MathCastle

AMC 12

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

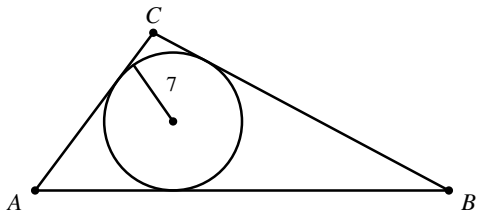
1. What is the value of $\arcsin\frac{4}{5} + \arcsin\frac{15}{17} + \arcsin\frac{77}{85}$? (Here arcsin takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.)

(A) $\frac{\pi}{2}$ **(B)** $\frac{2\pi}{3}$ **(C)** $\frac{3\pi}{4}$ **(D)** $\frac{5\pi}{6}$ **(E)** π

2. Call an arrangement $(p_1, p_2, \ldots, p_8)$ of the numbers $1, 2, \ldots, 8$ *extreme* if every entry after the first is either larger than all of the entries before it or smaller than all of the entries before it. For example, $(5, 4, 6, 3, 7, 8, 2, 1)$ is extreme. How many extreme arrangements have $p_1 = 4$?

(A) 21 **(B)** 35 **(C)** 64 **(D)** 70 **(E)** 128

3. In triangle ABC, $\tan\frac{\angle A}{2} = \frac{1}{2}$ and $\tan\frac{\angle B}{2} = \frac{1}{4}$. The circle inscribed in the triangle has radius 7. What is the area of triangle ABC?



(A) 336 **(B)** 343 **(C)** 384 **(D)** 392 **(E)** 672

4. Let $N = 12!$. For how many positive divisors d of N is $d + \frac{N}{d}$ *not* divisible by 4?

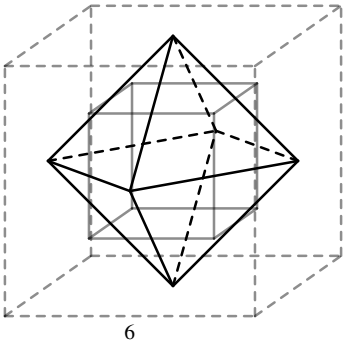
(A) 72 **(B)** 144 **(C)** 216 **(D)** 288 **(E)** 396

5. A triangle has two sides of lengths 10 and 17, and the altitude drawn to its third side has length 8. Consider all triangles with these properties (congruent triangles count once). What is the sum of their inradii?

(A) $\frac{3}{2}$ **(B)** 2 **(C)** $\frac{11}{4}$ **(D)** $\frac{7}{2}$ **(E)** $\frac{11}{2}$

6. A cube has edge length 6. The centres of its six faces are the vertices of a regular octahedron. A second cube has the same centre as the first, edges parallel to the first cube's edges, and edge length 3. What is the volume of the region that lies inside the octahedron or inside the second cube (or inside both)?

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(A) $\frac{81}{2}$ **(B)** 45 **(C)** $\frac{99}{2}$ **(D)** 63 **(E)** $\frac{135}{2}$

7. What is the area of the region consisting of all points (x, y) in the coordinate plane that satisfy $x^2 + y^2 \leq 4|x| + 2|y|$?

(A) 10π **(B)** $10\pi + 8$ **(C)** $20\pi - 16$ **(D)** $10\pi + 16$ **(E)** 20π

8. Let $r_1, r_2, \ldots, r_{2026}$ be the 2026 complex roots of the polynomial $p(x) = x^{2026} + 2x + 3$, listed with multiplicity. (None of them equals i or $-i$.) What is the value of $\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{2026}^2+1}$?

(A) -506 **(B)** -405 **(C)** 506 **(D)** 507 **(E)** 1012

9. A robot starts at the origin of a coordinate grid and makes 8 moves. Each move is one unit north, south, east or west, each direction chosen with probability $\frac{1}{4}$, independently of all other moves. What is the probability that after its 8 moves the robot is back at the origin?

(A) $\frac{1225}{16384}$ **(B)** $\frac{9}{64}$ **(C)** $\frac{6435}{32768}$ **(D)** $\frac{35}{128}$ **(E)** $\frac{1225}{4096}$

10. Ann rolls three fair six-sided dice and adds the three numbers showing. Ben does the same with three dice of his own. What is the probability that Ann's total is strictly greater than Ben's total?

(A) $\frac{361}{3888}$ **(B)** $\frac{3527}{7776}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{4249}{7776}$ **(E)** $\frac{3527}{3888}$

11. For how many ordered pairs (a, b) of integers with $2 \leq a \leq 100$ and $2 \leq b \leq 100$ is $\log_a b$ a rational number? (Pairs with $a = b$ are included.)

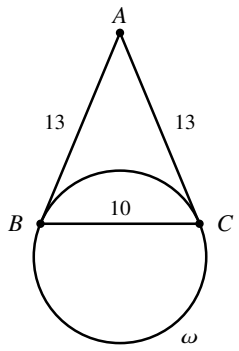
(A) 50 **(B)** 115 **(C)** 124 **(D)** 131 **(E)** 149

12. Let $\omega = \cos\frac{2\pi}{13} + i\sin\frac{2\pi}{13}$. What is the value of $\sum_{k=1}^{12} \frac{1}{1+\omega^k+\omega^{2k}}$?

(A) -8 **(B)** -4 **(C)** 0 **(D)** 3 **(E)** 4

13. Triangle ABC has $AB = AC = 13$ and $BC = 10$. Circle ω is tangent to line AB at B and to line AC at C. What is the distance from A to the nearest point of ω?

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(A) $\frac{65}{12}$ **(B)** 8 **(C)** $\frac{26}{3}$ **(D)** $\frac{119}{12}$ **(E)** $\frac{169}{12}$

14. The number 2027 is prime. For each positive integer j, let $S_j = 1^j + 2^j + 3^j + \cdots + 2026^j$. What is the remainder when $S_1 + S_2 + S_3 + \cdots + S_{10000}$ is divided by 2027?

(A) 0 **(B)** 4 **(C)** 2022 **(D)** 2023 **(E)** 2026

15. How many real numbers θ with $0 \le \theta < 2\pi$ satisfy $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin 10\theta = 0$?

(A) 10 **(B)** 11 **(C)** 19 **(D)** 20 **(E)** 21

16. What is the smallest positive integer n for which the last four digits of 7^n are 0001?

(A) 20 **(B)** 100 **(C)** 400 **(D)** 500 **(E)** 4000

17. Each of the 16 unit squares of a 4×4 grid is colored black or white, each color with probability $\frac{1}{2}$, independently of the others. What is the probability that no two black squares share a side?

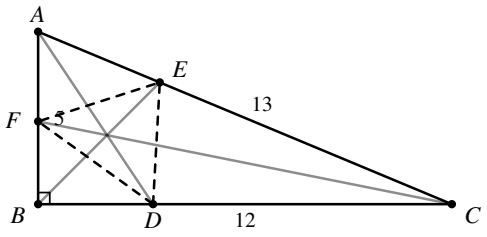
(A) $\frac{157}{32768}$ **(B)** $\frac{617}{32768}$ **(C)** $\frac{1681}{65536}$ **(D)** $\frac{227}{4096}$ **(E)** $\frac{1}{16}$

18. What is the remainder when $\left\lfloor \left(2+\sqrt{3}\right)^{2026} \right\rfloor$ is divided by 7? (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(A) 0 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 6

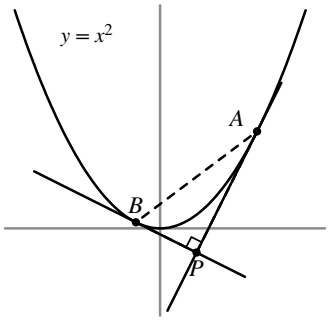
19. Triangle ABC has $AB = 5$, $BC = 12$ and $CA = 13$. The bisector of $\angle A$ meets side $\overline{BC}$ at D, the bisector of $\angle B$ meets side $\overline{CA}$ at E, and the bisector of $\angle C$ meets side $\overline{AB}$ at F. What is the area of triangle DEF?

