MathCastle

AMC 12

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Let $f(x) = \dfrac{3x-1}{x+1}$, defined for every real $x \neq -1$. For a positive integer n, write $f^{(n)}$ for f applied n times in a row, so $f^{(2)}(x) = f(f(x))$. There is exactly one real number x for which $f^{(10)}(x)$ is defined (no step ever feeds -1 into f) and $f^{(10)}(x) = 0$. What is this x?

(A) $\frac{4}{5}$ **(B)** $\frac{5}{6}$ **(C)** $\frac{10}{11}$ **(D)** $\frac{11}{12}$ **(E)** $\frac{7}{6}$

2. A 10×10 board has 100 squares. Two different squares are chosen at random, with every pair of squares equally likely. What is the probability that a chess knight standing on one of the two squares could reach the other in a single move? (A knight moves two squares horizontally and one vertically, or two squares vertically and one horizontally.)

(A) $\frac{8}{275}$ **(B)** $\frac{4}{75}$ **(C)** $\frac{16}{275}$ **(D)** $\frac{8}{125}$ **(E)** $\frac{32}{275}$

3. In triangle ABC it is known that $\sin A = \dfrac{3}{5}$ and $\sin B = \dfrac{7}{25}$, but not whether either angle is acute or obtuse. Consider every triangle consistent with this information. What is the sum of all the different possible values of $\cos C$?

(A) $-\frac{3}{5}$ **(B)** $\frac{42}{125}$ **(C)** $\frac{84}{125}$ **(D)** $\frac{117}{125}$ **(E)** $\frac{159}{125}$

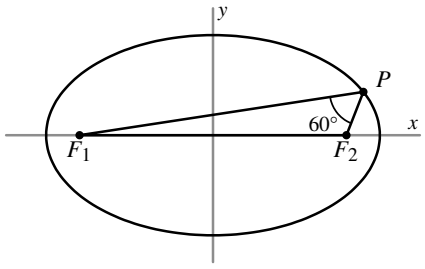
4. Let $N = 27000$. How many ordered pairs (a, b) of positive integers satisfy $a + b = N$ and have the property that the product ab is divisible by N?

(A) 0 **(B)** 14 **(C)** 29 **(D)** 30 **(E)** 59

5. Convex quadrilateral $ABCD$ has its vertices in that order around it, with $AB = BC = CD$. The interior angle at B is $\angle ABC = 90^\circ$ and the interior angle at C is $\angle BCD = 150^\circ$. What is the degree measure of the interior angle $\angle CDA$?

(A) 30 **(B)** 40 **(C)** 45 **(D)** 50 **(E)** 60

6. The ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ has foci F_1 and F_2. A point P on the ellipse is chosen so that $\angle F_1PF_2 = 60^\circ$. What is the radius of the circle inscribed in triangle F_1PF_2?



(A) $\frac{\sqrt{3}}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{2\sqrt{3}}{3}$ **(D)** $\frac{3\sqrt{3}}{4}$ **(E)** $3\sqrt{3}$

7. When the polynomial $(x+1)^{100}$ is divided by $x^2 - x + 1$, the remainder is a polynomial $R(x) = px + q$ with real coefficients p and q (possibly zero). What is q?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

(A) -3^{50} **(B)** $-\frac{1}{2}\cdot 3^{50}$ **(C)** 0 **(D)** 3^{50} **(E)** $2\cdot 3^{50}$

8. Seven rooks are placed on a 7×7 chessboard so that no two share a row or a column. The arrangement is called *mirror-proof* if reflecting the board across the diagonal running from the top-left corner to the bottom-right corner moves the set of occupied squares onto itself. How many mirror-proof arrangements are there?

(A) 64 **(B)** 76 **(C)** 128 **(D)** 232 **(E)** 256

9. A right prism has two parallel bases that are equilateral triangles, and three rectangular side faces perpendicular to the bases. Its height is chosen so that there is a sphere inside the prism tangent to all five of its faces. A second sphere passes through all six vertices of the prism. What is the ratio of the volume of the second sphere to the volume of the first?

(A) 8 **(B)** $6\sqrt{3}$ **(C)** $5\sqrt{5}$ **(D)** 12 **(E)** $6\sqrt{6}$

10. Let P be the product of all positive integers less than 98 that have no prime factor in common with 98. What is the remainder when P is divided by 98?

(A) 1 **(B)** 29 **(C)** 48 **(D)** 69 **(E)** 97

11. Find the sum of all real numbers x with $0\le x<2\pi$ that satisfy
$\sqrt{3}\sin x+\cos x=2\sin 3x$.

(A) $\frac{7\pi}{6}$ **(B)** $\frac{23\pi}{6}$ **(C)** 4π **(D)** 5π **(E)** 6π

12. Real numbers x and y satisfy $x^2+y^2=4$. As (x,y) varies over all such pairs, the product $(x+1)(y+1)$ has a largest value M and a smallest value m. What is $M-m$?

(A) $4\sqrt{2}$ **(B)** $3+2\sqrt{2}$ **(C)** $\frac{9}{2}+2\sqrt{2}$ **(D)** $3+4\sqrt{2}$ **(E)** $6+2\sqrt{2}$

13. Positive real numbers x, y and z satisfy
$$xyz=1,\qquad x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\qquad x^2+y^2+z^2=11.$$
What is the largest of the three numbers x, y, z?

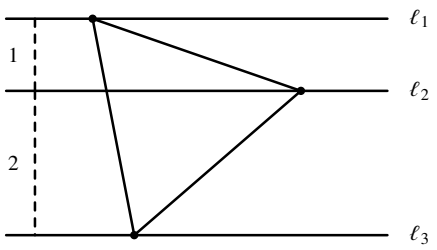
(A) $1+\sqrt{2}$ **(B)** $\sqrt{3}+\sqrt{2}$ **(C)** $\sqrt{11}$ **(D)** $2+\sqrt{3}$ **(E)** $1+2\sqrt{3}$

14. The Fibonacci numbers are $F_1=F_2=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n\ge 1$. There is exactly one positive real number x for which the infinite series
$F_1x+F_2x^2+F_3x^3+F_4x^4+\cdots$
converges and its sum equals 1. What is x?

(A) $\sqrt{5}-2$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3-\sqrt{5}}{2}$ **(E)** $\sqrt{2}-1$

15. Three parallel lines ℓ_1, ℓ_2, ℓ_3 lie in a plane, with ℓ_2 between the other two. The distance between ℓ_1 and ℓ_2 is 1, and the distance between ℓ_2 and ℓ_3 is 2. An equilateral triangle has one vertex on each of the three lines. What is the area of the triangle?

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.



(A) $\frac{7\sqrt{3}}{4}$ **(B)** $2\sqrt{3}$ **(C)** $\frac{9\sqrt{3}}{4}$ **(D)** $\frac{7\sqrt{3}}{3}$ **(E)** $3\sqrt{3}$

16. Call a positive integer *n good* if there is a complex number z with both $z^n = 1$ and $(z+1)^n = 1$. What is the sum of all good integers n with $1 \le n \le 100$?

