MathCastle

AMC 12

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

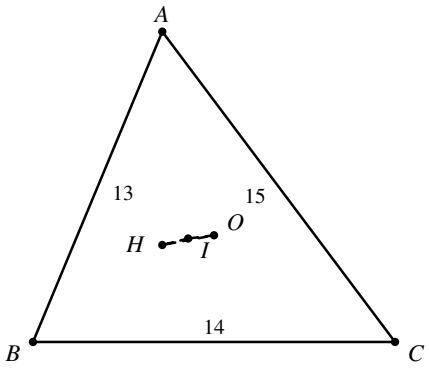
1. Among all complex numbers z satisfying $|z-3i|+|z-4|=5$, what is the smallest possible value of $|z|$?

(A) $\frac{12}{7}$ **(B)** $\frac{12}{5}$ **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** $\frac{144}{25}$

2. A word is any string of 7 letters, each of which is A, B or C. How many words have the property that every A in the word is immediately followed by a B? (For example, $CABBABC$ has the property, while $ABACBBB$ and $CCBBBBA$ do not.)

(A) 408 **(B)** 577 **(C)** 792 **(D)** 896 **(E)** 985

3. Triangle ABC has side lengths $AB = 13$, $BC = 14$ and $CA = 15$. Let I be the centre of its inscribed circle, H its orthocentre (the point where its three altitudes meet), and O the centre of the circle through its three vertices. What is the area of triangle IHO?

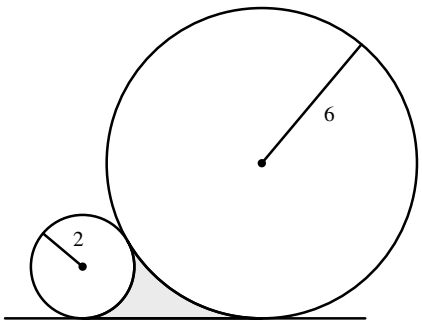


(A) $\frac{1}{48}$ **(B)** $\frac{1}{32}$ **(C)** $\frac{1}{24}$ **(D)** $\frac{1}{16}$ **(E)** $\frac{1}{8}$

4. How many ordered pairs (x, k), where k is an integer with $0 \le k \le 10$ and x is a real number, satisfy $\left|x^2 - 6|x| + 5\right| = k$?

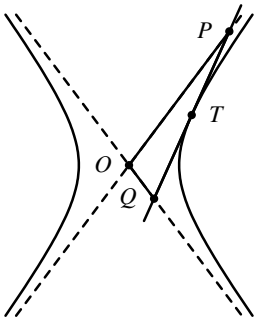
(A) 43 **(B)** 46 **(C)** 47 **(D)** 48 **(E)** 49

5. Two circles, of radii 2 and 6, are externally tangent (they touch at one point and neither lies inside the other). A line is tangent to both circles, touching them at two different points, with both circles on the same side of it. What is the area of the region bounded by this line and the two circles (the small curved-sided gap enclosed by the line and the arcs of the two circles)?



(A) $16\sqrt{3} - 8\pi$ **(B)** $16\sqrt{3} - \frac{22\pi}{3}$ **(C)** $16\sqrt{3} - \frac{20\pi}{3}$ **(D)** $32 - \frac{22\pi}{3}$ **(E)** $32\sqrt{3} - \frac{22\pi}{3}$

6. The point $T = \left(\frac{15}{4}, 3\right)$ lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. The line tangent to the hyperbola at T meets the two asymptotes of the hyperbola, the lines $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$, at points P and Q. Let O be the origin. What is the area of triangle OPQ?



(A) 6 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 24

7. How many angles θ with $0^\circ \le \theta < 360^\circ$ satisfy $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

8. Ten beads, 6 red and 4 blue, are threaded onto a string in a uniformly random order (all $\binom{10}{4}$ colour patterns equally likely). A bead is called *lonely* if no bead next to it has its colour. The two end beads each have only one neighbour; an end bead is lonely when its single neighbour has the other colour. What is the expected number of lonely beads?

(A) $\frac{32}{15}$ **(B)** $\frac{8}{3}$ **(C)** $\frac{72}{25}$ **(D)** $\frac{16}{5}$ **(E)** $\frac{16}{3}$

9. For a positive integer n, let $Z(n)$ be the number of zeros at the end of $n!$ written in base ten; for example $Z(10) = 2$ because $10! = 3628800$. How many integers k with $1 \le k \le 100$ are *not* equal to $Z(n)$ for any positive integer n?

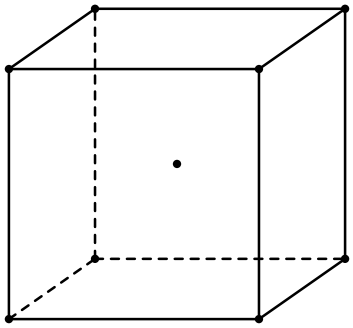
(A) 19 **(B)** 20 **(C)** 21 **(D)** 22 **(E)** 24

10. How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain more odd numbers than even numbers and have an even sum of elements?

(A) 190 **(B)** 193 **(C)** 196 **(D)** 315 **(E)** 386

11. Consider the nine points consisting of the eight vertices of a cube and the centre of the cube. Four of these nine points are chosen at random, all $\binom{9}{4} = 126$ choices of four points equally likely. What is the probability that the four chosen points lie in a single plane?

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(A) $\frac{2}{21}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{6}{35}$ **(D)** $\frac{5}{21}$ **(E)** $\frac{2}{7}$

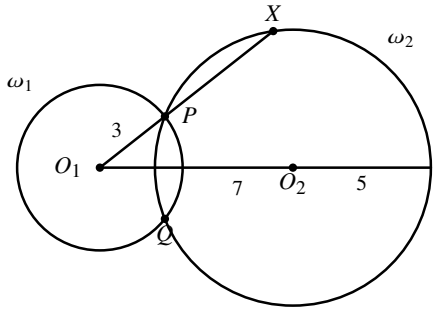
12. Compute $\displaystyle\sum_{k=0}^{12} \binom{12}{k} \cos^2(30k^\circ) = \binom{12}{0}\cos^2 0^\circ + \binom{12}{1}\cos^2 30^\circ + \binom{12}{2}\cos^2 60^\circ + \cdots + \binom{12}{12}\cos^2 360^\circ$.

(A) 729 **(B)** $\frac{3367}{2}$ **(C)** 2048 **(D)** 2132 **(E)** $\frac{4825}{2}$

13. A librarian places six different books on three identical, unlabelled shelves. Every shelf must receive at least one book, and the left-to-right order of the books on each shelf matters, but the shelves themselves cannot be told apart (so two placements that differ only by swapping the entire contents of two shelves count as the same). In how many different ways can the books be placed?

(A) 90 **(B)** 720 **(C)** 1200 **(D)** 2400 **(E)** 7200

14. Circle ω_1 has centre O_1 and radius 3, and circle ω_2 has centre O_2 and radius 5, where $O_1O_2 = 7$. The two circles meet at the points P and Q. The line O_1P meets ω_2 at a second point $X \neq P$. What is the length QX?