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(A) $\frac{60}{13}$ **(B)** 6 **(C)** $\frac{104}{17}$ **(D)** $\frac{20}{3}$ **(E)** $\frac{15}{2}$

20. Points A and B lie on the parabola $y = x^2$, and the line tangent to the parabola at A is perpendicular to the line tangent to the parabola at B. These two tangent lines meet at a point P. Over all such pairs A, B, what is the smallest possible area of triangle PAB?



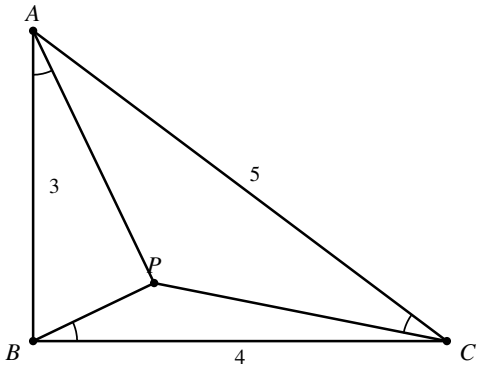
(A) $\frac{1}{16}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{2}$ **(E)** 1

21. A fair six-sided die is rolled repeatedly, and a running total of the rolls is kept. The rolling stops the first time the running total is a multiple of 5. What is the expected value of the running total when the rolling stops?

(A) $\frac{35}{2}$ **(B)** 20 **(C)** 21 **(D)** 25 **(E)** 35

22. Triangle ABC has $AB = 3$, $BC = 4$ and $CA = 5$. There is exactly one point P inside the triangle for which $\angle PAB = \angle PBC = \angle PCA$. What is $\tan \angle PAB$?

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(A) $\frac{7}{24}$ **(B)** $\frac{12}{25}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{3}{4}$

23. A cube has edge length 1. Consider the set of all points inside the cube that are closer to the center of the cube than to each of the cube's eight vertices. What is the volume of this set?

(A) $\frac{1}{6}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

24. How many integers n with $1 \le n \le 1000$ can be written in the form $n = x^3 + y^3 + z^3 - 3xyz$ for some integers x, y, z? (The integers x, y, z may be negative, zero, or equal to each other.)

(A) 667 **(B)** 700 **(C)** 778 **(D)** 889 **(E)** 1000

25. How many functions f from the set $\{1, 2, 3, 4, 5, 6, 7\}$ to itself satisfy $f(f(x)) \neq x$ for every x in the set?

(A) 1140 **(B)** 39570 **(C)** 138270 **(D)** 138690 **(E)** 279936

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Answer Key — AMC 12 — Hard Edition Practice Contest 4

1. (E) π

2. (B) 35

3. (A) 336

4. (D) 288

5. (E) $\frac{11}{2}$

6. (A) $\frac{81}{2}$

7. (D) $10\pi + 16$

8. (A) -506

9. (A) $\frac{1225}{16384}$

10. (B) $\frac{3527}{7776}$

11. (E) 149

12. (E) 4

13. (C) $\frac{26}{3}$

14. (D) 2023

15. (D) 20

16. (B) 100

17. (B) $\frac{617}{32768}$

18. (E) 6

19. (C) $\frac{104}{17}$

20. (C) $\frac{1}{4}$

21. (A) $\frac{35}{2}$

22. (B) $\frac{12}{25}$

23. (C) $\frac{1}{2}$

24. (C) 778

25. (D) 138690

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Solutions — AMC 12 — Hard Edition Practice Contest 4

1. Answer: (E) π**. The idea.** Combine the first two angles with the addition formulas, but watch the cosine: the sine of a sum does not tell you whether the sum is acute or obtuse. Here the sum of the first two angles is obtuse, and it is the supplement of the third.

Work it out. Let $\alpha = \arcsin\frac{4}{5}$ and $\beta = \arcsin\frac{15}{17}$, both acute, so $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{8}{17}$. Then $\sin(\alpha+\beta) = \frac{4}{5}\cdot\frac{8}{17} + \frac{3}{5}\cdot\frac{15}{17} = \frac{32+45}{85} = \frac{77}{85}$, $\cos(\alpha+\beta) = \frac{3}{5}\cdot\frac{8}{17} - \frac{4}{5}\cdot\frac{15}{17} = \frac{24-60}{85} = -\frac{36}{85} < 0$. So $\alpha+\beta$ lies between $\frac{\pi}{2}$ and π with sine $\frac{77}{85}$, which means $\alpha+\beta = \pi - \arcsin\frac{77}{85}$. Adding $\arcsin\frac{77}{85}$ gives exactly π.

Check. Numerically, $\alpha \approx 53.13°$, $\beta \approx 61.93°$ and $\arcsin\frac{77}{85} \approx 64.94°$; the total is $180.00°$. (Also $36^2 + 77^2 = 1296 + 5929 = 7225 = 85^2$, as it must.) Assuming $\alpha+\beta$ is acute would make the answer $2\arcsin\frac{77}{85} \approx 130°$, which is not even among the choices.

2. Answer: (B) 35**. The idea.** In an extreme arrangement, the first k entries always form a block of consecutive integers. Each new entry must stretch that block by one, either downward (a new record low) or upward (a new record high), so the arrangement is just a sequence of "down" and "up" steps.

Work it out. Suppose $p_1, \ldots, p_k$ are the consecutive integers from m to M. If p_{k+1} is larger than all of them, it cannot skip over $M+1$: otherwise $M+1$ would appear later and be neither larger than everything before it (it is less than p_{k+1}) nor smaller than everything before it (it is more than m). So $p_{k+1} = M+1$, and similarly a new low must be $m-1$. Starting from the block $\{4\}$, we must reach $\{1, \ldots, 8\}$, which takes exactly 3 down-steps (to reach 1) and 4 up-steps (to reach 8). Any order of these 7 steps works and gives a different arrangement, so the count is $\binom{7}{3} = 35$.

Check. Summing over all possible first entries j gives $\sum_{j=1}^{8}\binom{7}{j-1} = 2^7 = 128$ extreme arrangements in total, which matches the "read backwards" count: the last entry must be 1 or 8, then the last of the remaining must be an end of what is left, and so on, 2 choices at each of 7 steps. The example $(5,4,6,3,7,8,2,1)$ is the step sequence down, up, down, up, up, down, down from $\{5\}$.

3. Answer: (A) 336**. The idea.** The incircle touches the sides at points that cut each side into two tangent segments. From a vertex with angle θ, the right triangle formed by the vertex, the incenter and a point of tangency shows that the tangent length is $r\cot\frac{\theta}{2}$. So we need all three half-angle cotangents, and the third one comes from $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90°$.