(A) 432 **(B)** 816 **(C)** 1300 **(D)** 1683 **(E)** 2550

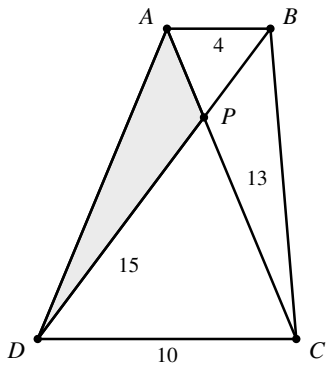
17. A jeweller has 20 identical gold coins and 4 identical purses. She puts every coin into one of the purses; some purses may stay empty. Since the coins are identical and the purses are identical, two ways count as the same if they give the same collection of purse contents (for example, $\{8, 7, 5, 0\}$ in any order is one way). In how many different ways can she do this?

(A) 44 **(B)** 64 **(C)** 74 **(D)** 108 **(E)** 1771

18. All angles are in degrees. What is the exact value of
$\tan 5^\circ \cdot \tan 55^\circ \cdot \tan 65^\circ \; + \; \tan 25^\circ \cdot \tan 35^\circ \cdot \tan 85^\circ$?

(A) 1 **(B)** 2 **(C)** $2\sqrt{3}$ **(D)** $\sqrt{3}+2$ **(E)** 4

19. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, with $AB = 4$ and $CD = 10$, and its diagonals have lengths $AC = 13$ and $BD = 15$. The diagonals meet at P. What is the area of triangle APD?



(A) $\frac{48}{7}$ **(B)** $\frac{60}{7}$ **(C)** 12 **(D)** $\frac{120}{7}$ **(E)** 21

20. For a positive integer n, let $\varphi(n)$ be the number of integers k with $1 \le k \le n$ and $\gcd(k, n) = 1$; for

example $\varphi(1) = 1$, $\varphi(9) = 6$ and $\varphi(12) = 4$. For how many integers n with $1 \le n \le 1000$ is $\varphi(n)$ a power of 2? (Here $1 = 2^0$ counts as a power of 2.)

(A) 10 **(B)** 34 **(C)** 51 **(D)** 53 **(E)** 54

21. Two towns A and B on a perfectly spherical planet both lie at latitude 45° north (latitude is the angle between the radius to a point and the plane of the equator). Their longitudes differ by θ, where $0° < \theta < 180°$. An aircraft flies the shortest route along the surface from A to B, and the most northerly point of that route is at latitude 60° north. What is $\cos\theta$?

(A) $-\frac{1}{3}$ **(B)** 0 **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

22. How many ordered pairs (a, b) of positive integers satisfy $\operatorname{lcm}(a, b) + \gcd(a, b) = a + b + 60$?

(A) 36 **(B)** 42 **(C)** 45 **(D)** 51 **(E)** 54

23. A token sits on one vertex of a regular decagon (a polygon with 10 vertices). Every minute it moves along an edge to one of the two neighbouring vertices, choosing each with probability $\frac{1}{2}$, independently of all earlier moves. The starting vertex counts as visited. What is the expected number of minutes until every one of the 10 vertices has been visited at least once?

(A) 45 **(B)** 50 **(C)** 55 **(D)** 81 **(E)** 90

24. A polynomial $P(x)$ of degree at most 10 satisfies $P(k) = 3^k$ for each of the eleven integers $k = 0, 1, 2, \ldots, 10$. What is $P(12)$?

(A) 502769 **(B)** 506865 **(C)** 525297 **(D)** 527345 **(E)** 531441

25. A panel has 4 lamps, all off, and 5 buttons. Each button is wired to toggle a fixed set of lamps (switching each of them on if off, off if on). The wiring is random: for each of the 20 (button, lamp) pairs, independently with probability $\frac{1}{2}$, that button toggles that lamp. What is the probability that every one of the $2^4 = 16$ on/off patterns of the lamps can be reached from all-off by pressing buttons?

(A) $\frac{315}{1024}$ **(B)** $\frac{15}{32}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{17}{32}$ **(E)** $\frac{9765}{16384}$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Answer Key — AMC 12 — Hard Edition Practice Contest 2

1. (B) $\frac{5}{6}$

2. (C) $\frac{16}{275}$

3. (B) $\frac{42}{125}$

4. (C) 29

5. (C) 45

6. (A) $\frac{\sqrt{3}}{3}$

7. (A) -3^{50}

8. (D) 232

9. (C) $5\sqrt{5}$

10. (E) 97

11. (D) 5π

12. (C) $\frac{9}{2} + 2\sqrt{2}$

13. (B) $\sqrt{3} + \sqrt{2}$

14. (E) $\sqrt{2} - 1$

15. (D) $\frac{7\sqrt{3}}{3}$

16. (B) 816

17. (D) 108

18. (E) 4

19. (D) $\frac{120}{7}$

20. (E) 54

21. (A) $-\frac{1}{3}$

22. (B) 42

23. (A) 45

24. (A) 502769

25. (E) $\frac{9765}{16384}$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Solutions — AMC 12 — Hard Edition Practice Contest 2

1. Answer: (B) $\frac{5}{6}$. The idea. The only fixed point of f is $x = 1$ (solve $x(x+1) = 3x - 1$, i.e. $(x-1)^2 = 0$). A Möbius map with a single fixed point becomes a plain translation once we measure distance from that fixed point on a reciprocal scale: look at $g = \dfrac{1}{x-1}$.

Work it out. We have $f(x) - 1 = \dfrac{3x - 1 - (x+1)}{x+1} = \dfrac{2(x-1)}{x+1}$, so $\dfrac{1}{f(x)-1} = \dfrac{x+1}{2(x-1)} = \dfrac{(x-1)+2}{2(x-1)} = \dfrac{1}{2} + \dfrac{1}{x-1}$. So every application of f adds exactly $\frac{1}{2}$ to $g = \frac{1}{x-1}$, and ten applications add 5. The target 0 has $g = \frac{1}{0-1} = -1$, so the starting point has $g = -1 - 5 = -6$, i.e. $x - 1 = -\frac{1}{6}$ and $x = \frac{5}{6}$.

Along the way g takes the values $-6, -\frac{11}{2}, -5, \ldots, -\frac{3}{2}, -1$. The forbidden input $x = -1$ corresponds to $g = -\frac{1}{2}$, which never appears, so all ten steps are defined. Since each step is a one-to-one map, this x is the only one.

Check. Directly: $f(\frac{5}{6}) = \frac{3/2}{11/6} = \frac{9}{11}$, and $\frac{1}{9/11-1} = -\frac{11}{2}$, exactly $-6 + \frac{1}{2}$ as predicted. Continuing, the values run $\frac{5}{6}, \frac{9}{11}, \frac{4}{5}, \frac{7}{9}, \frac{3}{4}, \frac{5}{7}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}, \frac{1}{3}, 0$.

