k+2})$ as well. Count the overlap of the two blocks for each k. Powers of 3 outrun powers of 2, so only a few values of k can contribute.

Work it out. $k = 0$: $[1, 3) \cap [2, 4)$ holds only $n = 2$, which is 1 value. $k = 1$: $[3, 9) \cap [4, 8)$ holds $4, \ldots, 7$, which is 4 values. $k = 2$: $[9, 27) \cap [8, 16)$ holds $9, \ldots, 15$, which is 7 values. $k = 3$: $[27, 81) \cap [16, 32)$ holds $27, \ldots, 31$, which is 5 values. $k = 4$: $[81, 243) \cap [32, 64)$ is empty. For every larger k the blocks stay apart, because $3^k \geq 2^{k+2}$ once $k \geq 4$: the left side triples at each step while the right side only doubles. The total is $1 + 4 + 7 + 5 = 17$.

Check. Spot checks. $n = 31$: $\lfloor \log_2 31 \rfloor = 4$ and $\lfloor \log_3 31 \rfloor = 3$. $n = 32$: the values are 5 and 3, which fails. $n = 8$: the values are 3 and 1, which fails, and that is why the $k = 2$ block starts at 9, not at 8. $n = 3$: the values are 1 and 1, which fails. Stopping after $k = 2$ gives 12. Missing $n = 2$ gives 16. Letting the $k = 3$ block run to 32 gives 18.

11. Answer: (E) 49**. The idea.** A sum is hard to factor, so pull out the common factor first: $101! =$

$101 \cdot 100!$, which turns the sum into a product. Then count the factors of 3 in each piece; for 100! use Legendre's formula, which counts multiples of 3, then of 9, of 27, of 81, each contributing one more 3.

Work it out. $100! + 101! = 100!\,(1 + 101) = 100! \cdot 102$. In 100! the number of factors of 3 is $\lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{9} \rfloor + \lfloor \frac{100}{27} \rfloor + \lfloor \frac{100}{81} \rfloor = 33 + 11 + 3 + 1 = 48$. And $102 = 2 \cdot 3 \cdot 17$ contributes exactly one more. So $k = 48 + 1 = 49$.

Check. The second factor must be exactly $1 + 101 = 102$, not 101: the two terms share all of 100!. The slips: forgetting the factor 102 gives 48; also dropping the $\lfloor \frac{100}{81} \rfloor$ term gives 47; counting only the multiples of 3 and 9 gives $33 + 11 = 44$; skipping the multiples of 27 gives $33 + 11 + 1 + 1 = 46$.

12. Answer: (E) $\frac{7}{27}$. The idea. Condition on which of the three numbers is the smallest. By symmetry each is the smallest with the same chance, so the answer is 3 times the probability that one particular number, a, is the smallest while the other two both land within $\frac{1}{3}$ above it. For a given a that is a simple product of two lengths. Only near the right end of the interval does the window get cut short.

Work it out. Write $r = \frac{1}{3}$. Fix a. The other two numbers must both fall in $[a, a + r]$ intersected with $[0, 1]$, a window of length $\min(r, 1 - a)$. So the probability for this a is $\min(r, 1 - a)^2$. Integrate over a: $\int_0^{1-r} r^2\, da + \int_{1-r}^1 (1 - a)^2\, da = r^2(1 - r) + \frac{r^3}{3}$. Multiplying by 3 gives $3r^2(1 - r) + r^3 = 3r^2 - 2r^3$. With $r = \frac{1}{3}$ this is $\frac{3}{9} - \frac{2}{27} = \frac{7}{27}$.

Check. The formula behaves correctly at the ends: $r = 0$ gives 0 and $r = 1$ gives $3 - 2 = 1$. The three events "a is the smallest", one for each number, overlap only with probability 0, so multiplying by 3 does not double count. The traps: squaring alone gives $r^2 = \frac{1}{9}$. Asking all three to lie in the fixed interval $[0, \frac{1}{3}]$ gives $\frac{1}{27}$. Dropping the corner integral near 1 gives $3r^2(1-r) = \frac{2}{9}$, and subtracting it instead of adding gives $\frac{2}{9} - \frac{1}{27} = \frac{5}{27}$.