Work it out. By the tangent addition formula, $\tan\frac{A+B}{2} = \frac{\frac{1}{2}+\frac{1}{4}}{1-\frac{1}{8}} = \frac{3/4}{7/8} = \frac{6}{7}$. Since $\frac{C}{2} = 90° - \frac{A+B}{2}$, we get $\tan\frac{C}{2} = \cot\frac{A+B}{2} = \frac{7}{6}$. The tangent lengths from A, B, C are therefore $7\cdot 2 = 14$, $7\cdot 4 = 28$, $7\cdot\frac{6}{7} = 6$. Each side is the sum of the two tangent lengths at its ends, so the semiperimeter is $s = 14+28+6 = 48$, and the area is $rs = 7\cdot 48 = 336$.

Check. The sides are $BC = 28+6 = 34$, $CA = 14+6 = 20$ and $AB = 14+28 = 42$. Heron's formula gives $\sqrt{48\cdot 14\cdot 28\cdot 6} = \sqrt{112896} = 336$. Using the full perimeter 96 in place of the semiperimeter gives $7\cdot 96 = 672$.

4. Answer: (D) 288**. The idea.** Only the powers of 2 matter. If d contains exactly 2^i, then $\frac{N}{d}$ contains exactly 2^{10-i}, since $N = 12! = 2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 11$. When two numbers carry *different*

powers of 2, their sum carries the smaller one; when they carry the *same* power, the sum picks up at least one extra factor of 2.

Work it out. Write $d = 2^i m$ and $\frac{N}{d} = 2^{10-i} m'$ with m, m' odd.

If $i \neq 5$, the exponents i and $10 - i$ differ, and the sum is exactly divisible by $2^{\min(i,10-i)}$. It fails to be divisible by 4 exactly when $\min(i, 10 - i) \leq 1$, that is, $i \in \{0, 1, 9, 10\}$.

If $i = 5$, the sum is $2^5(m + m')$ with $m + m'$ even, so it is divisible by 2^6, in particular by 4.

So we need $i \in \{0, 1, 9, 10\}$: 4 choices for the power of 2. The odd part of d is any divisor of $3^5 \cdot 5^2 \cdot 7 \cdot 11$, which has $6 \cdot 3 \cdot 2 \cdot 2 = 72$ divisors. The count is $4 \cdot 72 = 288$.

Check. Small cases: $d = 2$ gives $2 + \frac{N}{2}$, where $\frac{N}{2}$ is divisible by 2^9, so the sum is $2 \pmod 4$, as claimed; $d = 32$ gives $32 + \frac{N}{32}$ with $\frac{N}{32} = 32 \cdot (\text{odd})$, a multiple of 64. Out of all $11 \cdot 72 = 792$ divisors, 288 is the answer. Counting only the divisors where the sum is odd ($i = 0$ or 10) gives 144; allowing just one of the two endpoints of the range (say $i \in \{0, 1, 9\}$) gives 216; counting only the odd divisors gives 72.

5. Answer: (E) $\frac{11}{2}$. The idea. The altitude to the third side comes from the vertex where the sides 10 and 17 meet. Its foot H is fixed relative to the two other vertices only up to a choice: the two vertices can lie on the same side of H or on opposite sides. Both choices give genuine triangles, so there are two answers to combine.

Work it out. Let the apex be A with $AH = 8$, and let B, C be on the line through H with $AB = 10$, $AC = 17$. Then $HB = \sqrt{100 - 64} = 6$ and $HC = \sqrt{289 - 64} = 15$.

Opposite sides of H: $BC = 6+15 = 21$, area $\frac{1}{2} \cdot 21 \cdot 8 = 84$, semiperimeter $\frac{10+17+21}{2} = 24$, inradius $\frac{84}{24} = \frac{7}{2}$.

Same side of H: $BC = 15 - 6 = 9$, area $\frac{1}{2} \cdot 9 \cdot 8 = 36$, semiperimeter $\frac{10+17+9}{2} = 18$, inradius $\frac{36}{18} = 2$.

The sum is $\frac{7}{2} + 2 = \frac{11}{2}$.

Check. Heron's formula agrees: $\sqrt{24 \cdot 14 \cdot 7 \cdot 3} = \sqrt{7056} = 84$ and $\sqrt{18 \cdot 8 \cdot 1 \cdot 9} = \sqrt{1296} = 36$. Missing the obtuse case gives $\frac{7}{2}$, and dividing each area by the full perimeter instead of the semiperimeter gives $\frac{7}{4} + 1 = \frac{11}{4}$. Interestingly, both triangles have the same circumradius $\frac{10 \cdot 17}{2 \cdot 8} = \frac{85}{8}$, since $AB \cdot AC = 2R \cdot AH$ in any triangle.

6. Answer: (A) $\frac{81}{2}$. The idea. Neither solid contains the other: the octahedron's six tips poke out through the faces of the small cube, and the small cube's eight corners poke out through the faces of the octahedron. The union is the octahedron plus the eight corner pieces of the cube that stick out of it, and each of those is a small right-corner tetrahedron.

Work it out. Put the big cube at $[-3, 3]^3$. The face centres are $(\pm 3, 0, 0)$, $(0, \pm 3, 0)$, $(0, 0, \pm 3)$, so the octahedron is $|x| + |y| + |z| \leq 3$: two square pyramids of height 3 on a square with diagonals 6 (area 18), of total volume $2 \cdot \frac{1}{3} \cdot 18 \cdot 3 = 36$. The small cube is $[-\frac{3}{2}, \frac{3}{2}]^3$.

A point of the small cube lies outside the octahedron when $|x| + |y| + |z| > 3$. In the octant $x, y, z \geq 0$ put $u = \frac{3}{2} - x$, $v = \frac{3}{2} - y$, $w = \frac{3}{2} - z$; the condition becomes $u, v, w \geq 0$ with $u + v + w < \frac{3}{2}$, a corner tetrahedron with three perpendicular legs $\frac{3}{2}$ and volume $\frac{1}{6}\left(\frac{3}{2}\right)^3 = \frac{9}{16}$. The eight corners contribute $8 \cdot \frac{9}{16} = \frac{9}{2}$, so the union has volume $36 + \frac{9}{2} = \frac{81}{2}$.

Check. Compute the overlap from the octahedron's side. The part of the octahedron with $x > \frac{3}{2}$ is the pyramid with apex $(3,0,0)$, height $\frac{3}{2}$ and base the square $|y|+|z| \le \frac{3}{2}$ (diagonals 3, area $\frac{9}{2}$), of volume $\frac{1}{3}\cdot\frac{9}{2}\cdot\frac{3}{2} = \frac{9}{4}$; there are six such tips, $\frac{27}{2}$ in all. So the overlap is $36 - \frac{27}{2} = \frac{45}{2}$, which also equals $27 - \frac{9}{2}$ from the cube's side. Then the union is $36+27-\frac{45}{2} = \frac{81}{2}$. Adding the volumes without removing the overlap gives 63; using $\frac{1}{3}$ or $\frac{1}{2}$ instead of $\frac{1}{6}$ for a corner tetrahedron gives 45 or $\frac{99}{2}$.

7. Answer: (D) $10\pi + 16$**. The idea.** The region is symmetric in both axes, so find its part in the first quadrant and multiply by 4. There the inequality is a disk whose boundary passes through the origin, and because the two axes meet at a right angle there, the chord joining the other two axis crossings is a diameter.