(A) $\frac{55}{7}$ **(B)** 8 **(C)** $\sqrt{73}$ **(D)** $5\sqrt{3}$ **(E)** $\frac{\sqrt{4129}}{7}$

15. There is exactly one real number $x \geq 0$ satisfying $x = \sqrt{2 - \sqrt{2 + x}}$. What is x?

(A) $\frac{3-\sqrt{5}}{2}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** 1 **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

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16. Find the sum of the squares of all real numbers x that satisfy $\sqrt[3]{x+1} + \sqrt[3]{x-1} = \sqrt[3]{5x}$. (Here $\sqrt[3]{t}$ denotes the real cube root, which is defined for negative t as well.)

(A) $\frac{5}{4}$ **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** 5

17. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many ordered pairs (A, B) of subsets of S satisfy $A \cap B = \varnothing$ and $|A| = |B|$? (Either set may be empty; (A, B) and (B, A) count separately when $A \neq B$.)

(A) 393 **(B)** 554 **(C)** 1037 **(D)** 1106 **(E)** 1107

18. For how many primes $p < 100$ is there an integer n such that $n^4 + 1$ is divisible by p?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 11 **(E)** 12

19. A complex number z satisfies $|z^2 + 1| = |z|$. What is the largest possible value of $|z|$?

(A) $\frac{\sqrt{5}-1}{2}$ **(B)** 1 **(C)** $\sqrt{2}$ **(D)** $\frac{1+\sqrt{5}}{2}$ **(E)** 2

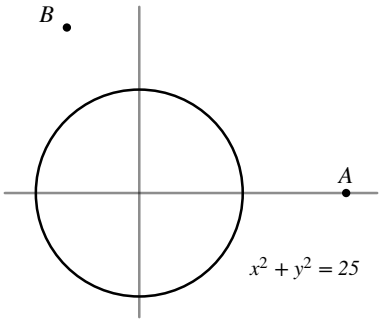
20. A polynomial $f(x)$ with real coefficients and degree at most 2 satisfies $|f(x)| \leq 1$ for every real x with $-1 \leq x \leq 1$. What is the largest possible value of $f(2)$?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

21. Compute $\tan^2 9^\circ + \tan^2 45^\circ + \tan^2 81^\circ + \tan^2 117^\circ + \tan^2 153^\circ$. (These are the angles $9^\circ + 36^\circ k$ for $k = 0, 1, 2, 3, 4$.)

(A) 5 **(B)** 25 **(C)** 35 **(D)** 45 **(E)** 65

22. Point P moves on the circle $x^2 + y^2 = 25$. Let $A = (10, 0)$ and $B = \left(-\frac{7}{2}, 8\right)$. What is the smallest possible value of $PA + 2\,PB$?



(A) 10 **(B)** $\frac{\sqrt{985}}{2}$ **(C)** 18 **(D)** 20 **(E)** 25

23. For each positive integer k, write down the integer $\lfloor k\sqrt{2} \rfloor$, the greatest integer not exceeding $k\sqrt{2}$. How many integers n with $1 \leq n \leq 1000$ never appear in this list?

(A) 292 **(B)** 293 **(C)** 294 **(D)** 414 **(E)** 707

24. How many subsets of $\{1, 2, 3, \ldots, 12\}$ (including the empty set) contain no two consecutive integers and no two elements whose sum is 13?

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(A) 99 **(B)** 238 **(C)** 239 **(D)** 377 **(E)** 729

25. A fair coin is flipped 12 times, producing a string of 12 letters, each H or T. Each of the eight patterns HHH, HHT, HTH, HTT, THH, THT, TTH, TTT is said to appear if it occurs at least once as three consecutive letters of the string. What is the expected number of the eight patterns that appear?

(A) $\frac{2391}{512}$ **(B)** $\frac{2939}{512}$ **(C)** $\frac{2977}{512}$ **(D)** $\frac{3015}{512}$ **(E)** $\frac{3095}{512}$

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Answer Key — AMC 12 — Hard Edition Practice Contest 3

1. (B) $\frac{12}{5}$

2. (A) 408

3. (D) $\frac{1}{16}$

4. (C) 47

5. (B) $16\sqrt{3} - \frac{22\pi}{3}$

6. (C) 12

7. (A) 7

8. (D) $\frac{16}{5}$

9. (A) 19

10. (A) 190

11. (E) $\frac{2}{7}$

12. (E) $\frac{4825}{2}$

13. (C) 1200

14. (A) $\frac{55}{7}$

15. (B) $\frac{\sqrt{5}-1}{2}$

16. (C) $\frac{5}{2}$

17. (E) 1107

18. (B) 6

19. (D) $\frac{1+\sqrt{5}}{2}$

20. (E) 7

21. (D) 45

22. (D) 20

23. (B) 293

24. (C) 239

25. (E) $\frac{3095}{512}$

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Solutions — AMC 12 — Hard Edition Practice Contest 3

1. Answer: (B) $\frac{12}{5}$**. The idea.** The equation looks like an ellipse with foci $3i$ and 4, but the distance between the foci is $|4 - 3i| = 5$, exactly the constant sum. By the triangle inequality $|z - 3i| + |z - 4| \geq |(z - 3i) - (z - 4)| = 5$ always, with equality only when z lies on the segment from $3i$ to 4. So the "ellipse" is flattened into that segment, and we want the point of the segment closest to the origin.

Work it out. In the plane, the segment joins $(0, 3)$ and $(4, 0)$ and lies on the line $\frac{x}{4} + \frac{y}{3} = 1$, i.e. $3x + 4y = 12$. The distance from the origin to this line is $\frac{|0+0-12|}{\sqrt{3^2+4^2}} = \frac{12}{5}$. The foot of that perpendicular is $\left(\frac{36}{25}, \frac{48}{25}\right)$, which does lie between the endpoints (both coordinates are positive), so the minimum over the segment is the distance to the line: $|z|_{\min} = \frac{12}{5}$.

Equivalently, the triangle with vertices 0, 4, $3i$ is a 3-4-5 right triangle, and the answer is its altitude to the hypotenuse, $\frac{3 \cdot 4}{5}$.

Check. Parametrise the segment as $z = (4 - 4t) + 3t\,i$, $0 \leq t \leq 1$: $|z|^2 = 16(1 - t)^2 + 9t^2$ is smallest at $t = \frac{16}{25}$, where it equals $\frac{144}{25}$, so $|z| = \frac{12}{5}$. Reporting $|z|^2$ gives the trap $\frac{144}{25}$; taking the nearer endpoint $3i$ gives 3; taking the midpoint $2 + \frac{3}{2}i$ gives $\frac{5}{2}$; the slip $\frac{3 \cdot 4}{3+4}$ gives $\frac{12}{7}$.

2. Answer: (A) 408**. The idea.** Glue each A to the B that must follow it. Then a valid word is exactly a sequence of three kinds of tiles, B and C of length 1 and AB of length 2, and the tiling is unique (every A starts an AB tile, and every other letter is its own tile). Counting tilings of length n gives a recursion by looking at the first tile.

Work it out. Let $f(n)$ be the number of valid words of length n. The first tile is B or C (leaving a valid word of length $n - 1$) or AB (leaving one of length $n - 2$), so $f(n) = 2f(n - 1) + f(n - 2)$, $f(0) = 1$, $f(1) = 2$. Then $f(2) = 5$, $f(3) = 12$, $f(4) = 29$, $f(5) = 70$, $f(6) = 169$, $f(7) = 408$.