2. Answer: (C) $\frac{16}{275}$. The idea. Counting knight moves square by square means handling corner, edge and interior squares separately. Instead, notice that the two ends of a knight move are opposite corners of a 2×3 or 3×2 block of squares, and each such block contains exactly two knight-move pairs (its two "diagonals"). So count blocks.

Work it out. A block 2 rows tall and 3 columns wide can be placed in $9 \cdot 8 = 72$ positions on a 10×10 board, and a block 3 rows tall and 2 wide also in $8 \cdot 9 = 72$ positions. Every knight-move pair lies in exactly one block (the smallest rectangle containing both squares), and each block holds 2 pairs, so there are $2(72 + 72) = 288$ unordered pairs a knight's move apart.

The number of unordered pairs of squares is $\binom{100}{2} = 4950$, so the probability is $\dfrac{288}{4950} = \dfrac{16}{275}$.

Check. Count ordered moves instead: interior squares (the central 6×6 region, 36 squares) have 8 moves each; summing the move counts of all 100 squares gives $576 = 2 \cdot 288$, as it must, since every unordered pair is counted from both ends. (Counting ordered pairs but dividing by unordered ones gives the trap answer $\frac{32}{275}$.)

3. Answer: (B) $\frac{42}{125}$. The idea. A sine value does not pin down an angle of a triangle: A could be acute or obtuse. Try all four sign choices for $\cos A$ and $\cos B$, then throw out any choice whose two angles already add to 180° or more. Finally $\cos C = -\cos(A + B)$.

Work it out. $\cos A = \pm\frac{4}{5}$ and $\cos B = \pm\frac{24}{25}$. Numerically $A \approx 36.9^\circ$ or 143.1°, and $B \approx 16.3^\circ$ or 163.7°.

Both acute ($36.9^\circ + 16.3^\circ$): $\cos(A+B) = \frac{4}{5} \cdot \frac{24}{25} - \frac{3}{5} \cdot \frac{7}{25} = \frac{96-21}{125} = \frac{3}{5}$, so $\cos C = -\frac{3}{5} = -\frac{75}{125}$.

A obtuse, B acute ($143.1^\circ + 16.3^\circ = 159.4^\circ < 180^\circ$, allowed): $\cos(A+B) = -\frac{4}{5} \cdot \frac{24}{25} - \frac{21}{125} = -\frac{117}{125}$, so $\cos C = \frac{117}{125}$.

A acute, B obtuse: $36.9^\circ + 163.7^\circ > 180^\circ$, impossible. (Exactly: B obtuse needs $A < 180^\circ - B$, i.e. $\sin A < \sin B$, which fails.) Both obtuse is impossible too.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

The sum of the possible values is $-\frac{75}{125} + \frac{117}{125} = \frac{42}{125}$.

Check. In the second triangle $C \approx 20.6°$, and $\cos 20.6° \approx 0.936 = \frac{117}{125}$. In the first, $C \approx 126.9°$ with cosine -0.6. Keeping the impossible case would add another $\frac{117}{125}$ and give $\frac{159}{125}$.

4. Answer: (C) 29**. The idea.** Since $b = N - a$, we have $ab = a(N-a) \equiv -a^2 \pmod N$. So the condition is simply $N \mid a^2$, and a square is divisible by N exactly when a carries half the exponent of each prime of N, rounded up.

Work it out. $N = 27000 = 2^3 \cdot 3^3 \cdot 5^3$. For $2^3 \mid a^2$ we need $2^2 \mid a$ (since a^2 has an even power of 2, at least 4); likewise $3^2 \mid a$ and $5^2 \mid a$. So the condition is $900 \mid a$.

The positive multiples of 900 below 27000 are $900, 1800, \ldots, 26100$, which is $\frac{27000}{900} - 1 = 29$ values of a, each giving one pair with $b = 27000 - a$ positive.

Check. Take $a = 900$, $b = 26100$: $ab = 900 \cdot 26100 = 2^4 3^4 5^4 \cdot 29$, divisible by $2^3 3^3 5^3$. And $a = 300$ fails: $300^2 = 2^4 3^2 5^4$ has only 3^2. Requiring $N \mid a$ (a common slip) would leave no pairs at all.

5. Answer: (C) 45**. The idea.** The $150° = 90° + 60°$ suggests an equilateral triangle. Build equilateral triangle BCE on side BC, inside the quadrilateral. Then E is at distance 1 (the common side length) from B and C, which creates two isosceles triangles, ABE and ECD, and it turns out that A, E, D are collinear.

Work it out. Let the common side length be 1 and let BCE be equilateral with E on the same side of BC as A and D. Then $BE = CE = 1$.

At B: $\angle ABE = 90° - 60° = 30°$ and $BA = BE$, so triangle ABE is isosceles with base angles $\angle BEA = \frac{180° - 30°}{2} = 75°$.

At C: $\angle ECD = 150° - 60° = 90°$ and $CE = CD$, so triangle ECD is an isosceles right triangle and $\angle CED = \angle CDE = 45°$.

Around E: $\angle AEB + \angle BEC + \angle CED = 75° + 60° + 45° = 180°$, so A, E, D lie on one line, with E between A and D. Hence $\angle CDA = \angle CDE = 45°$.

Check. Coordinates: $B = (0,0), C = (1,0), A = (0,1), D = (1+\frac{\sqrt3}{2}, \frac12)$. Then $\overrightarrow{DA} = (-1-\frac{\sqrt3}{2}, \frac12)$ and $\overrightarrow{DC} = (-\frac{\sqrt3}{2}, -\frac12)$; their dot product is $\frac{\sqrt3}{2} + \frac34 - \frac14 = \frac{1+\sqrt3}{2}$, and $|DA| = \sqrt{2+\sqrt3} = \frac{1+\sqrt3}{\sqrt2}$, so $\cos\angle CDA = \frac{1}{\sqrt2}$. The angle sum also checks: $\angle A = 360° - 90° - 150° - 45° = 75°$.

6. Answer: (A) $\frac{\sqrt3}{3}$**. The idea.** We never need P's coordinates. The ellipse fixes the sum $PF_1 + PF_2$, the foci fix F_1F_2, and the law of cosines at the 60° angle then fixes the product $PF_1 \cdot PF_2$, which gives the area. The inradius is area divided by semiperimeter, and the perimeter is already known.

Work it out. Here $a = 5$, $b = 3$, $c = \sqrt{25-9} = 4$, so $F_1F_2 = 8$ and $PF_1 + PF_2 = 2a = 10$. Write $u = PF_1$, $v = PF_2$. The law of cosines gives $64 = u^2 + v^2 - uv = (u+v)^2 - 3uv = 100 - 3uv$, so $uv = 12$. The area is $K = \frac12 uv \sin 60° = \frac12 \cdot 12 \cdot \frac{\sqrt3}{2} = 3\sqrt3$.