Work it out. For $x, y \ge 0$ the condition is $x^2-4x+y^2-2y \le 0$, that is $(x-2)^2+(y-1)^2 \le 5$: a disk of radius $\sqrt{5}$ centred at $(2,1)$. Its boundary meets the axes at $O = (0,0)$, $P = (4,0)$ and $Q = (0,2)$. Since $\angle POQ = 90^\circ$ is inscribed in the circle, $\overline{PQ}$ is a diameter (indeed $PQ = \sqrt{16+4} = 2\sqrt{5}$). The arc from P to Q on the far side from O stays in the first quadrant, while the arc through O leaves it. So the part of the disk in the first quadrant is the half-disk beyond $\overline{PQ}$ together with the right triangle OPQ: $\frac{1}{2}\cdot 5\pi + \frac{1}{2}\cdot 4\cdot 2 = \frac{5\pi}{2} + 4$. The four quadrants give $4\left(\frac{5\pi}{2}+4\right) = 10\pi + 16$.

Check. Subtract instead: the full disk has area 5π; the x-axis cuts off a segment on chord OP (distance 1 from the centre) and the y-axis one on chord OQ (distance 2). Their central half-angles α, β have $\cos\alpha = \frac{1}{\sqrt{5}}$, $\cos\beta = \frac{2}{\sqrt{5}}$, so $\alpha+\beta = \frac{\pi}{2}$, and the segments total $5(\alpha+\beta) - 1\cdot 2 - 2\cdot 1 = \frac{5\pi}{2} - 4$. The quadrant piece is $5\pi - \frac{5\pi}{2} + 4$, as before.

8. Answer: (A) -506**. The idea.** There are far too many roots for power sums, but sums of $\frac{1}{c-r_k}$ over all roots are easy: from $p(x) = \prod_k (x - r_k)$, taking the logarithmic derivative gives $\frac{p'(x)}{p(x)} = \sum_k \frac{1}{x-r_k}$. So split $\frac{1}{r^2+1}$ into partial fractions with poles at $\pm i$ and evaluate p'/p there.

Work it out. Since $r^2+1 = (r-i)(r+i)$, $\frac{1}{r^2+1} = \frac{1}{2i}\left(\frac{1}{r-i} - \frac{1}{r+i}\right)$. Let $w = \frac{p'(i)}{p(i)} = \sum_k \frac{1}{i-r_k}$. Then $\sum_k \frac{1}{r_k - i} = -w$, and because p has real coefficients, $\sum_k \frac{1}{r_k+i} = -\frac{p'(-i)}{p(-i)} = -\overline{w}$. Hence $\sum_k \frac{1}{r_k^2+1} = \frac{1}{2i}(-w + \overline{w}) = \frac{-2i\,\mathrm{Im}\,w}{2i} = -\mathrm{Im}\,w$. Now $2026 = 4\cdot 506 + 2$ and $2025 = 4\cdot 506 + 1$, so $i^{2026} = -1$ and $i^{2025} = i$. Thus $p(i) = -1 + 2i + 3 = 2 + 2i$ and $p'(i) = 2026\,i^{2025} + 2 = 2 + 2026i$, so $w = \frac{2+2026i}{2+2i} = \frac{(1+1013i)(1-i)}{2} = \frac{1014+1012i}{2} = 507 + 506i$. The sum is $-\mathrm{Im}\,w = -506$.

Check. The same steps for $x^n + 2x + 3$ with $n = 4m+2$ give $w = \frac{(1+(n/2)i)(1-i)}{2}$, so the sum is $-\frac{n-2}{4}$. Test $n = 2$, where it predicts 0: the roots of $x^2 + 2x + 3$ satisfy $r^2 + 1 = -2(r+1)$ with $r + 1 = \pm i\sqrt{2}$, so the two terms are $-\frac{1}{2}\left(\frac{1}{i\sqrt{2}} + \frac{1}{-i\sqrt{2}}\right) = 0$. Mistakes: treating i^{2026} as 1 gives $p(i) = 4 + 2i$, $w = 203 + 405i$ and the sum -405; a sign slip in $\sum \frac{1}{r_k - i} = -w$ gives 506; with that slip, taking the real part of w gives 507, and also dropping the $\frac{1}{2}$ in $\frac{1}{2i}$ gives 1012.

9. Answer: (A) $\frac{1225}{16384}$**. The idea.** Track the rotated coordinates $u = x+y$ and $v = x-y$. A move east changes (u,v) by $(+1,+1)$, west by $(-1,-1)$, north by $(+1,-1)$ and south by $(-1,+1)$. These are exactly the four sign patterns $(\pm 1, \pm 1)$, each with probability $\frac{1}{4}$, so u and v are two *independent* fair ± 1 walks.

Work it out. The robot is at the origin exactly when $u = v = 0$. A fair ± 1 walk is back at 0 after 8 steps with probability $\binom{8}{4}/2^8 = \frac{70}{256} = \frac{35}{128}$. By independence the answer is $\left(\frac{35}{128}\right)^2 = \frac{1225}{16384}$.

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Check. Count directly: with k moves east, k west, $4-k$ north and $4-k$ south, the number of orders is $\frac{8!}{k!\,k!\,(4-k)!\,(4-k)!} = \binom{8}{4}\binom{4}{k}^2$. Summing over k gives $70\sum_k \binom{4}{k}^2 = 70\cdot\binom{8}{4} = 4900$ walks out of $4^8 = 65536$, and $\frac{4900}{65536} = \frac{1225}{16384}$. Tracking only the x-coordinate gives $\binom{16}{8}/4^8 = \frac{6435}{32768}$, which is merely the chance of ending on the y-axis.

10. Answer: (B) $\frac{3527}{7776}$**. The idea.** By symmetry Ann wins exactly as often as Ben, so the answer is $\frac{1}{2}(1-P(\text{tie}))$. To count ties, turn each of Ben's dice upside down: replacing each of his numbers b by $7-b$ changes his total T to $21-T$, and a tie $S=T$ becomes $S+(21-T)=21$. So ties correspond one-to-one to rolls of six dice with total 21.

Work it out. Six dice total 21 exactly when the numbers $d_i-1\in\{0,\dots,5\}$ total 15. Stars and bars with inclusion-exclusion (subtracting cases where some $d_i-1\ge 6$) gives $\binom{20}{5}-6\binom{14}{5}+15\binom{8}{5} = 15504-12012+840 = 4332$. So $P(\text{tie}) = \frac{4332}{6^6} = \frac{4332}{46656} = \frac{361}{3888}$, and $P(\text{Ann wins}) = \frac{1}{2}\left(1-\frac{361}{3888}\right) = \frac{3527}{7776}$.

Check. The numbers of ways to roll totals $3,4,\dots,18$ with three dice are $1,3,6,10,15,21,25,27,27,25,21,15,10$. The number of ties is the sum of their squares, $2(1+9+36+100+225+441+625+729) = 4332$. Forgetting to halve gives $\frac{3527}{3888}$, and $\frac{4249}{7776}$ is the chance that Ann's total is at least Ben's.

11. Answer: (E) 149**. The idea.** $\log_a b = \frac{p}{q}$ with positive integers p,q means $a^p = b^q$. Comparing prime factorisations, this happens exactly when a and b are powers of one common integer. So sort $2,\dots,100$ into families $\{c, c^2, c^3, \dots\}$, one for each c that is not itself a perfect power; the good pairs are the pairs inside a family.