Check. Length 2 by hand: BB, BC, CB, CC, AB, which is 5. The count for length 8 is 985, an off-by-one trap. Allowing a final unmatched A adds $f(6)$ more words and gives 577; reading the condition as "every A is followed by a letter other than A" gives 896, and as "every A has a B on at least one side" gives 792.

3. Answer: (D) $\frac{1}{16}$**. The idea.** Put the triangle in coordinates with BC on the x-axis; the 13-14-15 triangle has the altitude 12 from A, so every centre has rational coordinates. The incentre is the side-length-weighted average of the vertices, the circumcentre lies on the perpendicular bisector of BC, and the orthocentre lies on the altitude from A.

Work it out. Take $B = (0, 0)$, $C = (14, 0)$, $A = (x, y)$ with $x^2 + y^2 = 169$ and $(x - 14)^2 + y^2 = 225$. Subtracting, $28x = 140$, so $x = 5$, $y = 12$: $A = (5, 12)$.

Incentre: with $a = BC = 14$, $b = CA = 15$, $c = AB = 13$, $I = \frac{aA+bB+cC}{a+b+c} = \frac{14(5,12)+15(0,0)+13(14,0)}{42} = \frac{(252,168)}{42} = (6, 4)$. (Check: the inradius is area over semiperimeter, $\frac{84}{21} = 4$.)

Circumcentre: $O = (7, k)$ with $OB = OA$: $49 + k^2 = 4 + (12 - k)^2$, so $24k = 99$ and $k = \frac{33}{8}$. Thus $O = \left(7, \frac{33}{8}\right)$.

Orthocentre: H is on the altitude $x = 5$. The altitude from B is perpendicular to CA, whose direction is $(-9, 12)$, so it is the line $-9x + 12y = 0$, i.e. $y = \frac{3}{4}x$. At $x = 5$: $H = \left(5, \frac{15}{4}\right)$.

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From I: $H - I = \left(-1, -\frac{1}{4}\right)$ and $O - I = \left(1, \frac{1}{8}\right)$. The area is $\frac{1}{2}\left|(-1)\cdot\frac{1}{8} - \left(-\frac{1}{4}\right)\cdot 1\right| = \frac{1}{2}\cdot\frac{1}{8} = \frac{1}{16}$.

Check. Euler's line: the centroid is $G = \left(\frac{19}{3}, 4\right)$ and indeed $H = 3G - 2O = \left(19 - 14, 12 - \frac{33}{4}\right) = \left(5, \frac{15}{4}\right)$. Using G in place of H gives triangle IGO with area $\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{8} = \frac{1}{48}$ (a trap); since G divides OH in the ratio $1:2$, the triangle IGO is exactly one third of IHO, consistent with $\frac{1}{16}$. Forgetting the $\frac{1}{2}$ gives $\frac{1}{8}$.

4. Answer: (C) 47**. The idea.** Do not solve eleven equations. Sketch the graph of $g(x) = \left|x^2 - 6|x| + 5\right|$ once; then the number of solutions for a given k is the number of times the horizontal line $y = k$ meets the graph. Because $g(-x) = g(x)$, it is enough to understand $x \geq 0$ and mirror it, taking care with the one point, $x = 0$, that is its own mirror image.

Work it out. For $x \geq 0$, $g(x) = |(x-1)(x-5)|$. The parabola $(x-1)(x-5)$ starts at 5 when $x = 0$, crosses zero at $x = 1$, dips to -4 at $x = 3$, and crosses zero again at $x = 5$. Taking the absolute value folds the dip up, so on $x \geq 0$ the graph falls from 5 to 0 on $[0, 1]$, rises to a peak of 4 at $x = 3$, falls to 0 at $x = 5$, and then rises forever.

Counting intersections with $y = k$ on $x \geq 0$: for $k = 0$, two points ($x = 1, 5$); for $k = 1, 2, 3$, four points (one on each of the four monotone pieces); for $k = 4$, three points (the peak $x = 3$ is touched once, plus one point on $[0, 1]$ and one beyond 5); for $k = 5$, two points, namely $x = 0$ and $x = 6$; for $6 \leq k \leq 10$, one point (beyond 5).

Now mirror. Every solution with $x > 0$ has a partner $-x$; the solution $x = 0$ (only at $k = 5$) does not. So the totals are 4 for $k = 0$; 8 each for $k = 1, 2, 3$; 6 for $k = 4$; $2 \cdot 2 - 1 = 3$ for $k = 5$; and 2 each for $k = 6, \ldots, 10$. The number of pairs is $4 + 3 \cdot 8 + 6 + 3 + 5 \cdot 2 = 47$.

Check. At $k = 5$: $x^2 - 6|x| + 5 = 5$ gives $|x|(|x| - 6) = 0$, so $x \in \{0, 6, -6\}$, and $x^2 - 6|x| + 5 = -5$ gives $|x|^2 - 6|x| + 10 = 0$, which has no real roots: three solutions, as claimed. At $k = 4$: $|x| = 3$ (double root of $|x|^2 - 6|x| + 9$) or $|x| = 3 \pm \sqrt{8}$, so $|x| \in \{3, 3 - 2\sqrt{2}, 3 + 2\sqrt{2}\}$ and six values of x. Counting $x = 0$ twice gives 48; forgetting it gives 46; counting the peak at $k = 4$ as two crossings gives 49.

5. Answer: (B) $16\sqrt{3} - \frac{22\pi}{3}$**. The idea.** Join the two centres and drop the radii to the two points of tangency. This makes a right trapezoid (both radii are perpendicular to the tangent line), and the gap is that trapezoid minus a sector of each circle. Everything hinges on the trapezoid's angles.

Work it out. Let the centres be A (radius 2) and B (radius 6), touching the line at S and T. Then $AB = 8$ and $AS \parallel BT$, both perpendicular to ST. Drop a perpendicular from A to BT: it has length ST and cuts off $6 - 2 = 4$ from BT, so $ST = \sqrt{8^2 - 4^2} = 4\sqrt{3}$.

The trapezoid $ASTB$ has parallel sides 2 and 6 and height $4\sqrt{3}$, so its area is $\frac{2+6}{2}\cdot 4\sqrt{3} = 16\sqrt{3}$.

In that right triangle with hypotenuse $AB = 8$ and leg 4 along BT, the angle at B has cosine $\frac{4}{8} = \frac{1}{2}$, so $\angle ABT = 60^\circ$ and hence $\angle BAS = 180^\circ - 60^\circ = 120^\circ$ (co-interior angles between the parallels AS and BT).

The two circles touch on segment AB, so the parts of the trapezoid not in the gap are exactly the sector of the large circle with angle 60° and the sector of the small circle with angle 120°: $\frac{60}{360}\pi\cdot 6^2 + \frac{120}{360}\pi\cdot 2^2 = 6\pi + \frac{4\pi}{3} = \frac{22\pi}{3}$. The gap has area $16\sqrt{3} - \frac{22\pi}{3}$.

Check. Numerically $16\sqrt{3} \approx 27.71$ and $\frac{22\pi}{3} \approx 23.04$, leaving about 4.67, a small positive area as it should be (a grid count of the region gives the same). The smaller circle gets the *larger* angle; giving both circles 60° is the trap $16\sqrt{3} - \frac{20\pi}{3}$. Taking $ST = AB = 8$ gives $32 - \frac{22\pi}{3}$.

