MathCastle

AMC 12

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

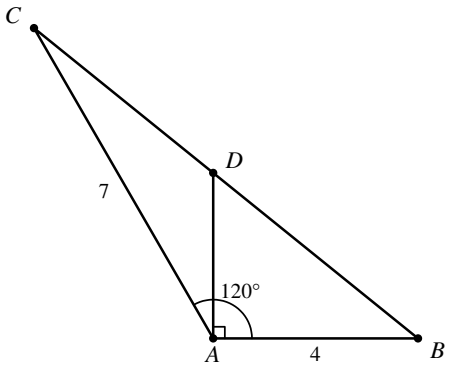
1. Real numbers x, y and z satisfy both $\cos x + \cos y + \cos z = 0$ and $\sin x + \sin y + \sin z = 0$. It turns out that the quantity $\cos^2 x + \cos^2 y + \cos^2 z$ takes the same value for every such triple. What is that value?

(A) 0 **(B)** $\frac{3}{4}$ **(C)** 1 **(D)** $\frac{3}{2}$ **(E)** 3

2. Real numbers a, b and c, each greater than 1, satisfy $\log_a(bc) = 2$ and $\log_b(ca) = 3$. What is the value of $\log_c(ab)$?

(A) $\frac{7}{5}$ **(B)** $\frac{7}{4}$ **(C)** $\frac{12}{5}$ **(D)** 5 **(E)** 6

3. In triangle ABC, $AB = 4$, $AC = 7$ and $\angle BAC = 120°$. The line through A perpendicular to AB meets side BC at the point D. What is the length AD?

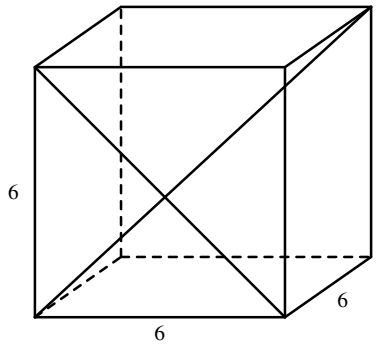


(A) $\frac{14\sqrt{3}}{11}$ **(B)** $\frac{28}{11}$ **(C)** $\frac{28\sqrt{3}}{15}$ **(D)** $2\sqrt{3}$ **(E)** $4\sqrt{3}$

4. For a positive integer n, let $\varphi(n)$ be the number of integers k with $1 \le k \le n$ that share no common factor greater than 1 with n. For example, $\varphi(10) = 4$, counting $1, 3, 7, 9$. How many positive integers n satisfy $\varphi(n) = 24$?

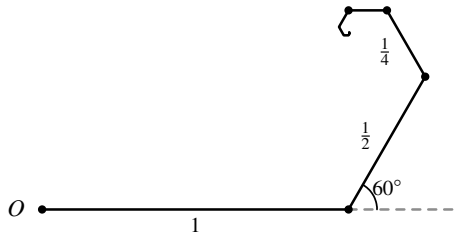
(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 12

5. A room is a cube with edges of length 6 metres. A taut wire runs along a space diagonal of the room, from one corner to the opposite corner through the middle of the room. A second taut wire runs along a diagonal of one of the walls (a face of the cube), chosen so that the two wires do not touch; in particular, it does not end at either corner used by the first wire. What is the shortest distance, in metres, between a point of the first wire and a point of the second wire?



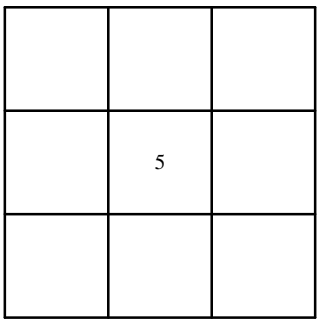
(A) $\frac{\sqrt{6}}{2}$ **(B)** $\sqrt{3}$ **(C)** 2 **(D)** $\frac{3\sqrt{2}}{2}$ **(E)** $\sqrt{6}$

6. A frog sits at a point O in a large flat field. It jumps 1 metre due east. Then it turns 60° to its left (counterclockwise) and jumps $\frac{1}{2}$ metre in its new direction; then it turns 60° left again and jumps $\frac{1}{4}$ metre; and so on forever: before each jump it turns 60° left, and each jump is half as long as the one before. The frog's positions get closer and closer to a point P. What is the distance OP, in metres?