Work it out. Every integer $n\ge 2$ is uniquely c^k with c not a perfect power, and if $a=c^i$, $b=c^j$ then $\log_a b = \frac{j}{i}$ is rational. Conversely, if $a^p = b^q$ then every prime exponent of b is $\frac{p}{q}$ times that of a, which forces a and b into the same family. The families with more than one member up to 100 are

$c=2$: $2,4,8,16,32,64$ (6 members, 36 ordered pairs); $\quad c=3$: $3,9,27,81$ (16 pairs);

$c=5,6,7,10$: $\{5,25\},\{6,36\},\{7,49\},\{10,100\}$ (4 pairs each, 16 in all).

The perfect powers up to 100 are $4,8,9,16,25,27,32,36,49,64,81,100$, twelve numbers, so there are $99-12=87$ families. Of these, $87-6=81$ have a single member and contribute one pair (c,c) each. The total is $36+16+16+81 = 149$.

Check. Pairs such as $(8,32)$ matter: $\log_8 32 = \frac{5}{3}$ is rational although neither number is a power of the other. Counting only pairs where one number is a power of the other gives 131; counting only integer logarithms gives 115; counting unordered pairs gives $21+10+12+81 = 124$; leaving out the pairs with $a=b$ gives 50.

12. Answer: (E) 4**. The idea.** Write $1+z+z^2 = \frac{z^3-1}{z-1}$, so each term is $\frac{z-1}{z^3-1}$ with $z=\omega^k$. Cubing permutes the twelve nontrivial 13th roots of unity, so rename $w=z^3$. Since $3\cdot 9 = 27\equiv 1 \pmod{13}$, we get $z=w^9$ back, and every term becomes a geometric sum in w.

Work it out. The numbers $z=\omega^k$, $1\le k\le 12$, are the 13th roots of unity other than 1. None of them is a cube root of unity, so $z^3\neq 1$ and no denominator vanishes. As z runs over these twelve roots, so does $w=z^3$ (because 3 is invertible mod 13), and $w^9 = z^{27} = z$. Hence $\sum_{k=1}^{12}\frac{z-1}{z^3-1} = \sum_w \frac{w^9-1}{w-1} = \sum_w \left(1+w+w^2+\cdots+w^8\right)$, the sum over the twelve nontrivial roots w.

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For each j with $1 \le j \le 8$, the powers w^j again run over all twelve nontrivial roots, whose sum is -1 (all thirteen roots sum to 0). So the total is $12 \cdot 1 + 8 \cdot (-1) = 4$.

Check. Pairing k with $13-k$: the term for $\overline{z}$ is the conjugate of the term for z, so the sum is real, as it should be; numerically it evaluates to 4.0000. A sign slip in $1+z+z^2 = \frac{z^3-1}{z-1}$ flips the answer to -4; letting the geometric sum run up to w^9 gives $12-9=3$; dropping the constant terms 1 gives -8.

13. Answer: (C) $\frac{26}{3}$**. The idea.** A radius to a point of tangency is perpendicular to the tangent line, so the centre O of ω lies on the perpendiculars to AB at B and to AC at C, which meet on the axis of symmetry, below BC. The point of ω nearest the outside point A lies on segment AO, so the distance is $AO - r$.

Work it out. Let M be the midpoint of BC, so $BM = 5$ and $AM = \sqrt{13^2 - 5^2} = 12$. Put $M = (0,0)$, $B = (-5,0)$, $C = (5,0)$, $A = (0,12)$, and $O = (0,-k)$.

Tangency at B means $OB \perp AB$: $\overrightarrow{BO} \cdot \overrightarrow{BA} = (5,-k)\cdot(5,12) = 25 - 12k = 0$, so $k = \frac{25}{12}$. Then $r = OB = \sqrt{25 + \frac{625}{144}} = \sqrt{\frac{4225}{144}} = \frac{65}{12}$.

Since AB is tangent at B, triangle ABO has a right angle at B: $AO = \sqrt{13^2 + r^2} = \sqrt{\frac{24336+4225}{144}} = \frac{169}{12}$ (also $12 + \frac{25}{12}$, as it should be). The nearest point of ω is where segment AO crosses it, at distance $AO - r = \frac{169-65}{12} = \frac{104}{12} = \frac{26}{3}$.

Check. That nearest point P is the midpoint of the arc BC facing A, and it is the incenter of triangle ABC: by the tangent-chord angle, $\angle PBC$ is half of $\angle ABC$, so BP bisects angle B. The inradius is $\frac{\text{area}}{s} = \frac{60}{18} = \frac{10}{3}$, so the incenter is at height $\frac{10}{3}$ on AM and its distance from A is $12 - \frac{10}{3} = \frac{26}{3}$, a match. The traps: $\frac{65}{12}$ is the radius, $\frac{169}{12}$ is AO, $\frac{119}{12} = 12 - \frac{25}{12}$ subtracts OM instead of r, and $8 = 13 - 5$ guesses from the sides.

14. Answer: (D) 2023**. The idea.** Modulo a prime p, the power sum $1^j + 2^j + \cdots + (p-1)^j$ is $\equiv 0$ unless $p-1$ divides j, when every term is 1 and the sum is $p - 1 \equiv -1$. So almost every S_j vanishes, and only the exponents that are multiples of 2026 contribute.

Work it out. Let $p = 2027$. By Fermat's little theorem $k^{2026} \equiv 1$ for $1 \le k \le 2026$, so k^j depends only on j modulo 2026.

If $2026 \mid j$, every term is 1 and $S_j \equiv 2026 \equiv -1 \pmod{p}$.

If $2026 \nmid j$, let r be the remainder of j modulo 2026, so $1 \le r \le 2025$ and $S_j \equiv S_r$. The polynomial $x^r - 1$ has at most $r < 2026$ roots modulo the prime p, so some a in $1, \ldots, 2026$ has $a^r \not\equiv 1$. Multiplying by a permutes the nonzero residues, so $a^r S_r = \sum (ak)^r \equiv S_r$, i.e. $(a^r - 1)S_r \equiv 0$, and since $a^r - 1 \not\equiv 0$ we get $S_r \equiv 0$.

The multiples of 2026 up to 10000 are $2026, 4052, 6078, 8104$ (the next, 10130, is too big), so the total is $\equiv 4 \cdot (-1) = -4 \equiv 2023 \pmod{2027}$.

Check. Try $p = 5$ and $j = 1, \ldots, 4$: $S_1 = 10 \equiv 0$, $S_2 = 30 \equiv 0$, $S_3 = 100 \equiv 0$, $S_4 = 354 \equiv 4 \equiv -1 \pmod 5$, exactly the pattern used. The traps: 4 (sign slip), 2026 (only one special exponent), 2022 (five multiples counted).

15. Answer: (D) 20**. The idea.** A sum of sines of equally spaced angles telescopes once you multiply by $2\sin\frac{\theta}{2}$, and it collapses to a product of two sines. A product is zero exactly when a

factor is, so the count is two easy lists of angles, minus overlaps, with one special angle ($\theta = 0$, where the multiplier $\sin\frac{\theta}{2}$ vanishes) checked by hand.

Work it out. Let $S = \sum_{k=1}^{10} \sin k\theta$. Since $2\sin\frac{\theta}{2}\sin k\theta = \cos\left(k-\frac{1}{2}\right)\theta - \cos\left(k+\frac{1}{2}\right)\theta$, the sum telescopes: $2\sin\frac{\theta}{2}\cdot S = \cos\frac{\theta}{2} - \cos\frac{21\theta}{2} = 2\sin 5\theta \sin\frac{11\theta}{2}$.