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6. Answer: (C) 12**. The idea.** Factor the hyperbola along its asymptotes: $\frac{x^2}{9}-\frac{y^2}{16} = \left(\frac{x}{3}-\frac{y}{4}\right)\left(\frac{x}{3}+\frac{y}{4}\right)$. In the coordinates $u = \frac{x}{3}-\frac{y}{4}$, $v = \frac{x}{3}+\frac{y}{4}$ the hyperbola is $uv = 1$, the asymptotes are the axes, and the problem becomes one about the curve $uv = 1$, where the answer does not depend on the point at all.

Work it out. For $uv = 1$, the tangent at (u_0, v_0) is $\frac{u}{2u_0}+\frac{v}{2v_0} = 1$: it cuts the axes at $(2u_0, 0)$ and $(0, 2v_0)$, so T is the midpoint of the cut-off piece and the triangle with the origin has uv-area $\frac{1}{2}\cdot 2u_0 \cdot 2v_0 = 2$.

Converting back: $x = \frac{3}{2}(u+v)$, $y = 2(v-u)$, a linear map with determinant $\left|\frac{3}{2}\cdot 2 - \frac{3}{2}\cdot(-2)\right| = 6$. Areas scale by 6, so the area of OPQ is $6 \cdot 2 = 12$.

Directly for our T: the tangent is $\frac{(15/4)x}{9}-\frac{3y}{16} = 1$, i.e. $\frac{5x}{12}-\frac{3y}{16} = 1$. On $y = \frac{4}{3}x$: $\frac{5x}{12}-\frac{x}{4} = \frac{x}{6} = 1$, so $P = (6, 8)$. On $y = -\frac{4}{3}x$: $\frac{5x}{12}+\frac{x}{4} = \frac{2x}{3} = 1$, so $Q = \left(\frac{3}{2}, -2\right)$. The area is $\frac{1}{2}\left|6\cdot(-2) - 8\cdot\frac{3}{2}\right| = \frac{1}{2}\cdot 24 = 12$.

Check. T is the midpoint of PQ: $\frac{6+3/2}{2} = \frac{15}{4}$ and $\frac{8-2}{2} = 3$. The tangent at the vertex $(3, 0)$ is $x = 3$, which cuts the asymptotes at $(3, \pm 4)$: area $\frac{1}{2}\cdot 8 \cdot 3 = 12$ again, which is ab for this hyperbola. The trap 6 forgets that P and Q lie on opposite sides of the x-axis.

7. Answer: (A) 7**. The idea.** Pair the terms symmetrically, θ with 4θ and 2θ with 3θ: every pair averages to $\frac{5\theta}{2}$, so sum-to-product gives both sides the same factor, and the equation factors into two simple pieces. Then watch for an angle that solves both pieces.

Work it out. Sum-to-product gives $\sin\theta + \sin 4\theta + \sin 2\theta + \sin 3\theta = 2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} + \cos\frac{\theta}{2}\right)$ and the same with $\cos\frac{5\theta}{2}$ in front for the cosines. Also $\cos\frac{3\theta}{2} + \cos\frac{\theta}{2} = 2\cos\theta\cos\frac{\theta}{2}$. So the equation is $4\cos\theta\cos\frac{\theta}{2}\left(\sin\frac{5\theta}{2} - \cos\frac{5\theta}{2}\right) = 0$.

Case 1: $\cos\theta = 0$: $\theta = 90°, 270°$.

Case 2: $\cos\frac{\theta}{2} = 0$ with $0° \le \frac{\theta}{2} < 180°$: $\theta = 180°$.

Case 3: $\tan\frac{5\theta}{2} = 1$ (here $\cos\frac{5\theta}{2} \neq 0$, else the sine would be ± 1). As θ runs over $[0°, 360°)$, $\frac{5\theta}{2}$ runs over $[0°, 900°)$ and equals $45° + 180°k$ for $k = 0, 1, 2, 3, 4$, giving $\theta = 18° + 72°k$: $18°, 90°, 162°, 234°, 306°$.

The angle $90°$ appears in both Case 1 and Case 3, so the distinct solutions are $18°, 90°, 162°, 180°, 234°, 270°, 306°$: seven of them.

Check. At $\theta = 90°$ the sines are $1, 0, -1, 0$ and the cosines $0, -1, 0, 1$, both summing to 0. At $\theta = 180°$ both sides are 0 as well. At $\theta = 0°$ the left side is 0 and the right side is 4, so $0°$ is not a solution. Counting $90°$ twice gives the trap 8; squaring $\sin\frac{5\theta}{2} = \cos\frac{5\theta}{2}$ (so solving $\tan\frac{5\theta}{2} = \pm 1$) lets in the spurious angles $54°, 126°, \ldots$ and gives 11.

8. Answer: (D) $\frac{16}{5}$**. The idea.** Linearity of expectation: the expected number of lonely beads is the sum, over the ten positions, of the probability that the bead in that position is lonely. Each probability only involves two or three positions, and the colours are drawn without replacement.

Work it out. *An end position* is lonely when it and its neighbour differ. The chance that two given positions show red then blue is $\frac{6}{10}\cdot\frac{4}{9}$, and blue then red is the same, so the probability is $2\cdot\frac{24}{90} = \frac{8}{15}$.

An interior position is lonely when both neighbours have the other colour. A red bead flanked by two blues: $\frac{6}{10}\cdot\frac{4}{9}\cdot\frac{3}{8} = \frac{1}{10}$. A blue bead flanked by two reds: $\frac{4}{10}\cdot\frac{6}{9}\cdot\frac{5}{8} = \frac{1}{6}$. Total $\frac{1}{10}+\frac{1}{6} = \frac{4}{15}$.

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There are 2 end positions and 8 interior ones, so the expected number is $2 \cdot \frac{8}{15} + 8 \cdot \frac{4}{15} = \frac{16+32}{15} = \frac{16}{5}$.

Check. Forgetting the ends gives $\frac{32}{15}$; treating the ends like interior beads gives $10 \cdot \frac{4}{15} = \frac{8}{3}$; colouring each bead independently red with probability $\frac{3}{5}$ (ignoring that the counts are fixed) gives $\frac{72}{25}$. A direct average over all 210 patterns confirms $\frac{16}{5}$; for instance the pattern RBRBRBRBRR has 8 lonely beads, while RRRRRRBBBB has none.

9. Answer: (A) 19**. The idea.** $Z(n)$ is the exponent of 5 in $n!$ (there are always more 2s), so it never decreases and changes only when n passes a multiple of 5. Going from $(n-1)!$ to $n!$ adds exactly $v_5(n)$ fives, the exponent of 5 in n. A jump by $v_5(n)$ skips $v_5(n) - 1$ values: one for each multiple of 25, one more for each multiple of 125, and so on. So count the skips that land at or below 100.

Work it out. By Legendre's formula $Z(n) = \lfloor n/5 \rfloor + \lfloor n/25 \rfloor + \lfloor n/125 \rfloor + \cdots$. Then $Z(400) = 80 + 16 + 3 = 99$ and $Z(405) = 100$, while the next skip happens at $n = 425$, where Z jumps from $Z(424) = 84 + 16 + 3 = 103$ to 105, beyond 100. So the missing values in $1, \ldots, 100$ come from the jumps at $n \le 400$.