(A) 1 **(B)** $\frac{2\sqrt{3}}{3}$ **(C)** $\frac{\sqrt{7}}{2}$ **(D)** $\sqrt{3}$ **(E)** 2

7. The numbers $1, 2, 3, \ldots, 9$ are written in the nine cells of a 3×3 grid, one number in each cell. The numbers must increase from left to right along every row and from top to bottom down every column, and the number 5 must be in the centre cell. How many different ways are there to fill the grid?



(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 18

8. The complex number z ranges over all complex numbers with $|z| = 1$. What is the largest possible value of $\left|z^3 - z^2 - z + 1\right|$?

(A) 2 **(B)** $2\sqrt{2}$ **(C)** 3 **(D)** $\frac{16\sqrt{3}}{9}$ **(E)** 4

9. Ten people sit around a round table, and each of them flips a fair coin once; the ten flips are independent. What is the probability that there are no three people sitting in consecutive seats around the table who all flip heads?

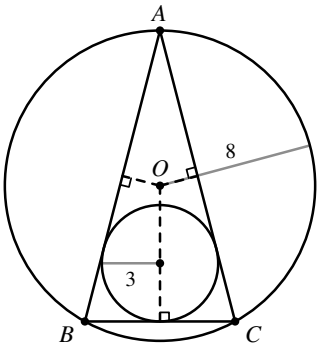
(A) $\frac{443}{1024}$ **(B)** $\frac{111}{256}$ **(C)** $\frac{7}{16}$ **(D)** $\frac{63}{128}$ **(E)** $\frac{1}{2}$

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10. Call a positive integer n a champion if n has more positive divisors than every positive integer smaller than n. For example, 1 is a champion (there is nothing smaller to beat), 4 is a champion since it has 3 divisors while $1, 2, 3$ have at most 2, but 5 is not. How many champions are there among $1, 2, 3, \ldots, 1000$?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

11. Triangle ABC has all three angles acute. Its inscribed circle has radius 3 and its circumscribed circle has radius 8. Let O be the centre of the circumscribed circle. What is the sum of the distances from O to the three lines BC, CA and AB?

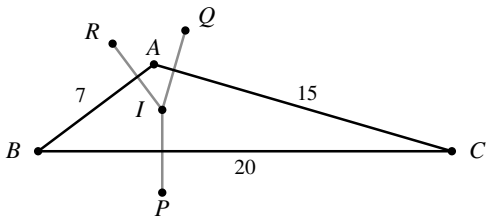


(A) 11 **(B)** 14 **(C)** 16 **(D)** 19 **(E)** 24

12. Real numbers a, b, c and d satisfy $a^2+b^2+c^2+d^2 = 1$. What is the largest possible value of $ab+bc+cd$?

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{1+\sqrt{5}}{4}$ **(E)** 1

13. Triangle ABC has side lengths $AB = 7$, $CA = 15$ and $BC = 20$. Let I be the center of the circle inscribed in the triangle. Reflect I across each of the three lines BC, CA and AB, obtaining three points P, Q and R. What is the area of triangle PQR?



(A) $\frac{42}{25}$ **(B)** $\frac{84}{25}$ **(C)** $\frac{168}{25}$ **(D)** 12 **(E)** $\frac{336}{25}$

14. Each of the integers $1, 2, 3, \ldots, 2026$ is coloured either red or blue. The colouring must satisfy two rules: whenever n and $2n$ are both at most 2026, they get different colours; and whenever n and $3n$ are both at most 2026, they get different colours. The number of colourings that satisfy both rules is 2^k. What is k?

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(A) 337 **(B)** 674 **(C)** 675 **(D)** 1013 **(E)** 1351

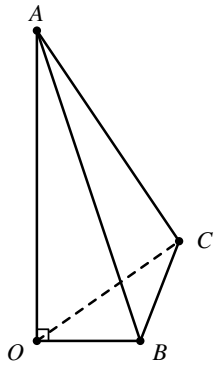
15. Compute the value of the infinite sum $\frac{1}{1^4+4} + \frac{2}{2^4+4} + \frac{3}{3^4+4} + \frac{4}{4^4+4} + \cdots$, whose nth term is $\frac{n}{n^4+4}$.