13. Answer: (B) $\frac{89}{2}$. The idea. Nobody can compute $\tan^{2026}(1°)$, so the terms must cancel in pairs. The angles $k°$ and $(90 - k)°$ are complementary, and $\tan(90° - x) = \dfrac{1}{\tan x}$, which turns one term of a pair into the "other half" of its partner.

Work it out. Fix k and write $t = \tan^{2026}(k°) > 0$. Then $\tan^{2026}((90 - k)°) = \dfrac{1}{t}$, so $a_k + a_{90-k} = \dfrac{1}{1+t} + \dfrac{1}{1+\frac{1}{t}} = \dfrac{1}{1+t} + \dfrac{t}{t+1} = 1$. The terms $k = 1, 2, \ldots, 44$ pair with $k = 89, 88, \ldots, 46$, giving 44 pairs that each add to 1. The middle term is $k = 45$, where $\tan 45° = 1$, so $a_{45} = \frac{1}{1+1} = \frac{1}{2}$. The sum is $44 + \frac{1}{2} = \frac{89}{2}$.

Check. The exponent 2026 played no role; any exponent gives the same total. With exponent 0 every term is $\frac{1}{1+1} = \frac{1}{2}$, and 89 halves make $\frac{89}{2}$, as found.

14. Answer: (C) 8. The idea. Move everything to one side and use sum-to-product: a difference of two sines factors into a product, and a product is zero exactly when one factor is. The only trap is counting a solution twice when both factors vanish.

Work it out. $\sin 5x - \sin x = 2\cos\left(\frac{5x+x}{2}\right)\sin\left(\frac{5x-x}{2}\right) = 2\cos 3x\, \sin 2x$. First factor: $\sin 2x = 0$ means $2x$ is a multiple of π, so $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$: four values. Second factor: $\cos 3x = 0$ means

$3x = \frac{\pi}{2} + m\pi$, so $x = \frac{\pi}{6} + \frac{m\pi}{3}$. In $[0, 2\pi)$ this gives $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{3\pi}{2}$, $\frac{11\pi}{6}$: six values. The values $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ appear in both lists, so the number of different solutions is $4 + 6 - 2 = 8$.

Check. Test the shared value $x = \frac{\pi}{2}$: $\sin\frac{5\pi}{2} = 1 = \sin\frac{\pi}{2}$, a genuine solution counted once. Test $x = \frac{\pi}{6}$: $\sin\frac{5\pi}{6} = \frac{1}{2} = \sin\frac{\pi}{6}$. Forgetting the overlap gives the tempting wrong answer 10.

15. Answer: (C) 7**. The idea.** The total is $1 + 2 + \cdots + 8 = 36$, so each group must sum to 18. Because the groups are unlabelled, name each split by the group that contains 8: that removes the double counting at the start instead of halving at the end.

Work it out. The group containing 8 needs its other numbers to sum to $18 - 8 = 10$, using numbers from 1 to 7. List those sets by size. Two numbers: $\{3, 7\}$, $\{4, 6\}$ (not $\{5, 5\}$). Three numbers: $\{1, 2, 7\}$, $\{1, 3, 6\}$, $\{1, 4, 5\}$, $\{2, 3, 5\}$. Four numbers: $\{1, 2, 3, 4\}$, the only one, since four different numbers sum to at least 10. One number would have to be 10, too big, and five numbers sum to at least 15. That is $2+4+1 = 7$ splits, for example $\{8, 3, 7\}$ against $\{1, 2, 4, 5, 6\}$.

Check. Count all subsets of $\{1, \ldots, 8\}$ with sum 18 instead: 3 of size three ($\{3, 7, 8\}$, $\{4, 6, 8\}$, $\{5, 6, 7\}$), 8 of size four, and 3 of size five (the complements of the size-three ones), 14 in all. Each split is counted twice, once from each group, giving $14/2 = 7$. Forgetting to halve gives the trap 14; allowing only groups of four gives $8/2 = 4$.

16. Answer: (E) $\frac{1372}{195}$**. The idea.** From B we see the circle two ways: along the tangent BM, and along the secant BPA. The power of the point B says these give the same product, and the same holds at C. No coordinates are needed.