The multiples of 25 up to 400 number 16, each skipping one value; the multiples of 125 among them, 125, 250, 375, skip one extra each. (No multiple of 625 is in range.) The count is 16+3 = 19.

Check. The first skipped value is 5: $Z(24) = 4$ but $Z(25) = 6$. At $n = 125$, Z jumps from 28 to 31, skipping both 29 and 30. The last skip in range is at $n = 400$, where Z goes from 97 to 99, skipping 98. Counting each extra skip at a multiple of 125 twice gives 22; guessing that one value in five is missed gives 20; running the count up to $n = 500$ (since $\frac{500}{5} = 100$) gives $20 + 4 = 24$.

10. Answer: (A) 190**. The idea.** Only the odd elements affect the parity of the sum: the sum is even exactly when the subset has an even number of odd elements. So split by a, the number of odd elements, and b, the number of even elements; the five odds and the five evens are chosen independently.

Work it out. We need a even and $b < a$. Since $a > b \ge 0$, $a \ne 0$, so $a \in \{2, 4\}$ (there are only five odd numbers).

$a = 2$: $\binom{5}{2} = 10$ ways to pick the odds, and $b \in \{0, 1\}$ gives $\binom{5}{0} + \binom{5}{1} = 6$ ways to pick the evens: 60 subsets.

$a = 4$: $\binom{5}{4} = 5$ ways, and $b \in \{0, 1, 2, 3\}$ gives $1 + 5 + 10 + 10 = 26$ ways: 130 subsets.

Total: $60 + 130 = 190$.

Check. Without the parity condition, the counts for $a = 1, 3, 5$ are $5 \cdot 1 = 5$, $10 \cdot 16 = 160$ and $1 \cdot 31 = 31$, i.e. 196 subsets with odd sum (a trap if parity is reversed), and $190 + 196 = 386$ subsets with more odds than evens in all (another trap). Assuming the parity splits evenly gives 193; allowing $b = a$ gives $10 \cdot 16 + 5 \cdot 31 = 315$.

11. Answer: (E) $\frac{2}{7}$**. The idea.** Split by whether the centre is chosen. Without it, we need the planes containing four cube vertices. With it, use the central symmetry: a plane through the centre that contains a vertex also contains the opposite vertex, so it contains vertices in opposite pairs.

Work it out. *Four vertices.* A plane that contains four vertices of the cube is either a face (6 of them) or a "diagonal rectangle" through two opposite edges (6 of them, one for each pair of opposite edges). No other plane works: if the four vertices do not lie on one face, some two of

them are opposite corners of a face or of the cube, and following the coordinates shows the plane must be one of the diagonal rectangles. Each such plane contains exactly 4 vertices, so there are 12 coplanar sets.

The centre and three vertices. Such a set is coplanar exactly when the plane through the three vertices passes through the centre O. That plane is then symmetric under the point reflection through O, so it also contains the vertices opposite the three chosen ones; it therefore contains at least four vertices and passes through O. Faces miss O, but all 6 diagonal rectangles pass through it. So the three vertices are 3 of the 4 vertices of a diagonal rectangle: $6 \cdot \binom{4}{3} = 24$ sets. (Different rectangles share at most two vertices, so nothing is counted twice.)

Total: $12 + 24 = 36$ of 126, a probability of $\frac{36}{126} = \frac{2}{7}$.

Check. With the cube $\{0, 2\}^3$ and centre $(1, 1, 1)$: the vertices $(0, 0, 0)$, $(2, 2, 0)$, $(2, 2, 2)$ lie on the rectangle $x = y$, which contains $(1, 1, 1)$. Ignoring the centre sets gives $\frac{12}{126} = \frac{2}{21}$; forgetting the diagonal rectangles among the vertices gives $\frac{30}{126} = \frac{5}{21}$; the answer for vertices alone would be $\frac{12}{70} = \frac{6}{35}$.

12. Answer: (E) $\frac{4825}{2}$**. The idea.** Power-reduce $\cos^2$ to turn the sum into $\sum \binom{12}{k}$ plus $\sum \binom{12}{k} \cos(60k^\circ)$. The second sum is the real part of a binomial expansion, $(1 + \omega)^{12}$ with $\omega = \cos 60^\circ + i \sin 60^\circ$, and $1 + \omega$ has a clean polar form.

Work it out. Since $\cos^2 x = \frac{1+\cos 2x}{2}$, $S = \frac{1}{2} \sum_{k=0}^{12} \binom{12}{k} + \frac{1}{2} \sum_{k=0}^{12} \binom{12}{k} \cos(60k^\circ) = \frac{1}{2} \cdot 2^{12} + \frac{1}{2} C$. For C: by the binomial theorem and de Moivre, $\sum_k \binom{12}{k} \omega^k = (1 + \omega)^{12}$, and C is its real part. Now $1 + \omega = \frac{3}{2} + \frac{\sqrt{3}}{2} i = \sqrt{3}\,(\cos 30^\circ + i \sin 30^\circ)$, so $(1 + \omega)^{12} = (\sqrt{3})^{12}\,(\cos 360^\circ + i \sin 360^\circ) = 729$. Thus $C = 729$ and $S = \frac{4096+729}{2} = \frac{4825}{2}$.

Check. The values $\cos^2(30k^\circ)$ cycle through $1, \frac{3}{4}, \frac{1}{4}, 0, \frac{1}{4}, \frac{3}{4}$ with period 6. Group the binomial coefficients $1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1$ by k mod 6: residue 0: $1+924+1 = 926$; residues 1, 5: $12+792+792+12 = 1608$ (weight $\frac{3}{4}$); residues 2, 4: $66+495+495+66 = 1122$ (weight $\frac{1}{4}$); residue 3: 440 (weight 0). So $S = 926 + 1206 + 280.5 = 2412.5 = \frac{4825}{2}$. Using $\cos^2 x = \frac{1-\cos 2x}{2}$ by mistake gives $\frac{3367}{2}$; stopping at C gives 729; dropping the weight-$\frac{1}{4}$ group in the check gives $926 + 1206 = 2132$.

13. Answer: (C) 1200**. The idea.** Put all six books in one row, then cut the row into three nonempty consecutive blocks: each block becomes one shelf, in its left-to-right order. This counts placements on three shelves listed in some order; because the shelves are identical and nonempty, every real placement is counted once for each of the 3! orders in which its three shelves can be listed.

Work it out. There are $6! = 720$ rows. A row of 6 books has 5 gaps, and choosing 2 of them to cut gives three nonempty blocks: $\binom{5}{2} = 10$ ways. So there are $720 \cdot 10 = 7200$ ordered lists (first shelf, second shelf, third shelf). Each placement on identical shelves appears as exactly $3! = 6$ of these lists, since the three shelves hold different (nonempty) sets of books. The answer is $7200/6 = 1200$.