The perimeter is $10 + 8 = 18$, so the semiperimeter is $s = 9$ and the inradius is $r = \frac{K}{s} = \frac{3\sqrt3}{9} = \frac{\sqrt3}{3}$.

Check. Such a P exists: $u + v = 10$, $uv = 12$ gives $u, v = 5 \pm \sqrt{13}$, both positive. The height of P above the axis is $\frac{2K}{8} = \frac{3\sqrt3}{4}$, which is less than 3, as it must be for a point of the ellipse. (Stopping at the area gives $3\sqrt3$; stopping at the height gives $\frac{3\sqrt3}{4}$.)

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

7. Answer: (A) -3^{50}**. The idea.** $(x+1)^{100} = Q(x)(x^2-x+1) + px + q$, so at a root ζ of x^2-x+1 we get $(\zeta+1)^{100} = p\zeta + q$. The roots are $\zeta = \cos 60^\circ \pm i \sin 60^\circ$, and $\zeta + 1$ has a clean polar form, so De Moivre does all the work. Expanding $(x+1)^{100}$ is hopeless; evaluating at ζ is easy.

Work it out. Take $\zeta = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, which satisfies $\zeta^2 = \zeta - 1$. Then $\zeta + 1 = \frac{3}{2} + \frac{\sqrt{3}}{2}i = \sqrt{3}\,(\cos 30^\circ + i \sin 30^\circ)$, so $(\zeta+1)^{100} = 3^{50}\,(\cos 3000^\circ + i \sin 3000^\circ)$. Now $3000^\circ = 8 \cdot 360^\circ + 120^\circ$, and $\cos 120^\circ + i \sin 120^\circ = \zeta^2 = \zeta - 1$. Hence $p\zeta + q = 3^{50}(\zeta - 1) = 3^{50}\zeta - 3^{50}$. Because p and q are real and ζ is not, we can match the two sides: $p = 3^{50}$ and $q = -3^{50}$. (The conjugate root gives the conjugate equation, so it agrees.)

Check. Try a small exponent the same way: $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$. Dividing by $x^2 - x + 1$ leaves $9x - 9$, and the recipe predicts $(\zeta+1)^4 = 9(\cos 120^\circ + i \sin 120^\circ) = 9(\zeta - 1)$. The exponent $4 \equiv 100 \pmod{12}$, so the pattern (a power of 3 times $x - 1$) is exactly the same, with 3^{50} in place of 3^2. A sign slip with 120° gives $+3^{50}$, and taking q to be just the real part of $(\zeta+1)^{100}$ gives $-\frac{1}{2} \cdot 3^{50}$.

8. Answer: (D) 232**. The idea.** Number rows and columns 1 to 7 so that the mirror diagonal consists of the squares (i, i). A non-attacking placement is a permutation σ: the rook in row i is in column $\sigma(i)$. Reflection sends square (i, j) to (j, i), i.e. replaces σ by its inverse. So we are counting permutations of 7 elements with $\sigma = \sigma^{-1}$: every rook is either on the diagonal or paired with its mirror image.

Work it out. Such a permutation splits $\{1, \ldots, 7\}$ into fixed points (rooks on the diagonal) and 2-cycles (a rook at (i, j) together with one at (j, i)). Choose the number k of mirror pairs, choose the $2k$ indices involved, and pair them up in $(2k-1)(2k-3)\cdots 1$ ways:

$k = 0$: 1. $\quad k = 1$: $\binom{7}{2} = 21$. $\quad k = 2$: $\binom{7}{4} \cdot 3 = 105$. $\quad k = 3$: $\binom{7}{6} \cdot 15 = 105$.

Total: $1 + 21 + 105 + 105 = 232$.

Check. The same count t_n satisfies $t_n = t_{n-1} + (n-1)t_{n-2}$ (element n is fixed, or paired with one of the other $n - 1$): $1, 1, 2, 4, 10, 26, 76, 232$. The answer 76 belongs to a 6×6 board.

9. Answer: (C) $5\sqrt{5}$**. The idea.** By symmetry both spheres are centred at the centre of the prism (midway between the bases, above the centroid of the base). The inscribed sphere touches the three side faces, so its radius is the inradius of the triangular base; touching both bases makes the height twice that radius. The circumscribed sphere then comes from one right triangle.

Work it out. Let the base have side s. Its inradius is $r = \frac{s}{2\sqrt{3}}$ and its circumradius is $\frac{s}{\sqrt{3}}$. The inscribed sphere has radius r, so the height is $h = 2r = \frac{s}{\sqrt{3}}$.

From the centre of the prism to a vertex: go $\frac{h}{2} = \frac{s}{2\sqrt{3}}$ vertically and $\frac{s}{\sqrt{3}}$ horizontally, so $R^2 = \frac{s^2}{3} + \frac{s^2}{12} = \frac{5s^2}{12}$, $\quad \frac{R^2}{r^2} = \frac{5s^2/12}{s^2/12} = 5$. So $\frac{R}{r} = \sqrt{5}$ and the volume ratio is $(\sqrt{5})^3 = 5\sqrt{5}$.

Check. With $s = 2\sqrt{3}$: $r = 1$, $h = 2$, and a vertex is at horizontal distance 2 and vertical distance 1 from the centre, so $R = \sqrt{5}$. A common slip is using the triangle's circumradius $\frac{s}{\sqrt{3}}$ alone for R, which gives a radius ratio of 2 and a volume ratio of 8.

10. Answer: (E) 97**. The idea.** The numbers in the product are exactly the residues k with $\gcd(k, 98) = 1$, and each such k has an inverse k' in the same set with $kk' \equiv 1 \pmod{98}$. Pair every number with its inverse: each pair multiplies to 1. The only numbers left unpaired are

those that are their own inverse, the solutions of $x^2 \equiv 1 \pmod{98}$. So the whole problem reduces to finding those.

Work it out. Since $98 = 2 \cdot 7^2$, the numbers in the product are the odd numbers below 98 that are not multiples of 7; there are $98 \cdot \frac{1}{2} \cdot \frac{6}{7} = 42$ of them.

Now solve $x^2 \equiv 1 \pmod{98}$, i.e. $98 \mid (x-1)(x+1)$. The factor 2 just says x is odd. For 49: the numbers $x-1$ and $x+1$ differ by 2, so 7 cannot divide both; hence all of 49 divides one of them, and $x \equiv 1$ or $x \equiv -1 \pmod{49}$. Combined with x odd, $x \equiv 1$ or $97 \pmod{98}$. So exactly two numbers, 1 and 97, are their own partners, and the other 40 split into 20 pairs with product 1 each. Therefore $P \equiv 1 \cdot 97 \equiv 97 \pmod{98}$.

Check. The pairing is really an inverse pairing: for example $3 \cdot 33 = 99 \equiv 1$ and $5 \cdot 59 = 295 = 3 \cdot 98 + 1$. Contrast a modulus like 15, where $x^2 \equiv 1$ has four solutions $1, 4, 11, 14$ whose product is $616 \equiv 1 \pmod{15}$; that is why the answer 1 is a trap here. Reducing only modulo 49 gives $-1 \equiv 48$, but P is a product of odd numbers, so it is odd and the remainder mod 98 is 97, not 48.