Work it out. M is the midpoint of BC, so $BM = CM = 7$. The power of B with respect to the circle is BM^2 (tangent) and also $BP \cdot BA$ (secant through P and A). So $BP \cdot 13 = 49$ and $BP = \frac{49}{13}$. In the same way $CQ \cdot CA = CM^2$, so $CQ = \frac{49}{15}$. Both are less than the side lengths, so P and Q really lie on the sides. Then $BP + CQ = \frac{49}{13} + \frac{49}{15} = \frac{49 \cdot 28}{195} = \frac{1372}{195}$.

Check. Put $B = (0, 0)$, $C = (14, 0)$, $A = (5, 12)$. The center is $(7, k)$ with $2^2 + (12 - k)^2 = k^2$, so $k = \frac{37}{6}$. The point of $\overline{AB}$ at distance $\frac{49}{13}$ from B is $\frac{49}{169}(5, 12)$; its squared distance from the center is $(7 - \frac{245}{169})^2 + (\frac{37}{6} - \frac{588}{169})^2 = (\frac{37}{6})^2$ after simplifying, so it is on the circle.

17. Answer: (E) 5**. The idea.** The rule can be rewritten as $a_{n+1} - 1 = a_n(a_n - 1)$. A product like $a_n(a_n - 1)$ splits into partial fractions, and that turns each $\frac{1}{a_n}$ into a difference of two consecutive terms of a new sequence, which telescopes.

Work it out. Let $d_n = a_n - 1$, so $d_{n+1} = a_n d_n$. Then $\frac{1}{d_n} - \frac{1}{d_{n+1}} = \frac{1}{d_n} - \frac{1}{a_n d_n} = \frac{a_n - 1}{a_n d_n} = \frac{1}{a_n}$. So $\frac{1}{a_1} + \cdots + \frac{1}{a_N} = \frac{1}{d_1} - \frac{1}{d_{N+1}}$. Since $d_1 = \frac{1}{5} > 0$ and $d_{n+1} = d_n(1 + d_n)$, the d_n increase, each step adds at least d_1^2, and so $d_n \to \infty$ and $\frac{1}{d_{N+1}} \to 0$. The sum is $\frac{1}{d_1} = 5$.

Check. Partial sums: $\frac{1}{a_1} = \frac{5}{6} \approx 0.833$; the next terms are $\frac{25}{31} \approx 0.806$, then about 0.77, 0.72, 0.65, 0.55, 0.40, 0.21, 0.05, then tiny. Adding gives about 4.99 and climbing toward 5. Starting the telescoping one term late gives $\frac{1}{d_2} = \frac{25}{6}$, and because the partial sums creep up so slowly, stopping early suggests 4 or $\frac{9}{2}$.

18. Answer: (D) 910**. The idea.** The tens digit of n^3 depends only on n^3 mod 100, and so only on the last two digits of n. Write those as $10a + b$ with digits a (tens) and b (units) and expand: the cube splits into a part that depends on b alone and a part that brings in a, and only its parity matters.

Work it out. $(10a+b)^3 = 1000a^3 + 300a^2b + 30ab^2 + b^3 \equiv 30ab^2 + b^3 \pmod{100}$. So the tens digit of n^3 has the same parity as $3ab^2 + \left\lfloor \frac{b^3}{10} \right\rfloor$, which has the parity of $ab^2 + \left\lfloor \frac{b^3}{10} \right\rfloor$ (the term $30ab^2$ adds $3ab^2$ tens, and the carries from b^3 add $\lfloor b^3/10 \rfloor$ tens). If b is even, ab^2 is even, so only b^3 matters: $0, 8, 64, 216, 512$ have tens-and-up parts $0, 0, 6, 21, 51$, so the tens digit is odd exactly for $b = 6$ and $b = 8$, whatever a is. If b is odd, ab^2 has the parity of a, and $1, 27, 125, 343, 729$ have tens-and-up parts $0, 2, 12, 34, 72$, all even. So the tens digit is odd exactly when a is odd. In each block of 100 consecutive integers that makes $2 \cdot 10 = 20$ values with $b \in \{6, 8\}$ plus $5 \cdot 5 = 25$ with b and a both odd: 45. The numbers 1 to 2000 form 20 such blocks, giving 900. For 2001 to 2026 (last two digits 01 to 26): $b \in \{6, 8\}$ gives 06, 08, 16, 18, 26, five numbers; a and b both odd gives 11, 13, 15, 17, 19, five more. Total $900 + 10 = 910$.