Check. Count by shelf sizes. Sizes $\{4, 1, 1\}$: choose the 4 books (15 ways) and order them (24), the two singletons are then determined: 360. Sizes $\{3, 2, 1\}$: $\binom{6}{3}\binom{3}{2} = 60$ ways to split, times $3! \cdot 2! = 12$ orders: 720. Sizes $\{2, 2, 2\}$: $\frac{1}{3!}\binom{6}{2}\binom{4}{2} = 15$ splits, times $2^3 = 8$: 120. Total $360 + 720 + 120 = 1200$. (Forgetting that the shelves are identical gives 7200; ignoring order on the shelves gives 90.)

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14. Answer: (A) $\frac{55}{7}$. The idea. Triangle O_1PO_2 has sides $3, 5, 7$, and the law of cosines shows its angle at P is $120°$. That angle tells you exactly where the line O_1P cuts ω_2 again, which gives O_1X. Then QX follows from the law of cosines in triangle O_1QX, because Q is the mirror image of P in the line O_1O_2.

Work it out. In triangle O_1PO_2: $\cos\angle O_1PO_2 = \frac{9+25-49}{2\cdot3\cdot5} = -\frac{1}{2}$, so $\angle O_1PO_2 = 120°$. Past P the line O_1P makes a $60°$ angle with PO_2, so the foot of the perpendicular from O_2 to this line lies $5\cos 60° = \frac{5}{2}$ beyond P. The chord PX of ω_2 is bisected by that foot, so $PX = 5$ and $O_1X = 3 + 5 = 8$.

Let $\theta = \angle PO_1O_2$; then $\cos\theta = \frac{9+49-25}{2\cdot3\cdot7} = \frac{11}{14}$. By symmetry in O_1O_2, $\angle QO_1X = \angle QO_1P = 2\theta$, with $\cos 2\theta = 2\cdot\frac{121}{196} - 1 = \frac{23}{98}$. So $QX^2 = 3^2 + 8^2 - 2\cdot3\cdot8\cdot\frac{23}{98} = 73 - \frac{552}{49} = \frac{3025}{49}$, and $QX = \frac{55}{7}$.

Check. Coordinates: $O_1 = (0,0)$, $O_2 = (7,0)$. Subtracting the circle equations gives $x = \frac{33}{14}$, so $P, Q = \left(\frac{33}{14}, \pm\frac{15\sqrt{3}}{14}\right)$. Then $X = \frac{8}{3}P = \left(\frac{44}{7}, \frac{20\sqrt{3}}{7}\right)$, and indeed $\left(\frac{44}{7} - 7\right)^2 + \frac{1200}{49} = \frac{25+1200}{49} = 25$. Finally $QX^2 = \left(\frac{55}{14}\right)^2 + \left(\frac{55\sqrt{3}}{14}\right)^2 = 4\left(\frac{55}{14}\right)^2$, so $QX = \frac{55}{7}$. (Stopping at $O_1X = 8$ is tempting; treating $\angle QO_1X$ as a right angle gives $\sqrt{73}$; a sign slip in the law of cosines, adding $\frac{552}{49}$ instead of subtracting, gives $\frac{\sqrt{4129}}{7}$; and using the $120°$ angle at P as the central angle QO_2X gives $5\sqrt{3}$.)

15. Answer: (B) $\frac{\sqrt{5}-1}{2}$. The idea. Squaring twice turns the equation into a quartic, but squaring also lets in solutions of the "wrong-sign" equations. The key is to notice that one of those intruders is easy to spot, namely $x = 2$ (it solves $x = \sqrt{2+\sqrt{2+x}}$, since $\sqrt{2+\sqrt{4}} = 2$). Every sign variant squares to the same quartic, so $x - 2$ is a factor, and the rest factors easily.

Work it out. Squaring gives $x^2 = 2 - \sqrt{2+x}$, so $\sqrt{2+x} = 2 - x^2$, which forces $x^2 \le 2$. Squaring again: $2 + x = (2-x^2)^2 = x^4 - 4x^2 + 4$, that is, $x^4 - 4x^2 - x + 2 = 0$. Dividing by $x - 2$ leaves $x^3 + 2x^2 - 1$, which has the root $x = -1$, and $x^3 + 2x^2 - 1 = (x+1)(x^2 + x - 1)$. So

$$x^4 - 4x^2 - x + 2 = (x-2)(x+1)(x^2+x-1),$$

with roots $2, -1$ and $\frac{-1\pm\sqrt{5}}{2}$. We need $0 \le x \le \sqrt{2}$, and only $x = \frac{\sqrt{5}-1}{2}$ qualifies. (The root 2 fails because it makes $2 - x^2$ negative.)

Check. With $x = \frac{\sqrt{5}-1}{2} \approx 0.618$: $2 + x = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$, so $\sqrt{2+x} = \frac{1+\sqrt{5}}{2}$; then $2 - \frac{1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2} = \left(\frac{\sqrt{5}-1}{2}\right)^2$, whose square root is x. A trigonometric view gives the same result: put $x = 2\cos t$ with $0 \le t \le \frac{\pi}{2}$. Then $\sqrt{2+x} = 2\cos\frac{t}{2}$ and $\sqrt{2-2\cos\frac{t}{2}} = 2\sin\frac{t}{4}$, so $\cos t = \sin\frac{t}{4}$, which gives $t = \frac{2\pi}{5}$ and $x = 2\cos 72° = \frac{\sqrt{5}-1}{2}$. Choosing 2 means keeping the extraneous root, and $\frac{3-\sqrt{5}}{2} = x^2$ is the value one square too early.

16. Answer: (C) $\frac{5}{2}$. The idea. With $a = \sqrt[3]{x+1}$ and $b = \sqrt[3]{x-1}$, cube both sides using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, and replace the $a + b$ inside by the right-hand side $\sqrt[3]{5x}$. That removes the sum and leaves a single product of cube roots, which cubes cleanly.

Work it out. Cubing gives $(x+1)+(x-1)+3ab\sqrt[3]{5x} = 5x$, so $ab\sqrt[3]{5x} = x$, i.e. $\sqrt[3]{(x^2-1)\cdot5x} = x$. Cubing: $5x(x^2-1) = x^3$, so $x(4x^2-5) = 0$, giving $x = 0$ or $x = \pm\frac{\sqrt{5}}{2}$.

Substituting $a + b = \sqrt[3]{5x}$ could in principle add false roots, so check each. $x = 0$: $1 + (-1) = 0$. For $x = \frac{\sqrt{5}}{2}$ the derivation is reversible here: if $c = \sqrt[3]{5x}$ and $a^3 + b^3 - c^3 + 3abc = 0$, then with $-c$

in place of c the identity $a^3+b^3+(-c)^3-3ab(-c) = \frac{1}{2}(a+b-c)\left[(a-b)^2+(b+c)^2+(a+c)^2\right]$ forces $a+b=c$ unless $a=b=-c$, and $a=b$ is impossible since $x+1 \neq x-1$. So all three roots are genuine, and the sum of squares is $0+\frac{5}{4}+\frac{5}{4}=\frac{5}{2}$.

Check. Numerically, $x = 1.1180$: $\sqrt[3]{2.1180} + \sqrt[3]{0.1180} \approx 1.2839 + 0.4903 = 1.7742$, and $\sqrt[3]{5.5902} \approx 1.7742$. The equation is odd in x, so $-\frac{\sqrt{5}}{2}$ works too.