11. Answer: (D) 5π. The idea. The left side is a single shifted sine: $\sqrt{3}\sin x + \cos x = 2\left(\sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}\right) = 2\sin\left(x + \frac{\pi}{6}\right)$. The equation becomes $\sin 3x = \sin\left(x + \frac{\pi}{6}\right)$, and two sines are equal exactly when the angles are equal or supplementary, up to multiples of 2π.

Work it out. *Equal angles:* $3x = x + \frac{\pi}{6} + 2\pi k$, so $x = \frac{\pi}{12} + \pi k$. In $[0, 2\pi)$: $x = \frac{\pi}{12}, \frac{13\pi}{12}$.

Supplementary angles: $3x = \pi - x - \frac{\pi}{6} + 2\pi k$, so $4x = \frac{5\pi}{6} + 2\pi k$ and $x = \frac{5\pi}{24} + \frac{\pi k}{2}$. In $[0, 2\pi)$: $x = \frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}, \frac{41\pi}{24}$.

The two lists share nothing (in units of $\frac{\pi}{24}$ they are 2, 26 and 5, 17, 29, 41), so there are six solutions with sum $\frac{(2+26)+(5+17+29+41)}{24}\pi = \frac{28+92}{24}\pi = 5\pi$.

Check. $x = \frac{17\pi}{24}$: $x + \frac{\pi}{6} = \frac{21\pi}{24}$ and $3x = \frac{51\pi}{24} \equiv \frac{3\pi}{24}$, and $\frac{21\pi}{24} + \frac{3\pi}{24} = \pi$, so the sines agree. There can be no more than six roots, since $\sin 3x - \sin(x + \frac{\pi}{6})$ is a trigonometric polynomial of degree 3. Forgetting the supplementary family gives $\frac{7\pi}{6}$; keeping only it gives $\frac{23\pi}{6}$.

12. Answer: (C) $\frac{9}{2} + 2\sqrt{2}$. The idea. The product is symmetric in x and y, so write it in terms of $s = x + y$. The constraint gives xy in terms of s, turning the problem into a one-variable quadratic in s. The catch is the range of s: real x, y on the circle force $s^2 \le 2(x^2 + y^2) = 8$, and whether the vertex of the quadratic lies inside that range decides where the minimum is.

Work it out. From $s^2 = x^2 + y^2 + 2xy = 4 + 2xy$ we get $xy = \frac{s^2-4}{2}$, so $(x+1)(y+1) = xy + s + 1 = \frac{s^2-4}{2} + s + 1 = \frac{(s+1)^2-3}{2}$. Every s with $-2\sqrt{2} \le s \le 2\sqrt{2}$ occurs (x, y are the real roots of $t^2 - st + \frac{s^2-4}{2}$, whose discriminant $8 - s^2 \ge 0$).

The vertex $s = -1$ lies inside the range, so $m = -\frac{3}{2}$. The maximum is at the endpoint farther from -1, namely $s = 2\sqrt{2}$: $M = \frac{(2\sqrt{2}+1)^2-3}{2} = \frac{6+4\sqrt{2}}{2} = 3 + 2\sqrt{2}$.

Therefore $M - m = 3 + 2\sqrt{2} + \frac{3}{2} = \frac{9}{2} + 2\sqrt{2}$.

Check. M occurs at $x = y = \sqrt{2}$: $(\sqrt{2}+1)^2 = 3 + 2\sqrt{2}$. m occurs when $x + y = -1$ and $xy = -\frac{3}{2}$, e.g. $x = \frac{-1+\sqrt{7}}{2}$, $y = \frac{-1-\sqrt{7}}{2}$: then $x^2 + y^2 = 1 + 3 = 4$ and $(x+1)(y+1) = -\frac{3}{2} - 1 + 1 = -\frac{3}{2}$. Using the endpoint $s = -2\sqrt{2}$ as the minimum instead gives $3 - 2\sqrt{2}$ and the trap answer $4\sqrt{2}$.

13. Answer: (B) $\sqrt{3} + \sqrt{2}$. The idea. Because $xyz = 1$, the reciprocal sum is $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{xy+yz+zx}{xyz} =$

$xy + yz + zx$. So the middle condition says $e_1 = e_2$ (sum equals sum of pairwise products), and together with $e_3 = 1$ that is exactly what makes $(x-1)(y-1)(z-1)$ vanish.

Work it out. Expanding, $(x-1)(y-1)(z-1) = xyz-(xy+yz+zx)+(x+y+z)-1 = 1-e_2+e_1-1 = 0$. So one of the numbers equals 1; by symmetry say $x = 1$.

Then $yz = 1$ and $y^2 + z^2 = 11 - 1 = 10$, so $(y+z)^2 = 10 + 2 = 12$ and $y + z = 2\sqrt{3}$ (positive). Hence y and z are the roots of $t^2 - 2\sqrt{3}\,t + 1 = 0$, namely $t = \sqrt{3} \pm \sqrt{2}$.

The three numbers are 1, $\sqrt{3}-\sqrt{2} \approx 0.318$ and $\sqrt{3}+\sqrt{2} \approx 3.146$, so the largest is $\sqrt{3}+\sqrt{2}$. (They are forced up to order: if two of them were 1, all three would be 1, and $1+1+1 \neq 11$.)

Check. Product: $1 \cdot (\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) = 3 - 2 = 1$. Squares: $1 + (5+2\sqrt{6}) + (5-2\sqrt{6}) = 11$. Sum $1 + 2\sqrt{3}$; reciprocals $1 + (\sqrt{3}-\sqrt{2}) + (\sqrt{3}+\sqrt{2}) = 1 + 2\sqrt{3}$. The distractor $1 + 2\sqrt{3}$ is the sum $x + y + z$ (which follows from $e_1^2 - 2e_1 = 11$ alone), not the largest number.

14. Answer: (E) $\sqrt{2} - 1$. The idea. Let $S(x)$ be the sum. The recursion $F_{n+2} = F_{n+1} + F_n$ says that $S - xS - x^2S$ collapses: almost every coefficient becomes $F_n - F_{n-1} - F_{n-2} = 0$. That gives S in closed form, and the condition $S = 1$ becomes a quadratic.

Work it out. Where the series converges,

$S - xS - x^2S = F_1x + (F_2 - F_1)x^2 + \sum_{n\geq 3}(F_n - F_{n-1} - F_{n-2})x^n = x,$

so $S(x) = \dfrac{x}{1 - x - x^2}$. Setting this equal to 1 gives $x = 1 - x - x^2$, i.e. $x^2 + 2x - 1 = 0$, whose positive root is $x = \sqrt{2} - 1 \approx 0.414$.