Check. $16^3 = 4096$ (tens digit 9, odd: $b = 6$), $13^3 = 2197$ (tens 9, odd: $a = 1$, $b = 3$), $23^3 = 12167$ (tens 6, even: $a = 2$ is even). Counting only $b \in \{6, 8\}$ gives 405; counting only the odd-b case gives 505; dropping the last 26 numbers gives 900.

19. Answer: (D) 84**. The idea.** "Parallel to BC" just says that G and H are at the same height above line BC. The centroid is at one third of the height of A. The orthocenter's height comes from a pair of similar right triangles, and the condition turns into a clean relation between the base angles: $\tan B \tan C = 3$.

Work it out. Let D be the foot of the altitude from A and $h = AD$. The medians meet at a third of the way up each one, so G is at height $\frac{h}{3}$. H lies on AD. The altitude from B is perpendicular to AC, so it makes the angle $90° - C$ with BC, and in right triangle BDH we get $HD = BD \tan(90° - C) = \frac{BD}{\tan C} = \frac{BD \cdot DC}{h}$, using $\tan C = \frac{h}{DC}$. Equal heights: $\frac{BD \cdot DC}{h} = \frac{h}{3}$, so $h^2 = 3\, BD \cdot DC$, which says $\tan B \tan C = \frac{h}{BD} \cdot \frac{h}{DC} = 3$. With $\tan B = 2$ we get $\tan C = \frac{3}{2}$. Both base angles are acute, so D lies between B and C, with $BD = \frac{h}{2}$ and $DC = \frac{h}{3/2} = \frac{2h}{3}$. Then $\frac{h}{2} + \frac{2h}{3} = \frac{7h}{6} = 14$, so $h = 12$, and the area is $\frac{1}{2} \cdot 14 \cdot 12 = 84$.

Check. Put $B = (0, 0)$, $C = (14, 0)$, $A = (6, 12)$, so $\tan B = \frac{12}{6} = 2$. The altitude from A is $x = 6$; the altitude from B is perpendicular to $\overrightarrow{AC} = (8, -12)$, so it is the line $8x - 12y = 0$, which meets $x = 6$ at $H = (6, 4)$. The centroid is $G = \left(\frac{0+14+6}{3}, \frac{12}{3}\right) = \left(\frac{20}{3}, 4\right)$: the same height 4, and a different point. Flipping $\tan C$ to $\frac{h}{DC} = \frac{2}{3}$ by mistake gives $DC = \frac{3h}{2}$, $h = 7$ and the wrong answer 49.

20. Answer: (B) $\frac{1024}{231}$**. The idea.** For a count T that takes values $0, 1, 2, \ldots$, the expected value is $E[T] = \sum_{k \geq 0} P(T > k)$. Here "$T > k$" just says the first k socks all have different colours, which is easy to compute.

Work it out. $P(T > k)$ is the chance that the first k socks are all different colours. The first sock can be anything (12 of 12), the second must avoid one colour (10 of 11), the third must avoid two (8 of 10), and so on: $P(T > 0) = P(T > 1) = 1$, $P(T > 2) = \frac{10}{11}$, $P(T > 3) = \frac{10 \cdot 8}{11 \cdot 10} = \frac{8}{11}$, $P(T > 4) = \frac{8}{11} \cdot \frac{6}{9} = \frac{16}{33}$, $P(T > 5) = \frac{16}{33} \cdot \frac{4}{8} = \frac{8}{33}$, $P(T > 6) = \frac{8}{33} \cdot \frac{2}{7} = \frac{16}{231}$, and $P(T > 7) = 0$ since there are only 6 colours. Adding: $2 + \frac{18}{11} + \frac{24}{33} + \frac{16}{231} = 2 + \frac{378+168+16}{231} = 2 + \frac{562}{231} = \frac{1024}{231} \approx 4.43$.