17. Answer: (E) 1107**. The idea.** A pair of disjoint subsets is the same as giving each element one of three labels: in A, in B, or in neither. Encode the labels as x, x^{-1}, 1; then the product over all elements records $|A|-|B|$ as the exponent of x, so we want the constant term of $(x+1+x^{-1})^8$. It is also easy to count directly by the common size k.

Work it out. If $|A| = |B| = k$, choose A in $\binom{8}{k}$ ways and then B from the remaining $8-k$ elements in $\binom{8-k}{k}$ ways. Summing: $k=0$: 1; $k=1$: $8\cdot 7 = 56$; $k=2$: $28\cdot 15 = 420$; $k=3$: $56\cdot 10 = 560$; $k=4$: $70\cdot 1 = 70$. Total $1+56+420+560+70 = 1107$.

Check. The central trinomial coefficients (constant terms of $(x+1+x^{-1})^n$) are $1, 1, 3, 7, 19, 51, 141, 393, 1107$ for $n = 0, \ldots, 8$, satisfying $na_n = (2n-1)a_{n-1} + 3(n-1)a_{n-2}$: $8a_8 = 15\cdot 393 + 21\cdot 141 = 5895 + 2961 = 8856$, so $a_8 = 1107$. (Dropping the $k=4$ term gives 1037; reading $a_7 = 393$ off the list instead of a_8 is an off-by-one slip.)

18. Answer: (B) 6**. The idea.** If p is odd and $p \mid n^4+1$, then $n^4 \equiv -1$ and $n^8 \equiv 1 \pmod p$. So the multiplicative order of n modulo p divides 8 but not 4: it is exactly 8. By Fermat the order divides $p-1$, so $p \equiv 1 \pmod 8$. Conversely, for such p a primitive root g gives $n = g^{(p-1)/8}$ of order 8, and then n^4 is a square root of 1 other than 1, i.e. $n^4 \equiv -1$.

Work it out. The prime 2 works ($n=1$ gives 2). The primes below 100 that are 1 mod 8 are $17, 41, 73, 89, 97$ (the other candidates $9, 25, 33, 49, 57, 65, 81$ are composite). That is $1+5=6$ primes.

Check. Explicit witnesses: $2^4+1 = 17$; $3^4+1 = 82 = 2\cdot 41$; $10^4+1 = 10001 = 73\cdot 137$; $12^4+1 = 20737 = 89\cdot 233$; $33^4+1 = 1185922 = 97\cdot 12226$. And a prime like 5 fails: fourth powers mod 5 are only 0 and 1, never 4. (Using the n^2+1 rule $p \equiv 1 \pmod 4$ instead would give 12; forgetting $p=2$ gives 5.)

19. Answer: (D) $\frac{1+\sqrt{5}}{2}$**. The idea.** The triangle inequality bounds $|z^2+1|$ from below by $|z|^2-1$. Combined with the equation this gives a quadratic inequality in $r=|z|$, and the bound is reached when z^2 points in the direction opposite to 1.

Work it out. Let $r = |z|$. Then $r = |z^2+1| \geq |z^2|-1 = r^2-1$, so $r^2-r-1 \leq 0$ and $r \leq \frac{1+\sqrt{5}}{2}$. Equality in $|z^2+1| \geq |z^2|-1$ needs z^2 to be a negative real number, so take $z = ir$: then $z^2+1 = 1-r^2$ and we need $r^2-1 = r$, which holds for $r = \frac{1+\sqrt{5}}{2}$. So the maximum is $\frac{1+\sqrt{5}}{2}$.

Check. Writing $z = re^{i\theta}$, $|z^2+1|^2 = r^4+2r^2\cos 2\theta+1 = r^2$, so $\cos 2\theta = \frac{r^2-r^4-1}{2r^2}$, which must be at least -1: $r^4-3r^2+1 \leq 0$, i.e. $r^2 \leq \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$. The smallest possible $|z|$ is $\frac{\sqrt{5}-1}{2}$, a tempting wrong pick.

20. Answer: (E) 7**. The idea.** A polynomial of degree at most 2 is determined by its values at $-1, 0, 1$, so $f(2)$ is a fixed linear combination of $f(-1)$, $f(0)$, $f(1)$. Each of those is between -1 and 1, which bounds $f(2)$ by the sum of the absolute values of the weights.

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Work it out. Lagrange interpolation at $-1, 0, 1$ evaluated at $x = 2$: the basis polynomials are $\frac{x(x-1)}{2}$, $-(x+1)(x-1)$, $\frac{x(x+1)}{2}$, which at $x = 2$ equal 1, -3, 3. So $f(2) = f(-1) - 3f(0) + 3f(1) \le 1 + 3 + 3 = 7$. Equality needs $f(-1) = 1$, $f(0) = -1$, $f(1) = 1$, which is $f(x) = 2x^2 - 1$. This polynomial has minimum -1 at 0 and maximum 1 at ± 1 on the interval, so $|f| \le 1$ there, and $f(2) = 7$.

Check. Try the finite-difference view: for a quadratic, $f(2) - 3f(1) + 3f(0) - f(-1) = 0$ (third differences vanish), the same identity. Natural guesses such as $f(x) = x^2$ give only 4, and $f(x) = x$ gives 2; the extremal $2x^2 - 1 = \cos(2 \arccos x)$ is the Chebyshev polynomial, which swings fully between ± 1 three times on $[-1, 1]$.

21. Answer: (D) 45**. The idea.** Each angle $\theta = 9^\circ + 36^\circ k$ has $5\theta = 45^\circ + 180^\circ k$, so $\tan 5\theta = 1$. Writing $\tan 5\theta$ as a rational function of $t = \tan\theta$ turns this into a quintic whose five roots are exactly our five tangents (they are distinct, since the angles differ by less than 180°). Then Vieta gives the sum of squares without finding any root.

Work it out. From $(1 + it)^5$, $\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$. Setting this equal to 1: $t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$. So $e_1 = \sum t = 5$ and $e_2 = \sum_{i<j} t_i t_j = -10$. Hence $\sum t^2 = e_1^2 - 2e_2 = 25 + 20 = 45$.

Check. $t = 1$ (45°) is a root: $1 - 5 - 10 + 10 + 5 - 1 = 0$. Dividing out $t - 1$ leaves $t^4 - 4t^3 - 14t^2 - 4t + 1$, whose roots have $\sum t^2 = 16 + 28 = 44$, and $44 + 1^2 = 45$. Numerically $\tan^2 81^\circ \approx 39.86$, $\tan^2 9^\circ \approx 0.025$, $\tan^2 117^\circ \approx 3.85$, $\tan^2 153^\circ \approx 0.26$, and with 1 the total is ≈ 45.0.

22. Answer: (D) 20**. The idea.** The weight 2 spoils the usual "straight line is shortest" argument. Fix it by trading A for a point A' inside the circle with $PA = 2\,PA'$ for every P on the circle (an Apollonius circle). Then $PA + 2\,PB = 2(PA' + PB)$, an unweighted sum, which is smallest when P lies on segment $A'B$.