Convergence: $F_n \leq \varphi^n$ with $\varphi = \frac{1+\sqrt{5}}{2}$ (induction), so the terms are at most $(\varphi x)^n$ and the series converges whenever $\varphi x < 1$, i.e. $x < \frac{\sqrt{5}-1}{2} \approx 0.618$. Our $x \approx 0.414$ is inside this range, so the manipulation is legitimate. For $x \geq \frac{\sqrt{5}-1}{2}$ the terms satisfy $F_n x^n \geq \varphi^{n-2}x^n \geq \varphi^{-2}$, which do not tend to 0, so no other positive x can work.

Check. Numerically, with $x = 0.41421$: $x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots \approx 0.414 + 0.172 + 0.142 + 0.088 + 0.061 + 0.040 + 0.027 + \cdots$, and the tail (ratio about 0.67) brings the total to 1. The value $\frac{1}{3}$ gives only $\frac{3}{5}$, and $\frac{3-\sqrt{5}}{2}$ (the square of the convergence radius) gives about 0.81.

15. Answer: (D) $\frac{7\sqrt{3}}{3}$. The idea. An equilateral triangle is a vertex plus a 60° rotation: if A is one vertex, the vertex C is the image of B under a 60° rotation about A. Put A on the middle line and let the rotation carry the line ℓ_1 onto a line that must meet ℓ_3.

Work it out. Use coordinates with ℓ_2 the x-axis, ℓ_1 the line $y = 1$ and ℓ_3 the line $y = -2$, and put the vertex on ℓ_2 at the origin A. Let $B = (t, 1)$ be the vertex on ℓ_1. Rotating B by 60° clockwise about A gives

$C = \left(\frac{t}{2} + \frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2}t + \frac{1}{2}\right).$

This must lie on $y = -2$: $-\frac{\sqrt{3}}{2}t + \frac{1}{2} = -2$, so $t = \frac{5}{\sqrt{3}}$. (The counter-clockwise rotation gives $t = -\frac{5}{\sqrt{3}}$, a mirror image with the same size.) The side length satisfies $s^2 = AB^2 = t^2 + 1 = \frac{25}{3} + 1 = \frac{28}{3}$, so the area is

$\frac{\sqrt{3}}{4} \cdot \frac{28}{3} = \frac{7\sqrt{3}}{3}.$

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Check. $C = \left(\frac{5}{2\sqrt{3}} + \frac{\sqrt{3}}{2}, -2\right) = \left(\frac{4}{\sqrt{3}}, -2\right)$, so $AC^2 = \frac{16}{3} + 4 = \frac{28}{3}$ and $BC^2 = \left(\frac{1}{\sqrt{3}}\right)^2 + 3^2 = \frac{28}{3}$: all three sides agree. In general the side satisfies $s^2 = \frac{4}{3}(a^2 + ab + b^2)$ for gaps a and b; here $\frac{4}{3} \cdot 7$. Guessing that the side is the total width 3 gives $\frac{9\sqrt{3}}{4}$, which is too small.

16. Answer: (B) 816**. The idea.** Both conditions say something about absolute values first: $z^n = 1$ forces $|z| = 1$ and $(z+1)^n = 1$ forces $|z+1| = 1$. That pins z down to just two points, and then we only have to ask which powers kill them.

Work it out. The point z lies on the unit circle centred at 0 and on the unit circle centred at -1. These circles meet where the real part is $-\frac{1}{2}$, so $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i = e^{\pm 2\pi i/3}$. Then $z + 1 = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i = e^{\pm \pi i/3}$.

Now $z = e^{\pm 2\pi i/3}$ is a primitive cube root of unity, so $z^n = 1$ exactly when $3 \mid n$; and $z+1 = e^{\pm \pi i/3}$ is a primitive sixth root of unity, so $(z+1)^n = 1$ exactly when $6 \mid n$. Both hold exactly when $6 \mid n$.

The good $n \le 100$ are $6, 12, \ldots, 96$, and their sum is $6(1 + 2 + \cdots + 16) = 6 \cdot 136 = 816$.

Check. For $n = 3$: $z^3 = 1$ but $(z+1)^3 = e^{\pm \pi i} = -1$, so 3 is not good (the trap answer 1683 adds all multiples of 3). For $n = 6$: $z^6 = 1$ and $(z+1)^6 = e^{\pm 2\pi i} = 1$, so 6 is good.

17. Answer: (D) 108**. The idea.** We are counting partitions of 20 into at most 4 parts. Listing them directly is error-prone, but turning the Young diagram on its side (conjugation) swaps "at most 4 parts" for "every part is at most 4", and partitions with parts from $\{1, 2, 3, 4\}$ are easy to count layer by layer.

Work it out. Write a partition of 20 into at most 4 parts as rows of dots, largest row first. Reading the columns instead gives a partition of 20 in which every part is at most 4, and this correspondence is one-to-one. So we count ways to write $20 = 4d + 3c + 2b + a$ with $d, c, b, a \ge 0$.

For fixed d and c, the remainder $r = 20 - 4d - 3c$ is split into 2s and 1s in $\lfloor r/2 \rfloor + 1$ ways. Summing over c:

$d = 0$: $r = 20, 17, 14, 11, 8, 5, 2$ give $11 + 9 + 8 + 6 + 5 + 3 + 2 = 44$.

$d = 1$: $r = 16, 13, 10, 7, 4, 1$ give $9 + 7 + 6 + 4 + 3 + 1 = 30$.

$d = 2$: $r = 12, 9, 6, 3, 0$ give $7 + 5 + 4 + 2 + 1 = 19$.

$d = 3$: $r = 8, 5, 2$ give $5 + 3 + 2 = 10$. $d = 4$: $r = 4, 1$ give $3 + 1 = 4$. $d = 5$: $r = 0$ gives 1.

The total is $44 + 30 + 19 + 10 + 4 + 1 = 108$.

Check. Count by the number of non-empty purses instead: one purse 1 way, two purses 10 ways ($20 = a + b$ with $a \ge b \ge 1$), three purses 33 ways (the nearest integer to $20^2/12$), four purses 64 ways; $1 + 10 + 33 + 64 = 108$. Forgetting the empty purses gives 64; treating the purses as different gives $\binom{23}{3} = 1771$, and dividing that by 4! is not valid because many distributions have repeated amounts.

18. Answer: (E) 4**. The idea.** Each product has the shape $\tan x \cdot \tan(60^\circ - x) \cdot \tan(60^\circ + x)$: with $x = 5^\circ$ we get $5^\circ, 55^\circ, 65^\circ$, and with $x = 25^\circ$ we get $25^\circ, 35^\circ, 85^\circ$. That product is exactly $\tan 3x$, the tangent triple-angle formula in disguise.