Check. The same method with n pairs gives $\frac{4^n}{\binom{2n}{n}}$: for one pair $\frac{4}{2} = 2$ (obviously right: she must pull both socks), for two pairs $\frac{16}{6} = \frac{8}{3}$ (she stops at the 2nd sock with probability $\frac{1}{3}$, else at the 3rd: $\frac{2}{3} + 2 = \frac{8}{3}$). For six pairs $\frac{4096}{924} = \frac{1024}{231}$.

21. Answer: (B) 1592137**. The idea.** Factor the quadratic over its complex roots, $x^2 - 3x + 3 =$

$(x-\alpha)(x-\beta)$. A product of $(\alpha-\omega)$ over all 13th roots of unity ω is just $\alpha^{13}-1$, because $x^{13}-1=\prod(x-\omega)$. Then deal with the missing root $\omega=1$.

Work it out. The roots are $\alpha,\beta=\frac{3\pm i\sqrt{3}}{2}$, with $\alpha=\sqrt{3}\,(\cos 30^\circ+i\sin 30^\circ)$ and $\beta=\overline{\alpha}$. Over all 13 roots of unity (including 1), $\prod_\omega(\omega^2-3\omega+3)=\prod_\omega(\omega-\alpha)(\omega-\beta)=\prod_\omega(\alpha-\omega)\prod_\omega(\beta-\omega)=(\alpha^{13}-1)(\beta^{13}-1)$, where the two signs $(-1)^{13}$ cancel. The factor for $\omega=1$ is $1-3+3=1$, so the requested product is also $(\alpha^{13}-1)(\beta^{13}-1)=|\alpha^{13}-1|^2$. By de Moivre, $\alpha^{13}=3^{13/2}(\cos 390^\circ+i\sin 390^\circ)=729\sqrt{3}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=\frac{2187}{2}+\frac{729\sqrt{3}}{2}i$. So $|\alpha^{13}-1|^2=\left(\frac{2185}{2}\right)^2+\frac{3\cdot 729^2}{4}=\frac{4774225+1594323}{4}=1592137$.

Check. In general $|\alpha^{13}-1|^2=|\alpha|^{26}-2\,\mathrm{Re}(\alpha^{13})+1=3^{13}-2187+1=3^{13}-3^7+1$, which is $1594323-2187+1=1592137$. The answer is a positive integer, as it must be: the product is symmetric in the roots of unity, and conjugate factors pair up.

22. Answer: (C) 30**. The idea.** Three cevians AP, BP, CP through one point satisfy the trigonometric form of Ceva's theorem: $\sin\angle PAB\cdot\sin\angle PBC\cdot\sin\angle PCA=\sin\angle PAC\cdot\sin\angle PBA\cdot\sin\angle PCB$. Five of the six angles are known in terms of one unknown, and a product-to-sum identity settles it.

Work it out. $\angle BAC=180^\circ-70^\circ-80^\circ=30^\circ$, so $\angle PAC=20^\circ$ and $\angle PBC=70^\circ-30^\circ=40^\circ$. Let $x=\angle PCB$; then $\angle PCA=80^\circ-x$. Ceva gives $\sin 10^\circ\sin 40^\circ\sin(80^\circ-x)=\sin 20^\circ\sin 30^\circ\sin x=\frac{1}{2}\sin 20^\circ\sin x$. Try $x=30^\circ$: the left side is $\sin 10^\circ\sin 40^\circ\cos 40^\circ=\sin 10^\circ\cdot\frac{1}{2}\sin 80^\circ=\frac{1}{2}\sin 10^\circ\cos 10^\circ=\frac{1}{4}\sin 20^\circ$, and the right side is $\frac{1}{2}\sin 20^\circ\cdot\frac{1}{2}=\frac{1}{4}\sin 20^\circ$. They agree. It is the only solution: as x increases from 0° to 80°, $\sin(80^\circ-x)$ decreases and $\sin x$ increases, so $\frac{\sin(80^\circ-x)}{\sin x}$ is strictly decreasing and takes each value once. So $\angle PCB=30^\circ$.