Work it out. Try $A' = \left(\frac{5}{2}, 0\right)$. For $P = (x, y)$ with $x^2 + y^2 = 25$: $PA^2 = x^2 + y^2 - 20x + 100 = 125 - 20x$ and $PA'^2 = x^2 + y^2 - 5x + \frac{25}{4} = \frac{125}{4} - 5x$, so $PA^2 = 4\,PA'^2$ and $PA = 2\,PA'$. (In general the point on ray OA at distance $\frac{r^2}{OA} = \frac{25}{10}$ from the centre works, with ratio $\frac{OA}{r} = 2$.) Hence

$$PA + 2\,PB = 2(PA' + PB) \ge 2\,A'B = 2\sqrt{6^2 + 8^2} = 20.$$

Equality needs P on segment $A'B$. That segment really does cross the circle: $OA' = \frac{5}{2} < 5$ puts A' inside, and $OB^2 = \frac{49}{4} + 64 > 25$ puts B outside. So the minimum 20 is attained.

Check. The crossing point is $A' + s(-6, 8)$ with $100s^2 - 30s - \frac{75}{4} = 0$, $s \approx 0.608$, giving $P \approx (-1.150, 4.866)$; there $PA \approx 12.166$ and $PB \approx 3.917$, and $12.166 + 7.834 = 20.000$. A nearby point such as $P = (0, 5)$ gives $\sqrt{125} + 2\sqrt{21.25} \approx 20.40$. The bound $PA + PB \ge AB = \frac{\sqrt{985}}{2} \approx 15.69$ is true but not reachable with the weight 2, and forgetting to double $A'B$ gives 10.

23. Answer: (B) 293**. The idea.** Rayleigh's (Beatty's) theorem: if $\alpha, \beta > 1$ are irrational with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences $\lfloor k\alpha \rfloor$ and $\lfloor k\beta \rfloor$ ($k \ge 1$) together contain every positive integer exactly once. With $\alpha = \sqrt{2}$, $\beta = \frac{1}{1 - 1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} - 1} = 2 + \sqrt{2}$. So the missed integers are exactly $\lfloor k(2 + \sqrt{2}) \rfloor$.

Work it out. Count $k \ge 1$ with $\lfloor k(2 + \sqrt{2}) \rfloor \le 1000$, i.e. $k(2 + \sqrt{2}) < 1001$, i.e. $k < \frac{1001}{3.41421\ldots} = 293.19\ldots$. So $k = 1, \ldots, 293$: 293 missed integers. Careful at the edge: $293(2 + \sqrt{2}) = 586 +$

$293\sqrt{2} \approx 586 + 414.36 = 1000.36$, so the 293rd missed integer is exactly 1000, which is inside the range (using $k(2+\sqrt{2}) \le 1000$ would wrongly give 292).

Check. Count the hits instead: $\lfloor k\sqrt{2} \rfloor \le 1000$ iff $k\sqrt{2} < 1001$ iff $k \le 707$ (as $707\sqrt{2} \approx 999.85$ and $708\sqrt{2} \approx 1001.26$). These 707 values are distinct because $\sqrt{2} > 1$, so $1000 - 707 = 293$ are missed. Small case: $\lfloor k\sqrt{2} \rfloor = 1, 2, 4, 5, 7, 8, 9, \ldots$ misses $3, 6, 10, \ldots = \lfloor 3.41 \rfloor, \lfloor 6.83 \rfloor, \lfloor 10.24 \rfloor$.

24. Answer: (C) 239**. The idea.** Draw the "conflict graph": join two numbers if they may not both be chosen. Put k above $13 - k$ for $k = 1, \ldots, 6$. The top row $1, 2, \ldots, 6$ and the bottom row $12, 11, \ldots, 7$ are paths (consecutive numbers), each column $\{k, 13 - k\}$ is an edge (sum 13), and the only other consecutive pair, 6 and 7, is already the last column. So the graph is a 2×6 ladder, and we count its independent sets column by column.

Work it out. Each column is in one of three states: empty (E), top only (T), bottom only (B); both is forbidden. Adjacent columns may not both use the same row. Let e_n, t_n, b_n count valid choices of the first n columns ending in each state. Then $e_{n+1} = e_n + t_n + b_n$, $t_{n+1} = e_n + b_n$, $b_{n+1} = e_n + t_n$. Starting from $(e, t, b) = (1, 1, 1)$:

$n = 1$: total 3; $n = 2$: $(3, 2, 2)$, total 7; $n = 3$: $(7, 5, 5)$, total 17; $n = 4$: $(17, 12, 12)$, total 41; $n = 5$: $(41, 29, 29)$, total 99; $n = 6$: $(99, 70, 70)$, total 239.

Check. The totals obey $a_n = 2a_{n-1} + a_{n-2}$: $2 \cdot 99 + 41 = 239$. Dropping the sum condition would give the Fibonacci count $F_{14} = 377$; dropping the consecutive condition gives $3^6 = 729$.

25. Answer: (E) $\frac{3095}{512}$**. The idea.** By linearity, the expected number of patterns that appear is $\sum_P \Pr(P \text{ appears}) = 8 - \sum_P \frac{a_P}{2^{12}}$, where a_P counts the strings of length 12 that avoid P. The counts are not all equal: they depend on how P overlaps itself. Swapping H and T, or reading the string backwards, does not change a_P, so only three counts are needed: HHH (like TTT), HTH (like THT), and HHT (like TTH, and, reversed and swapped, HTT and THH). Let $a(n)$, $b(n)$, $c(n)$ count strings of length n avoiding HHH, HHT, HTH. All three start at 2, 4, 7 for $n = 1, 2, 3$.

Work it out. *Avoiding HHH:* the string starts with T, HT, or HHT, and the rest is again an avoider, so $a(n) = a(n-1) + a(n-2) + a(n-3)$: $13, 24, 44, 81, 149, 274, 504, 927, 1705$.

Avoiding HHT: starting with T or HT leaves an avoider of length $n - 1$ or $n - 2$. Starting with HH is only possible if every later letter is H (the first T after an HH would complete HHT). So $b(n) = b(n-1) + b(n-2) + 1$: $12, 20, 33, 54, 88, 143, 232, 376, 609$.

Avoiding HTH: starting with T leaves $c(n-1)$ strings. Starting with HTT, the rest is a free avoider (no HTH can straddle TT): $c(n-3)$. Starting with HH, drop the first H; what remains is an avoider of length $n-1$ that starts with H, and there are $c(n-1) - c(n-2)$ of those. HTH itself is forbidden. So $c(n) = 2c(n-1) - c(n-2) + c(n-3)$: $12, 21, 37, 65, 114, 200, 351, 616, 1081$.

Therefore

$$E = 8 - \frac{2 \cdot 1705 + 2 \cdot 1081 + 4 \cdot 609}{4096} = 8 - \frac{8008}{4096} = 8 - \frac{1001}{512} = \frac{3095}{512} \approx 6.04.$$

Check. The ordering makes sense: a self-overlapping pattern such as HHH has its occurrences clumped together, so it is the one most often missed (1705 misses), while patterns with no self-overlap are missed least often (609). Every pattern has the same expected number of occurrences, $10/8$, so these differences come entirely from clumping. Treating all eight patterns like HHH gives $\frac{2391}{512}$; treating them all like HTH gives $\frac{3015}{512}$; swapping the HTH and HHT counts (so the weight 4 falls on 1081) gives $\frac{2977}{512}$; using the HHH count in place of the HTH count gives $\frac{2939}{512}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.