Work it out. Let $t = \tan x$. Using $\tan 60^\circ = \sqrt{3}$,

$$\tan(60^\circ - x)\tan(60^\circ + x) = \frac{\sqrt{3} - t}{1 + \sqrt{3}\,t} \cdot \frac{\sqrt{3} + t}{1 - \sqrt{3}\,t} = \frac{3 - t^2}{1 - 3t^2}.$$

Multiplying by t gives $\dfrac{3t - t^3}{1 - 3t^2}$, which is the triple-angle formula for $\tan 3x$. Therefore

$\tan 5^\circ \tan 55^\circ \tan 65^\circ = \tan 15^\circ = 2 - \sqrt{3}$, and $\tan 25^\circ \tan 35^\circ \tan 85^\circ = \tan 75^\circ = 2 + \sqrt{3}$.

The sum is $(2 - \sqrt{3}) + (2 + \sqrt{3}) = 4$.

Check. Numerically $\tan 5^\circ \approx 0.08749$, $\tan 55^\circ \approx 1.42815$, $\tan 65^\circ \approx 2.14451$, with product $\approx 0.26795 = 2 - \sqrt{3}$. Multiplying the two products instead of adding gives $\tan 15^\circ \tan 75^\circ = 1$, a tempting wrong answer.

19. Answer: (D) $\frac{120}{7}$**. The idea.** Two separate facts. First, sliding diagonal $\overline{AC}$ along the bases turns the trapezoid into a triangle with sides 13, 15 and $4 + 10$, with the same height and the same area. Second, the diagonals cut the trapezoid into four triangles whose areas are fixed by the ratio of the bases.

Work it out. Translate $\overline{AC}$ by the vector $\overrightarrow{AB}$ to $\overline{BC'}$, where C' lies on line CD beyond C with $CC' = AB = 4$. Triangle BDC' has sides $BD = 15$, $BC' = 13$ and $DC' = 10 + 4 = 14$, and its height from B is the height of the trapezoid; its base equals the sum of the bases. So both have area equal to the 13-14-15 triangle: by Heron, $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$ (height 12).

Now split by the diagonals. Triangles PAB and PCD are similar with ratio $AB : CD = 2 : 5$, so $AP : PC = BP : PD = 2 : 5$. If $[PCD] = 25k$ then $[PAB] = 4k$, and each leg triangle has area $10k$ (for example $[APD] : [CPD] = AP : PC = 2 : 5$). The total is $4k + 10k + 10k + 25k = 49k = 84$, so $k = \frac{12}{7}$ and

$$[APD] = 10k = \frac{120}{7}.$$

Check. Coordinates: $D = (0, 0), C = (10, 0), A = (5, 12), B = (9, 12)$ satisfy $AC = \sqrt{25 + 144} = 13$ and $BD = \sqrt{81 + 144} = 15$. Then $P = D + \frac{5}{7}(B - D) = \left(\frac{45}{7}, \frac{60}{7}\right)$ and $[APD] = \frac{1}{2}\left|5 \cdot \frac{60}{7} - 12 \cdot \frac{45}{7}\right| = \frac{1}{2} \cdot \frac{240}{7} = \frac{120}{7}$. Splitting 84 into four equal parts gives 21, and taking a seventh of it gives 12 (which is the height, not an area); the other two pieces are $\frac{48}{7}$ and $\frac{300}{7}$.

20. Answer: (E) 54**. The idea.** φ is multiplicative, $\varphi(n) = \prod \varphi(p^a)$ with $\varphi(p^a) = p^{a-1}(p - 1)$. A product is a power of 2 only if every factor is, so the question becomes: which prime powers have $\varphi(p^a)$ a power of 2? The answer forces the odd part of n to be a product of distinct primes of the form $2^k + 1$.

Work it out. For $p = 2$, $\varphi(2^a) = 2^{a-1}$ is always a power of 2. For odd p, the factor p^{a-1} is odd, so we need $a = 1$ and $p - 1 = 2^k$: $p = 2^k + 1$. If k had an odd factor $m > 1$, say $k = mj$, then $2^j + 1$ would divide $2^{mj} + 1$ and p would not be prime; so k is a power of 2 and p is a Fermat prime $3, 5, 17, 257, \ldots$ (the next, 65537, is far beyond 1000).

So $\varphi(n)$ is a power of 2 exactly when $n = 2^a m$ with m a product of distinct primes from $\{3, 5, 17, 257\}$. The possible $m \le 1000$ are $1, 3, 5, 15, 17, 51, 85, 255, 257, 771$ (and $3 \cdot 5 \cdot 17 \cdot 257$, $5 \cdot 257$ are too big). For each, count $a \ge 0$ with $2^a m \le 1000$:

$m = 1$: 10; 3: 9; 5: 8; 15: 7; 17: 6; 51: 5; 85: 4; 255: 2; 257: 2; 771: 1.

Total: $10 + 9 + 8 + 7 + 6 + 5 + 4 + 2 + 2 + 1 = 54$.

Check. Spot checks: $\varphi(510) = \varphi(2)\varphi(3)\varphi(5)\varphi(17) = 1 \cdot 2 \cdot 4 \cdot 16 = 128$ and $\varphi(771) = 2 \cdot 256 = 512$ are powers of 2, while $\varphi(9) = 6$ and $\varphi(7) = 6$ are not (9 repeats a prime, $7 - 1$ is not a power of 2). Leaving out $n = 1$ gives 53; forgetting 257 gives 51; using only 3 and 5 gives 34; counting only odd n gives 10.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

21. Answer: (A) $-\frac{1}{3}$**. The idea.** The shortest route is the minor arc of the great circle through A and B (the circle cut out by the plane through A, B and the centre). By the mirror symmetry that swaps A and B, the route is highest exactly halfway, directly above the midpoint of the chord AB. So the peak latitude can be read off that midpoint.

Work it out. Take the sphere to have radius 1, the north pole on the z-axis, and put the towns at

$$A = (\cos 45^\circ, 0, \sin 45^\circ), \quad B = (\cos 45^\circ \cos\theta, \cos 45^\circ \sin\theta, \sin 45^\circ)\,.$$

The midpoint of chord AB has height $z = \sin 45^\circ$ and horizontal distance from the axis $\rho = \cos 45^\circ \cos\frac{\theta}{2}$ (its horizontal part is the midpoint of two horizontal vectors of length $\cos 45^\circ$ at angle θ). The peak of the route is the point of the sphere straight out from the centre through this midpoint, so its latitude L satisfies

$$\tan L = \frac{z}{\rho} = \frac{\tan 45^\circ}{\cos(\theta/2)} = \frac{1}{\cos(\theta/2)}.$$

With $L = 60^\circ$: $\cos\frac{\theta}{2} = \frac{1}{\sqrt{3}}$, so $\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$.

Check. Then $\theta \approx 109.5^\circ$. The peak point is $(\cos 45^\circ \cos\frac{\theta}{2}, \ldots)$ rescaled: its direction has height-to-horizontal ratio $\sqrt{3} = \tan 60^\circ$, as required. Sanity check at the extremes: for θ near 0 the formula gives $\tan L \approx 1$ (the route hugs latitude 45°), and as $\theta \to 180^\circ$ it gives $L \to 90^\circ$ (the route goes over the pole). Someone who thinks the route stays on the 45° parallel cannot reach 60° at all.