Check. Numerically, $\sin 10^\circ\sin 40^\circ\sin 50^\circ\approx 0.17365\cdot 0.64279\cdot 0.76604\approx 0.08551$ and $\frac{1}{4}\sin 20^\circ\approx 0.08551$. Also then $\angle PCA=50^\circ$ and $\angle APC=180^\circ-20^\circ-50^\circ=110^\circ$, and the three angles around P are 140°, 110° and 110°, which add to 360° as they must.

23. Answer: (D) 10**. The idea.** If a prime q divides n, then $2^n\equiv -1\pmod q$, so $2^{2n}\equiv 1$ but $2^n\not\equiv 1$. The order of 2 modulo q (the least e with $2^e\equiv 1$) therefore divides $2n$ but not n, and it also divides $q-1$. Playing these conditions off against each other pins down the prime factors of n.

Work it out. 2^n+1 is odd, so n is odd. Let e_q be the order of 2 mod a prime $q\mid n$; since n is odd, $e_q=2d$ with d odd and $d\mid n$, and $e_q\mid q-1$, so every prime factor of d is less than q. Smallest prime p of n: d has no prime factor of n below p, so $d=1$, $e_p=2$, $p\mid 2^2-1=3$. So $p=3$ (unless $n=1$). Powers of 3: for odd n, lifting the exponent gives $v_3(2^n+1)=v_3(2+1)+v_3(n)=1+v_3(n)$, so $3^a\mid 2^n+1$ whenever $3^a\mid n$. Hence $n=1,3,9,27,81,243,729$ all work: 7 numbers. A second prime q: its d can only involve the prime 3, so $d=3^b$ with $3^b\mid n$ and $q\mid 2^{3^b}+1$; also $q\equiv 1\pmod{2\cdot 3^b}$. For $b\le 1$, $2^{3^b}+1\in\{3,9\}$ gives nothing new. For $b=2$: $2^9+1=513=3^3\cdot 19$, so $q=19$. For $b=3$: $q\equiv 1\pmod{54}$ and $n\ge 27q$ forces $q\le 75$, but 55 is not prime. For $b\ge 4$: $n\ge 81\cdot 163>2026$. So the only other prime is 19. Now $2^{18}\equiv 1\pmod{19}$ with order exactly 18, so $19\mid 2^n+1$ exactly when $n\equiv 9\pmod{18}$, i.e. n odd and $9\mid n$. The candidates $3^a\cdot 19$ with $a\ge 2$ are $171, 513, 1539$ (next 4617 is too big); $19^2\cdot 9$ and $171\cdot 23$ already exceed 2026, so no third prime fits. Total: $7+3=10$.

Check. $n=171$: $2^{171}+1$ is divisible by 9 (since $v_3=1+2=3$) and by 19 (since $171=9\cdot 19\equiv 9\pmod{18}$). By contrast $n=57=3\cdot 19$ fails: $57\equiv 3\pmod{18}$, so $2^{57}\equiv 2^3=8\not\equiv -1\pmod{19}$.

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24. Answer: (C) 1026169**. The idea.** Two facts. (1) The doubling identity $\csc^2 x + \sec^2 x = \frac{1}{\sin^2 x \cos^2 x} = 4\csc^2 2x$ relates sums at step $\frac{\pi}{2m}$ to sums at step $\frac{\pi}{m}$. (2) For odd n, the numbers $\cot^2 \frac{k\pi}{n}$ are roots of a polynomial read off from de Moivre, so their sum is a ratio of binomial coefficients. Put $T(n) = \sum_{k=1}^{n-1} \csc^2 \frac{k\pi}{n}$ and $m = 1013$ (odd), so $2026 = 2m$.