For $0 < \theta < 2\pi$ we have $\sin\frac{\theta}{2} \neq 0$, so $S = 0$ exactly when $\sin 5\theta = 0$ or $\sin\frac{11\theta}{2} = 0$.

$\sin 5\theta = 0$ gives $\theta = \frac{k\pi}{5}$, $k = 1, \ldots, 9$: nine values. $\sin\frac{11\theta}{2} = 0$ gives $\theta = \frac{2j\pi}{11}$, $j = 1, \ldots, 10$: ten values. They never coincide: $\frac{k}{5} = \frac{2j}{11}$ would need $11k = 10j$, so $10 \mid k$, impossible for $1 \le k \le 9$.

Finally $\theta = 0$ must be checked directly (the identity divides by $\sin 0$ there): every term is $\sin 0 = 0$, so it works. Total $9 + 10 + 1 = 20$.

Check. Try $\theta = \pi$ (from the first list): $\sin k\pi = 0$ for every k, so $S = 0$. Try $\theta = \frac{2\pi}{11}$: the ten angles $\frac{2k\pi}{11}$ together with 0 are the eleven equally spaced angles around the circle, whose sines sum to 0. The traps: 21 counts $\theta = 0$ once in each list, and 19 throws $\theta = 0$ away because the formula breaks there.

16. Answer: (B) 100**. The idea.** $7^4 = 2401$ is already 1 more than a multiple of 100, so powers of it can be expanded with the binomial theorem, and modulo 10^4 only the first two terms survive.

Work it out. First, n must be a multiple of 4: the last digit of 7^n cycles $7, 9, 3, 1$ with period 4, and it is 1 only when $4 \mid n$. Write $n = 4k$. Then $7^{4k} = 2401^k = (1 + 2400)^k = 1 + 2400k + \binom{k}{2}2400^2 + \cdots$. Every term from $2400^2 = 5{,}760{,}000$ on is a multiple of 10,000, so $7^{4k} \equiv 1 + 2400k \pmod{10^4}$.

We need $10^4 \mid 2400k$, i.e. $100 \mid 24k$, i.e. $25 \mid 6k$, i.e. $25 \mid k$. The least such k is 25, so $n = 100$.

Check. The formula $7^{4k} \equiv 1 + 2400k$ predicts $7^{20} \equiv 1 + 12000 \equiv 2001$ and $7^{40} \equiv 4001$; indeed $2401^2 = 5764801$ ends in 4801 and $7^{20} = 2401^5$ ends in 2001. Traps: 20 only gets the last three digits to 001; 500 is the exponent that works for every number coprime to 10 (and $4000 = \varphi(10^4)$), but neither is the least for 7; 400 comes from requiring $100 \mid k$ and ignoring the factor 24.

17. Answer: (B) $\frac{617}{32768}$**. The idea.** Count the good colorings row by row. A single row is allowed only if it has no two adjacent blacks, and two consecutive rows are compatible exactly when they have no black in the same column. Keep, for each possible current row, the number of ways to fill the grid down to it; each new row adds up the counts of the rows compatible with it.

Work it out. Write a row as a string of 0s (white) and 1s (black). The rows with no two adjacent 1s are the eight strings 0000, 1000, 0100, 0010, 0001, 1010, 0101, 1001.

After one row each of the eight has count 1. For the next row, a string's count is the sum over the strings with no 1 in a common position: 0000 is compatible with all 8; 1000 and 0001 with 5 each; 0100 and 0010 with 6 each; 1010 and 0101 with 4 each; 1001 with 3. Two rows: $8 + 2\cdot 5 + 2\cdot 6 + 2\cdot 4 + 3 = 41$.

Repeating with these counts as weights, three rows give counts 41, 29, 29, 31, 31, 23, 23, 20 (in the order 0000; 1000, 0001; 0100, 0010; 1010, 0101; 1001), total 227. Four rows give 227, 155, 155, 173, 173, 124, 124, total 1234. For example the count for 0000 is the whole previous total, and the count for 1001 adds the previous counts of 0000, 0100, 0010: $41 + 31 + 31 = 103$.

There are $2^{16} = 65536$ equally likely colorings, so the probability is $\frac{1234}{65536} = \frac{617}{32768}$.

Check. The same method on a 2×2 grid gives $3 + 2 + 2 = 7$ (the rows are 00, 10, 01), matching a direct list: the empty grid, 4 single blacks and 2 diagonal pairs, and on a 3×3 grid the known count

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63. The traps: $\frac{1}{16} = \frac{8^4}{2^{16}}$ ignores the vertical condition, $\frac{227}{4096}$ stops at three rows, $\frac{1681}{65536} = \left(\frac{41}{256}\right)^2$ treats the top two rows and the bottom two rows as independent, and $\frac{157}{32768} = \frac{314}{65536}$ also forbids diagonal neighbors.

18. Answer: (E) 6**. The idea.** Pair $2+\sqrt{3}$ with its conjugate $2-\sqrt{3}$. The sums $T_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$ are integers obeying a linear recurrence, and since $0 < 2-\sqrt{3} < 1$, the power $(2+\sqrt{3})^n$ is just below the integer T_n. So the floor is $T_{2026} - 1$, and a recurrence can be reduced modulo 7, where it is periodic.

Work it out. $2 \pm \sqrt{3}$ are the roots of $x^2 = 4x - 1$, so $T_{n+2} = 4T_{n+1} - T_n$, with $T_0 = 2$ and $T_1 = 4$. All T_n are integers.

Because $0 < 2-\sqrt{3} < 1$, we have $0 < (2-\sqrt{3})^{2026} < 1$, so $(2+\sqrt{3})^{2026} = T_{2026} - (2-\sqrt{3})^{2026}$ lies strictly between $T_{2026} - 1$ and T_{2026}. Its floor is $T_{2026} - 1$.

Modulo 7 the sequence runs $T_0, T_1, \ldots = 2, 4, 0, 3, 5, 3, 0, 4, 2, 4, \ldots$; since $(T_8, T_9) \equiv (T_0, T_1)$, it repeats with period 8. As $2026 = 8 \cdot 253 + 2$, $T_{2026} \equiv T_2 \equiv 0$, and the floor is $\equiv 0 - 1 \equiv 6 \pmod 7$.

Check. Small case: $(2+\sqrt{3})^2 = 7 + 4\sqrt{3} \approx 13.93$, with floor $13 = T_2 - 1 = 14 - 1$, and $13 \equiv 6 \pmod 7$, matching the pattern for exponents $\equiv 2 \pmod 8$. The trap 0 forgets the -1 (taking the floor to be T_{2026}), and 4 comes from assuming period 6.

19. Answer: (C) $\frac{104}{17}$**. The idea.** Triangle DEF is what is left of ABC after cutting off three corner triangles, and each corner triangle shares an angle with ABC, so its area is $[ABC]$ times the product of two side fractions. The angle bisector theorem supplies every fraction.

Work it out. $5^2 + 12^2 = 13^2$, so the right angle is at B and $[ABC] = \frac{1}{2} \cdot 5 \cdot 12 = 30$. Write $a = BC = 12$, $b = CA = 13$, $c = AB = 5$.