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{2}$

16. Compute the sum $\cos^6 0^\circ + \cos^6 1^\circ + \cos^6 2^\circ + \cdots + \cos^6 179^\circ$, which has 180 terms, one for each whole number of degrees from 0° to 179°. (Here $\cos^6 x$ means $(\cos x)^6$.)

(A) 45 **(B)** $\frac{225}{4}$ **(C)** 60 **(D)** $\frac{135}{2}$ **(E)** 90

17. Tetrahedron $OABC$ has right angles $\angle AOB = \angle BOC = \angle COA = 90^\circ$. Face OAB has area 3, face OBC has area 4, and the face opposite O, triangle ABC, has area 13. What is the volume of the tetrahedron?

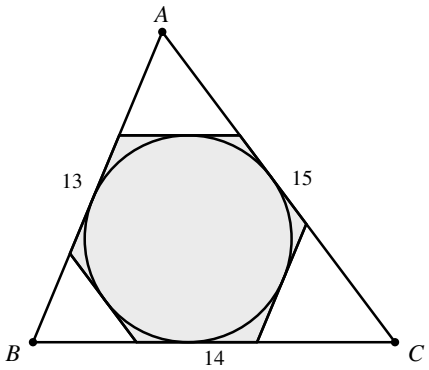


(A) 4 **(B)** $4\sqrt{2}$ **(C)** $8\sqrt{2}$ **(D)** $12\sqrt{2}$ **(E)** $24\sqrt{2}$

18. Ten guests leave their coats at a party. At the end of the evening the coats are handed back completely at random, one coat per guest, with all 10! ways of matching coats to guests equally likely. Let X be the number of guests who receive their own coat. What is the expected value of X^3?

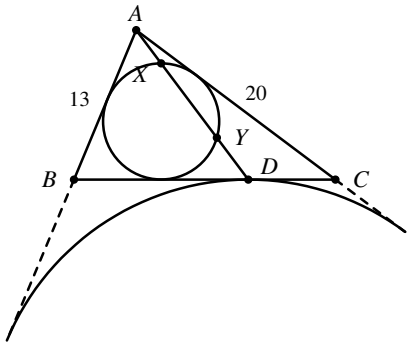
(A) 1 **(B)** 2 **(C)** $\frac{7}{2}$ **(D)** 4 **(E)** 5

19. Triangle ABC has side lengths $AB = 13$, $BC = 14$ and $CA = 15$. For each side of the triangle, draw the line parallel to that side and tangent to the triangle's inscribed circle, other than the line containing the side itself. Each of these three lines cuts a small triangle off one corner of ABC. Removing the three small triangles leaves a hexagon. What is the area of the hexagon?



(A) $\frac{596}{21}$ **(B)** $\frac{1168}{21}$ **(C)** 56 **(D)** 60 **(E)** $\frac{224}{3}$

20. Triangle ABC has $AB = 13$, $CA = 20$ and $BC = 21$. The excircle opposite A (the circle outside the triangle tangent to side $\overline{BC}$ and to the extensions of sides $\overline{AB}$ and $\overline{AC}$) touches $\overline{BC}$ at the point D. The line AD passes through the interior of the triangle's inscribed circle, meeting it at two points X and Y. What is the length XY?



(A) $\frac{56}{15}$ **(B)** 7 **(C)** $\frac{112}{15}$ **(D)** 8 **(E)** $\frac{28}{3}$

21. For real numbers a and b strictly between -1 and 1, define $a \star b = \dfrac{a+b}{1+ab}$. This operation is associative, and the result again lies strictly between -1 and 1. For $k = 2, 3, \ldots, 10$ let $x_k = \dfrac{2k}{k^2+1}$, so $x_2 = \frac{4}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{8}{17}$, and so on up to $x_{10} = \frac{20}{101}$. What is $x_2 \star x_3 \star x_4 \star \cdots \star x_{10}$?