22. Answer: (B) 42**. The idea.** Pull out the gcd. With $g = \gcd(a, b)$, $a = gx$, $b = gy$ and $\gcd(x, y) = 1$, the lcm is gxy, and the equation factors like Simon's trick: $g(xy - x - y + 1) = 60$, i.e. $g(x-1)(y-1) = 60$.

Work it out. Put $u = x - 1$, $v = y - 1$; both are at least 1 (if $x = 1$ the left side is 0). Each pair (a, b) corresponds to exactly one triple (g, u, v) of positive integers with $g\,u\,v = 60$ and $\gcd(u + 1, v + 1) = 1$, and conversely.

Without the coprimality condition, the triples with product $60 = 2^2 \cdot 3 \cdot 5$ number $\binom{4}{2} \cdot 3 \cdot 3 = 54$ (distribute each prime's exponent among three places).

Remove the bad ones, where $u + 1$ and $v + 1$ share a prime q, i.e. $u \equiv v \equiv -1 \pmod q$ with $uv \mid 60$.

$q = 2$: u, v both odd, so $uv \mid 15$: that is $3 \cdot 3 = 9$ ordered pairs (u, v), each giving one g.

$q = 3$: $u, v \in \{2, 5, 20\}$ (the divisors of 60 that are 2 mod 3) with $uv \mid 60$: $(2, 2)$, $(2, 5)$, $(5, 2)$: 3 pairs.

$q \geq 5$: now $u, v \geq q - 1 \geq 4$ are divisors of 60 that are $\equiv -1 \pmod q$, and $uv \mid 60$ fails every time: $q = 5$ needs $u = v = 4$ ($16 \nmid 60$), $q = 7$ needs $u, v \in \{6, 20\}$, and for $q \geq 11$ already $uv \geq 100 > 60$. The $q = 2$ and $q = 3$ lists do not overlap, since every $q = 3$ pair contains the even number 2.

So $54 - 9 - 3 = 42$.

Check. $(u, v, g) = (1, 2, 30)$ gives $x = 2$, $y = 3$, $(a, b) = (60, 90)$: $180 + 30 = 210 = 60 + 90 + 60$. A bad triple such as $(1, 1, 60)$ would give $x = y = 2$, not coprime: $(120, 120)$ has $120 + 120 \neq 300$. Forgetting coprimality gives 54; removing only one family gives 45 or 51.

23. Answer: (A) 45**. The idea.** The visited vertices always form an unbroken arc, and right after

a new vertex is reached the token sits at one end of that arc. The wait for the next new vertex is then a gambler's-ruin exit time on a path, which has a clean formula.

Work it out. Suppose k vertices have been visited ($1 \le k \le 9$) and the token has just arrived at an end of the visited arc. Number the arc $1, 2, \ldots, k$ with the token at 1, and call the two unvisited vertices beyond the ends 0 and $k+1$ (when $k = 9$ these are the same vertex, which changes nothing). The next new vertex is reached when the walk first hits 0 or $k+1$.

For a fair ± 1 walk on $0, 1, \ldots, N$, the expected time E_i to hit an end from i satisfies $E_0 = E_N = 0$ and $E_i = 1 + \frac{1}{2}(E_{i-1} + E_{i+1})$, so the second differences are constant, -2, and $E_i = i(N-i)$. Here $N = k+1$ and $i = 1$, so the expected wait is $1 \cdot k = k$.

Adding the waits for $k = 1, 2, \ldots, 9$: $1 + 2 + \cdots + 9 = 45$.

Check. For a triangle the same argument gives $1 + 2 = 3$: one minute to reach a second vertex, then from an end of a 2-vertex arc the token reaches the last vertex each minute with probability $\frac{1}{2}$, taking 2 minutes on average. In general an n-gon takes $\frac{n(n-1)}{2}$ minutes.

24. Answer: (A) 502769**. The idea.** Write $3^k = (1+2)^k = \sum_j \binom{k}{j} 2^j$. Each $\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}$ is a polynomial of degree j, so cutting the sum off at $j = 10$ gives a polynomial of degree at most 10 that still equals 3^k at $k = 0..10$. That polynomial must be P, and it can be evaluated anywhere.

Work it out. Let $Q(x) = \sum_{j=0}^{10} \binom{x}{j} 2^j$. For an integer $0 \le k \le 10$ the missing terms have $j > k$, where $\binom{k}{j} = 0$, so $Q(k) = \sum_{j=0}^{k} \binom{k}{j} 2^j = 3^k$. Two polynomials of degree at most 10 that agree at 11 points are equal, so $P = Q$.

Then $P(12) = \sum_{j=0}^{10} \binom{12}{j} 2^j = 3^{12} - \binom{12}{11} 2^{11} - \binom{12}{12} 2^{12} = 531441 - 24576 - 4096 = 502769$.

Check. The same reasoning gives $P(11) = 3^{11} - 2^{11} = 175099$. Finite differences confirm the structure: the j-th forward difference of 3^k at 0 is 2^j, and a degree-10 polynomial is exactly the Newton series truncated after $j = 10$. The one-step rule $3^{12} - 2^{12} = 527345$ is what you get if you forget that two terms are now missing, and $3 \cdot P(11) = 525297$ wrongly assumes P still triples each step.

25. Answer: (E) $\frac{9765}{16384}$**. The idea.** Work mod 2: pressing a button twice undoes it, so the reachable patterns are the sums of subsets of the 5 button vectors in $\{0,1\}^4$, and all 16 are reachable exactly when the 4×5 wiring matrix has rank 4. Building the 5 columns one at a time is messy (a column may be wasted), but rank can be read from the rows: we need the 4 rows, which are independent uniform vectors in $\{0,1\}^5$, to be linearly independent mod 2.

Work it out. Choose the lamp rows one at a time. Row 1 must avoid the zero vector: probability $1 - \frac{1}{32}$. Given independent rows $1..i$, they span 2^i vectors, and row $i+1$ must avoid all of them: probability $1 - \frac{2^i}{32}$. Hence

$$\frac{31}{32} \cdot \frac{30}{32} \cdot \frac{28}{32} \cdot \frac{24}{32} = \frac{31}{32} \cdot \frac{15}{16} \cdot \frac{7}{8} \cdot \frac{3}{4} = \frac{9765}{16384} \approx 0.596.$$

Check. With only 4 buttons the same count gives $\frac{15}{16} \cdot \frac{14}{16} \cdot \frac{12}{16} \cdot \frac{8}{16} = \frac{315}{1024}$, so the fifth button roughly doubles the chance, as it should. Union bound: a dependency means some nonzero combination of lamp rows vanishes; each of the 15 combinations vanishes with probability $\frac{1}{32}$, so the failure chance is at most $\frac{15}{32}$ and the answer is at least $\frac{17}{32}$ (not equal to it, because the failure events overlap). Mistaking that bound's $\frac{15}{32}$ for the success chance is a common slip; it is the most the failure chance can be.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.