The bisector theorem splits each side in the ratio of the adjacent sides: $BD = \frac{ac}{b+c} = \frac{10}{3}$, $DC = \frac{ab}{b+c} = \frac{26}{3}$; $CE = \frac{ab}{a+c} = \frac{156}{17}$, $EA = \frac{bc}{a+c} = \frac{65}{17}$; $AF = \frac{bc}{a+b} = \frac{13}{5}$, $FB = \frac{ac}{a+b} = \frac{12}{5}$.

Corner at A: $[AFE] = \frac{AF}{AB} \cdot \frac{AE}{AC} \cdot 30 = \frac{13}{25} \cdot \frac{5}{17} \cdot 30 = \frac{78}{17}$. Corner at B: $[BDF] = \frac{BD}{BC} \cdot \frac{BF}{BA} \cdot 30 = \frac{5}{18} \cdot \frac{12}{25} \cdot 30 = 4 = \frac{68}{17}$. Corner at C: $[CDE] = \frac{CD}{CB} \cdot \frac{CE}{CA} \cdot 30 = \frac{13}{18} \cdot \frac{12}{17} \cdot 30 = \frac{260}{17}$.

So $[DEF] = 30 - \frac{78+68+260}{17} = \frac{510-406}{17} = \frac{104}{17}$.

Check. Coordinates: $B = (0,0)$, $A = (0,5)$, $C = (12,0)$. Then $D = (\frac{10}{3}, 0)$, $F = (0, \frac{12}{5})$, and E divides CA with $CE : EA = 12 : 5$, so $E = \left(\frac{60}{17}, \frac{60}{17}\right)$ (it lies on $y = x$, the bisector of the right angle). The shoelace formula gives $\frac{1}{2}\left|\frac{10}{3} \cdot \frac{60}{17} - \frac{60}{17} \cdot 0 + \frac{60}{17} \cdot \frac{12}{5} - 0 \cdot \frac{60}{17} + 0 - \frac{10}{3} \cdot \frac{12}{5}\right| = \frac{1}{2}\left(\frac{200}{17} + \frac{144}{17} - 8\right) = \frac{104}{17}$. The trap $\frac{20}{3}$ comes from taking $AE = AB = 5$ (so CA splits $5 : 8$) instead of $AE : EC = AB : BC = 5 : 12$; $\frac{15}{2}$ is the medial triangle.

20. Answer: (C) $\frac{1}{4}$**. The idea.** The tangent at $x = a$ is $y = 2ax - a^2$ (no calculus needed: $x^2 - (2ax - a^2) = (x-a)^2$ has a double root at a). Everything about the tangent triangle then comes out in terms of a and b, and its area turns out to depend only on $|b - a|$.

Work it out. Let $A = (a, a^2)$ and $B = (b, b^2)$. The tangents have slopes $2a$ and $2b$, so perpendicularity means $4ab = -1$.

The tangents meet where $2ax - a^2 = 2bx - b^2$, i.e. $x = \frac{a+b}{2}$, and then $y = ab$. So $P = \left(\frac{a+b}{2}, ab\right)$.

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From A: $B - A = (b-a)(1, a+b)$ and $P - A = (b-a)\left(\frac{1}{2}, a\right)$. The area is half the absolute determinant: $[PAB] = \frac{1}{2}(b-a)^2\left|a - \frac{a+b}{2}\right| = \frac{|b-a|^3}{4}$.

With $ab = -\frac{1}{4}$, $(b-a)^2 = (a+b)^2 - 4ab = (a+b)^2 + 1 \geq 1$, with equality when $a = -b$, i.e. $a = \frac{1}{2}$, $b = -\frac{1}{2}$. So the least area is $\frac{1^3}{4} = \frac{1}{4}$.

Check. With $a = \frac{1}{2}$, $b = -\frac{1}{2}$: $A = (\frac{1}{2}, \frac{1}{4})$, $B = (-\frac{1}{2}, \frac{1}{4})$, $P = (0, -\frac{1}{4})$ (on the directrix, as perpendicular tangents always meet). Base $AB = 1$, height $\frac{1}{2}$, area $\frac{1}{4}$.

21. Answer: (A) $\frac{35}{2}$. The idea. Two classical facts combine. First, the number of rolls T: the total modulo 5 moves around the five residues, and each step's shift has the same distribution from every residue, so in the long run every residue is visited equally often, one fifth of the time; the average gap between visits to 0 is therefore 5, and $E[T] = 5$. Second, Wald's identity: the final total is the sum of T rolls, and its expectation is $E[T]$ times the average roll.

Work it out. Number of rolls. Each roll adds $1, 2, 3, 4, 0, 1$ modulo 5 (for faces 1 to 6). From any residue the next residue has the same shifted distribution, so the transition table is doubly stochastic and the uniform distribution is stationary. A chain spending a fraction $\frac{1}{5}$ of its time at 0 returns to 0 after 5 steps on average, so $E[T] = 5$.

The final total. Write $S_T = X_1 + X_2 + \cdots + X_T$ where X_k is the kth roll. Then $S_T = \sum_{k\geq 1} X_k \cdot [T \geq k]$. The event $T \geq k$ (no stop in the first $k-1$ rolls) depends only on $X_1, \ldots, X_{k-1}$, so it is independent of X_k, and $E\left[X_k[T \geq k]\right] = \frac{7}{2}P(T \geq k)$. Summing, $E[S_T] = \frac{7}{2}\sum_{k\geq 1} P(T \geq k) = \frac{7}{2}E[T] = \frac{7}{2} \cdot 5 = \frac{35}{2}$.

Check. Solve for $E[T]$ directly. Let h_r be the expected number of further rolls from residue r, with $h_0 = 0$ after the start. The four equations $h_r = 1 + \frac{1}{6}\sum_{d=1}^{6} h_{(r+d) \bmod 5}$ give $h_1 = \frac{1554}{311}$, $h_2 = \frac{1548}{311}$, $h_3 = \frac{1512}{311}$, $h_4 = \frac{1296}{311}$. From the start, the first roll lands on residues $1, 2, 3, 4, 0, 1$, so $E[T] = 1 + \frac{1}{6}(2h_1 + h_2 + h_3 + h_4) = 1 + \frac{7464}{1866} = 1 + 4 = 5$. Note the answer is not a multiple of 5 even though every possible final total is: an expectation need not be a possible value. The traps: $21 = \frac{7}{2} \cdot 6$ assumes six rolls on average, 20 rounds up to a multiple of 5, $25 = 5 \cdot 5$ multiplies $E[T]$ by 5 instead of by the average roll, and 35 uses 7 (a roll plus its opposite face) in place of $\frac{7}{2}$.

22. Answer: (B) $\frac{12}{25}$. The idea. Call the common angle ω. The right angle at B does most of the work: $\angle PBA = 90^\circ - \omega$, so in triangle APB the angles at A and B add to 90° and $\angle APB = 90^\circ$. That gives BP in terms of ω, and the law of sines in triangle BPC gives a second expression.

Work it out. The right angle is at B ($3^2 + 4^2 = 5^2$). In right triangle APB (right angle at P), $BP = AB\sin\angle PAB = 3\sin\omega$.

In triangle BPC: $\angle PBC = \omega$ and $\angle PCB = \angle C - \angle PCA = C - \omega$, so $\angle BPC = 180^\circ - C$. The law of sines gives $\frac{BP}{\sin(C-\omega)} = \frac{BC}{\sin C}$, i.e. $3\sin\omega\sin C = 4\sin(C-\omega)$.