(A) $\frac{1512}{1513}$ **(B)** $\frac{4355}{4357}$ **(C)** $\frac{3024}{3025}$ **(D)** $\frac{3025}{3026}$ **(E)** $\frac{12099}{12101}$

22. For each integer a with $1 \le a \le 36$, let $m(a)$ be the smallest positive integer m such that $a^m - 1$ is divisible by 37. For example, $m(1) = 1$, and $m(36) = 2$ because $36^2 - 1 = 35 \cdot 37$. What is $m(1) + m(2) + m(3) + \cdots + m(36)$?

(A) 666 **(B)** 671 **(C)** 703 **(D)** 1296 **(E)** 1911

23. Ten boxes are numbered 1 through 10, and each box has its own key. The ten keys are shuffled and one key is locked inside each box, with all 10! placements equally likely. For a box i, its *loop* is found as follows: open box i, then open the box whose key was just found, and keep going until the key of box i turns up; the loop of i is the set of boxes opened along the way (including box i). What is the probability that box 2 is in the loop of box 1 *and* box 4 is in the loop of box 3?

(A) $\frac{7}{24}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{5}{12}$ **(E)** $\frac{1}{2}$

24. A solid regular tetrahedron with all six edges of length 1 is held in sunlight, which travels in parallel rays, above a flat screen perpendicular to the rays. The tetrahedron can be turned to any orientation. What is the largest possible area of its shadow on the screen?

(A) $\frac{\sqrt{3}}{4}$ **(B)** $\frac{\sqrt{2}}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\sqrt{3}}{3}$ **(E)** $\frac{\sqrt{2}}{2}$

25. For a nonnegative integer n, let $s(n)$ be the number of 1s in the base-2 representation of n; for example

$s(0) = 0$ and $s(13) = s(1101_2) = 3$. Each integer n from 0 to 31 is given the sign $(-1)^{s(n)}$. Compute $\sum_{n=0}^{31}(-1)^{s(n)}\, n^6 = 0^6 - 1^6 - 2^6 + 3^6 - 4^6 + 5^6 + 6^6 - 7^6 + \cdots - 31^6$.

(A) -79994880 **(B)** -23592960 **(C)** -22855680 **(D)** -11796480 **(E)** -11427840

Answer Key — AMC 12 — Hard Edition Practice Contest 1

1. (D) $\frac{3}{2}$

2. (A) $\frac{7}{5}$

3. (C) $\frac{28\sqrt{3}}{15}$

4. (D) 10

5. (E) $\sqrt{6}$

6. (B) $\frac{2\sqrt{3}}{3}$

7. (E) 18

8. (D) $\frac{16\sqrt{3}}{9}$

9. (A) $\frac{443}{1024}$

10. (D) 15

11. (A) 11

12. (D) $\frac{1+\sqrt{5}}{4}$

13. (E) $\frac{336}{25}$

14. (C) 675

15. (C) $\frac{3}{8}$

16. (B) $\frac{225}{4}$

17. (B) $4\sqrt{2}$

18. (E) 5

19. (B) $\frac{1168}{21}$

20. (C) $\frac{112}{15}$

21. (A) $\frac{1512}{1513}$

22. (B) 671

23. (A) $\frac{7}{24}$

24. (C) $\frac{1}{2}$

25. (E) -11427840

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Solutions — AMC 12 — Hard Edition Practice Contest 1

1. Answer: (D) $\frac{3}{2}$. The idea. Read the two equations as one statement about complex numbers: the unit vectors e^{ix}, e^{iy}, e^{iz} add up to 0. Three unit vectors with zero sum form a closed equilateral triangle when laid head to tail, so they point 120° apart. Then the doubled angles also point 120° apart, and their cosines cancel too.

Work it out. Let $u = e^{ix}$, $v = e^{iy}$, $w = e^{iz}$, so $u + v + w = 0$. Because $|u| = 1$ we have $\bar{u} = \frac{1}{u}$, and conjugating gives $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 0$, that is, $uv + vw + wu = 0$. Hence $u^2 + v^2 + w^2 = (u+v+w)^2 - 2(uv+vw+wu) = 0$. The real part of this says $\cos 2x + \cos 2y + \cos 2z = 0$. Now $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, so $\cos^2 x + \cos^2 y + \cos^2 z = \frac{3}{2} + \frac{1}{2}(\cos 2x + \cos 2y + \cos 2z) = \frac{3}{2}$.