Work it out. Split $T(2m)$ into odd k (our sum S) and even $k = 2j$: the even terms are $\csc^2 \frac{j\pi}{m}$, $j = 1, \ldots, m-1$, so $S = T(2m) - T(m)$. Next, pair $k = j$ with $k = j + m$ in $T(2m)$ ($1 \le j \le m-1$): $\csc^2 \frac{j\pi}{2m} + \csc^2\left(\frac{j\pi}{2m} + \frac{\pi}{2}\right) = \csc^2 \frac{j\pi}{2m} + \sec^2 \frac{j\pi}{2m} = 4\csc^2 \frac{j\pi}{m}$. The leftover $k = m$ term is $\csc^2 \frac{\pi}{2} = 1$. So $T(2m) = 4T(m) + 1$ and $S = 3T(m) + 1$. Now $T(n)$ for odd $n = 2h + 1$. For $\theta = \frac{k\pi}{n}$, $(\cos\theta + i\sin\theta)^n = \pm 1$ is real, so the imaginary part of $(\cot\theta + i)^n$ vanishes: $\binom{n}{1}c^{n-1} - \binom{n}{3}c^{n-3} + \cdots = 0$ with $c = \cot\theta$. As a polynomial in c^2 of degree h, its roots are $\cot^2 \frac{k\pi}{n}$, $k = 1, \ldots, h$ (all different), and their sum is $\binom{n}{3}/\binom{n}{1} = \frac{(n-1)(n-2)}{6}$. Using $\csc^2 = 1 + \cot^2$ and the symmetry $k \leftrightarrow n - k$: $T(n) = 2\left(h + \frac{(n-1)(n-2)}{6}\right) = (n-1) + \frac{(n-1)(n-2)}{3} = \frac{n^2-1}{3}$. Therefore $S = 3 \cdot \frac{m^2-1}{3} + 1 = m^2 = 1013^2 = 1026169$.

Check. Small case $m = 1$: the sum is the single term $\csc^2 \frac{\pi}{2} = 1 = 1^2$. Case $m = 3$ (denominator 6): $\csc^2 30^\circ + \csc^2 90^\circ + \csc^2 150^\circ = 4 + 1 + 4 = 9 = 3^2$. The distractor $1368225 = T(2026)$ is the sum over all k.

25. Answer: (B) $\frac{451}{113}$**. The idea.** Let A' be the point of the circumcircle diametrically opposite A, and M the midpoint of BC. Two facts line everything up. First, $BHCA'$ is a parallelogram ($BH \perp AC$ and $A'C \perp AC$, so $BH \parallel A'C$; likewise $CH \parallel A'B$), so its diagonals bisect each other: M is the midpoint of HA'. Second, X is on both circles, so $\angle AXH = 90^\circ$ (diameter AH) and $\angle AXA' = 90^\circ$ (diameter AA'); hence X, H, A' are collinear. So X lies on line MH, and AX is simply the distance from A to the line MH.

Work it out. Put $B = (0,0)$, $C = (52,0)$. From $x^2 + y^2 = 53^2$ and $(x-52)^2 + y^2 = 51^2$ we get $A = (28, 45)$. The altitude from A is $x = 28$; the altitude from B is perpendicular to $\overrightarrow{AC} = (24, -45)$, i.e. $24x = 45y$, so $H = \left(28, \frac{224}{15}\right)$. With $M = (26, 0)$: $\overrightarrow{MA} = (2, 45)$ and $\overrightarrow{MH} = \left(2, \frac{224}{15}\right)$, whose length is $\sqrt{4 + \frac{50176}{225}} = \sqrt{\frac{51076}{225}} = \frac{226}{15}$. The distance from A to line MH is the cross product divided by MH: $AX = \frac{\left|2 \cdot \frac{224}{15} - 45 \cdot 2\right|}{226/15} = \frac{902/15}{226/15} = \frac{902}{226} = \frac{451}{113}$.

Check. Another route: $MH = MA' = \frac{226}{15}$, and the power of M gives $MX \cdot MA' = MB \cdot MC = 676$, so $MX = \frac{5070}{113}$ and $XH = MX - MH = \frac{50512}{1695}$. With $AH = 45 - \frac{224}{15} = \frac{451}{15} = \frac{50963}{1695}$, the right triangle AXH gives $AX^2 = \frac{50963^2 - 50512^2}{1695^2} = \frac{451 \cdot 101475}{1695^2} = \frac{451^2 \cdot 225}{1695^2}$, so $AX = \frac{451 \cdot 15}{1695} = \frac{451}{113}$ again. The option $\frac{451}{15}$ is AH itself (confusing X with the far end of the diameter), $\frac{902}{113}$ comes from mis-halving MH as $\frac{113}{15}$, and $\frac{451}{226}$ from doubling it.

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