Here $\sin C = \frac{3}{5}$ and $\cos C = \frac{4}{5}$. So $\frac{9}{5}\sin\omega = 4\left(\frac{3}{5}\cos\omega - \frac{4}{5}\sin\omega\right)$, i.e. $9\sin\omega = 12\cos\omega - 16\sin\omega$, so $25\sin\omega = 12\cos\omega$ and $\tan\omega = \frac{12}{25}$.

Check. For any triangle this angle satisfies $\cot\omega = \cot A + \cot B + \cot C$. Here $\cot A = \frac{3}{4}$, $\cot B = 0$, $\cot C = \frac{4}{3}$, sum $\frac{25}{12}$, so $\tan\omega = \frac{12}{25}$. It is also less than $\tan C = \frac{3}{4}$ and $\tan A = \frac{4}{3}$, as it must be since ω is part of each angle.

23. Answer: (C) $\frac{1}{2}$. The idea. Being closer to the center O than to a vertex V is a half-space condition bounded by the perpendicular bisector plane of OV. With coordinates, the eight conditions

combine into a single octahedron, and the answer is the part of that octahedron inside the cube.

Work it out. Put the center at the origin, so the vertices are $V = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ and $|V|^2 = \frac{3}{4}$. For a point X, $|X|^2 < |X - V|^2$ is equivalent to $2\,X \cdot V < |V|^2$, i.e. $X \cdot V < \frac{3}{8}$. With $X = (x, y, z)$ this reads $\pm x \pm y \pm z < \frac{3}{4}$ for each choice of signs, which together say $|x| + |y| + |z| < \frac{3}{4}$.

So the set is the octahedron $|x| + |y| + |z| < \frac{3}{4}$ cut by the cube $|x|, |y|, |z| \le \frac{1}{2}$. The octahedron has volume $\frac{4}{3}\left(\frac{3}{4}\right)^3 = \frac{9}{16}$. It reaches $\frac{3}{4}$ along each axis, so it pokes $\frac{1}{4}$ beyond each of the six faces. The piece beyond the face $x = \frac{1}{2}$ is a square pyramid of height $\frac{1}{4}$ whose base, $|y| + |z| \le \frac{1}{4}$ in the plane $x = \frac{1}{2}$, is a square of area $2\left(\frac{1}{4}\right)^2 = \frac{1}{8}$ (it fits inside the face). Each tip has volume $\frac{1}{3} \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{1}{96}$.

Volume $= \frac{9}{16} - 6 \cdot \frac{1}{96} = \frac{54}{96} - \frac{6}{96} = \frac{1}{2}$.

Check. The tips cut off are disjoint (each lies beyond a different face), and the tip pyramids' bases $|y| + |z| \le \frac{1}{4}$ stay well inside the faces, so no edge of the cube interferes. The trap $\frac{9}{16}$ forgets to clip the octahedron to the cube.

24. Answer: (C) 778**. The idea.** The expression factors as $x^3+y^3+z^3-3xyz = s \cdot Q$ with $s = x+y+z$ and $Q = x^2 + y^2 + z^2 - xy - yz - zx$. Nearly equal triples make Q tiny, which produces lots of values, and working modulo 3 shows which values can never appear.

Work it out. Note $Q = \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]$.

Constructions, for any integer k: $(k, k, k+1)$ has $s = 3k+1$, $Q = 1$, value $3k+1$. $(k, k+1, k+1)$ has $s = 3k+2$, $Q = 1$, value $3k+2$. $(k-1, k, k+1)$ has $s = 3k$, $Q = 3$, value $9k$. So every n not divisible by 3, and every multiple of 9, is attainable.

Obstruction: $Q = s^2 - 3(xy + yz + zx) \equiv s^2 \pmod 3$. If $3 \mid n = sQ$, then $3 \mid s$ or $3 \mid Q \equiv s^2$; either way $3 \mid s$, and then $Q \equiv s^2 \equiv 0$ too. So $9 \mid n$. Numbers that are $\equiv 3$ or $6 \pmod 9$ are impossible.

In $1, \ldots, 1000$ there are 111 numbers $\equiv 3$ $(3, 12, \ldots, 999)$ and 111 numbers $\equiv 6$ $(6, 15, \ldots, 996)$ modulo 9, so the count is $1000 - 222 = 778$.

Check. 3 is impossible: we would need $s \cdot Q = 3$ with $3 \mid s$ and $3 \mid Q$. 9 comes from the triple $(0, 1, 2)$, which has $s = 3$, $Q = 3$ and value $0 + 1 + 8 - 0 = 9$, as claimed. The trap 667 forgets the multiples of 9; 889 removes only one of the two impossible residue classes; 1000 misses the obstruction entirely.

25. Answer: (D) 138690**. The idea.** Draw an arrow $x \to f(x)$. Following arrows from any point eventually enters a cycle, so the picture is a set of cycles with rooted trees draining into them. A point with $f(f(x)) = x$ must lie on a cycle of length 1 or 2 (a point off the cycles never returns to itself), so the condition says exactly: every cycle has length at least 3. Count by the set of cyclic points, then count the ways to hang the remaining points as trees, which is where the real idea is needed.

Work it out. Let k be the number of cyclic points and $m = 7 - k$ the rest. Choose the cyclic points ($\binom{7}{k}$ ways), arrange them into cycles of length ≥ 3 (D_k ways), then choose the arrows of the other m points so that they create no new cycle ($T_{k,m}$ ways).

Cycle arrangements: $D_3 = 2$, $D_4 = 3! = 6$, $D_5 = 4! = 24$, $D_6 = 5! + \frac{1}{2}\binom{6}{3} \cdot 2 \cdot 2 = 120 + 40 = 160$ (a 6-cycle, or two 3-cycles), $D_7 = 6! + \binom{7}{3} \cdot 2 \cdot 3! = 720 + 420 = 1140$ (a 7-cycle, or a 3-cycle and a 4-cycle).

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Attachments: $T_{k,m} = k \cdot 7^{m-1}$ (for $m \geq 1$; $T_{7,0} = 1$). The arrows out of the m free points, with no cycle among them, form a forest on all 7 points in which every tree is rooted at one of the k cyclic points. By the rooted-forest form of Cayley's formula, the number of forests on n labelled vertices with k specified roots is $k\, n^{n-k-1}$. So $T_{3,4} = 3 \cdot 7^3 = 1029$, $T_{4,3} = 4 \cdot 49 = 196$, $T_{5,2} = 5 \cdot 7 = 35$, $T_{6,1} = 6$.

Sum: $\binom{7}{3}\cdot 2\cdot 1029 + \binom{7}{4}\cdot 6\cdot 196 + \binom{7}{5}\cdot 24\cdot 35 + \binom{7}{6}\cdot 160\cdot 6 + 1140 = 72030+41160+17640+6720+1140 = 138690$.

Check. Small cases of the forest formula by hand: one free point has $n-1 = k$ choices ($k \cdot n^0 = k$), and two free points u, v with $n = k+2$: of the n^2 maps, $2n-1$ send u or v to itself and one more is $u \to v \to u$, leaving $n^2 - 2n = kn$, as the formula says. Also a brute-force run over all $7^7 = 823543$ functions agrees. The traps: 1140 counts only permutations; 39570 lets the free points point only at cycle points (k^m instead of kn^{m-1}); 138270 forgets the $3 + 4$ split in D_7; $279936 = 6^7$ only rules out fixed points.

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