Check. Take $x = 0$, $y = 120°$, $z = 240°$: the cosines are $1, -\frac{1}{2}, -\frac{1}{2}$ (sum 0), the sines $0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$ (sum 0), and the squares of the cosines add to $1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$.

2. Answer: (A) $\frac{7}{5}$. The idea. Adding 1 to each logarithm turns it into a ratio with a common numerator: $1 + \log_a(bc) = \log_a(abc) = \frac{\ln(abc)}{\ln a}$. The reciprocals $\frac{\ln a}{\ln(abc)}, \frac{\ln b}{\ln(abc)}, \frac{\ln c}{\ln(abc)}$ add to exactly 1.

Work it out. Let $t = \log_c(ab)$. Then $\frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+t} = \frac{\ln a + \ln b + \ln c}{\ln(abc)} = 1$. So $\frac{1}{1+t} = 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$, which gives $1 + t = \frac{12}{5}$ and $t = \frac{7}{5}$.

Check. Solve directly with $\ln a = 1$, $\ln b = p$, $\ln c = q$: the conditions say $p + q = 2$ and $q + 1 = 3p$, so $p = \frac{3}{4}$, $q = \frac{5}{4}$ (both positive, so $b, c > 1$). Then $\log_c(ab) = \frac{1+\frac{3}{4}}{\frac{5}{4}} = \frac{7}{5}$. The trap answer $\frac{12}{5}$ forgets to subtract the 1 at the end, and $\frac{7}{4} = \ln(ab)$ forgets to divide by $\ln c$.

3. Answer: (C) $\frac{28\sqrt{3}}{15}$. The idea. The segment AD splits the 120° angle at A into a right angle $\angle BAD = 90°$ and $\angle DAC = 30°$. Triangles ABD and ADC together make up triangle ABC, and each of the three areas is $\frac{1}{2} \cdot (\text{side}) \cdot (\text{side}) \cdot \sin(\text{angle at } A)$. Adding the two small areas gives one linear equation in the unknown AD.

Work it out. Let $AD = x$. Then $[ABD] + [ADC] = [ABC]$ reads $\frac{1}{2} \cdot 4x \sin 90° + \frac{1}{2} \cdot 7x \sin 30° = \frac{1}{2} \cdot 4 \cdot 7 \sin 120°$, that is, $2x + \frac{7x}{4} = 7\sqrt{3}$. So $\frac{15x}{4} = 7\sqrt{3}$ and $x = \frac{28\sqrt{3}}{15}$.

Check. Put $A = (0,0)$, $B = (4,0)$, $C = \left(-\frac{7}{2}, \frac{7\sqrt{3}}{2}\right)$. The line through A perpendicular to AB is the y-axis. Along BC the x-coordinate $4 - \frac{15}{2}t$ vanishes at $t = \frac{8}{15}$, where $y = \frac{8}{15} \cdot \frac{7\sqrt{3}}{2} = \frac{28\sqrt{3}}{15}$. Since $0 < t < 1$, D really is on side BC. Treating $\sin 30°$ as 1 gives the trap $\frac{14\sqrt{3}}{11}$; the angle-bisector length $\frac{4 \cdot 7}{4+7} = \frac{28}{11}$ answers a different question; and $4\sqrt{3} = AB \tan 60°$ wrongly takes $\angle ABC = 60°$, when in fact right triangle ABD gives $AD = AB \tan \angle ABC = 4 \cdot \frac{7\sqrt{3}}{15}$, the same answer again.

4. Answer: (D) 10. The idea. If $n = \prod p^e$, then $\varphi(n) = \prod p^{e-1}(p-1)$. Every prime p dividing n contributes a factor $p - 1$, which must divide 24, so $p \in \{2, 3, 5, 7, 13\}$ (the primes with $p - 1 \in \{1, 2, 3, 4, 6, 8, 12, 24\}$). Also $p^2 \mid n$ puts a factor p into $\varphi(n)$, so only 2 and 3 can appear squared. The rest is careful bookkeeping.

Work it out. Write $n = 2^a m$ with m odd; the odd part m uses only $3, 5, 7, 13$, with φ-factors $\varphi(3) = 2$, $\varphi(9) = 6$, $\varphi(5) = 4$, $\varphi(7) = 6$, $\varphi(13) = 12$ (and 27 is out since $\varphi(27) = 18$). The

odd m and their φ values that divide 24: 1: 1, 3: 2, 5: 4, 7: 6, 9: 6, 13: 12, 15: 8, 21: 12, 35: 24, 39: 24, 45: 24 (other products such as 63, 65, 91 give 36, 48, 72). We need $\varphi(2^a)\varphi(m) = 24$ with $\varphi(2^a) \in \{1, 1, 2, 4, 8, \ldots\}$ for $a = 0, 1, 2, 3, 4$.

$\varphi(m) = 24$: $m \in \{35, 39, 45\}$, $a \in \{0, 1\}$: 35, 70, 39, 78, 45, 90 (six numbers).

$\varphi(m) = 12$: need $\varphi(2^a) = 2$, $a = 2$: $4 \cdot 13 = 52$, $4 \cdot 21 = 84$.

$\varphi(m) = 6$: need $\varphi(2^a) = 4$, $a = 3$: $8 \cdot 7 = 56$, $8 \cdot 9 = 72$.

$\varphi(m) \in \{1, 2, 4, 8\}$ would need $\varphi(2^a) \in \{24, 12, 6, 3\}$, impossible (these are not powers of 2). Total: $6 + 2 + 2 = 10$.

Check. The list is 35, 39, 45, 52, 56, 70, 72, 78, 84, 90; for instance $\varphi(72) = \varphi(8)\varphi(9) = 4 \cdot 6 = 24$ and $\varphi(84) = \varphi(4)\varphi(3)\varphi(7) = 2 \cdot 2 \cdot 6 = 24$. A computer search up to 5000 (far beyond the bound $\varphi(n) \geq \sqrt{n/2}$ forces) finds exactly these ten.

5. Answer: (E) $\sqrt{6}$. The idea. The two wires lie on skew lines. The shortest segment between skew lines is perpendicular to both, and its length is the projection of any connecting vector onto the common normal $\vec{u} \times \vec{v}$.

Work it out. Coordinates: the space diagonal runs from $(0, 0, 0)$ to $(6, 6, 6)$, direction $\vec{u} = (1, 1, 1)$. Every wall contains one end of the space diagonal; in the floor $z = 0$ the diagonal avoiding $(0, 0, 0)$ runs from $(6, 0, 0)$ to $(0, 6, 0)$, direction $\vec{v} = (-1, 1, 0)$. (By the symmetry of the cube about its space diagonal, all six admissible wall diagonals give the same answer.) Then $\vec{u} \times \vec{v} = (1 \cdot 0 - 1 \cdot 1,\ 1 \cdot (-1) - 1 \cdot 0,\ 1 \cdot 1 - 1 \cdot (-1)) = (-1, -1, 2)$, of length $\sqrt{6}$. The vector from $(0, 0, 0)$ to $(6, 0, 0)$ is $(6, 0, 0)$, and its dot product with the normal is -6. The distance is $\frac{|-6|}{\sqrt{6}} = \sqrt{6}$.

The closest points really lie on the wires, not just on the extended lines: for $X = (s, s, s)$ and $Y = (6 - t, t, 0)$, the vector $X - Y = (s - 6 + t,\ s - t,\ s)$ is perpendicular to $(1, 1, 1)$ when $3s - 6 = 0$ and to $(-1, 1, 0)$ when $6 - 2t = 0$. So $s = 2$, $t = 3$: the points $(2, 2, 2)$ and $(3, 3, 0)$, both inside their segments, are $\sqrt{1 + 1 + 4} = \sqrt{6}$ apart.

Check. Project along the space diagonal: the cube's shadow is a regular hexagon whose vertices are the images of the six corners off the diagonal, and the diagonal itself shrinks to the centre. The wall diagonal projects onto a long chord joining alternate hexagon vertices; its distance from the centre is half the hexagon's circumradius. The circumradius is the distance from $(6, 0, 0)$ to the diagonal, $6\sqrt{2/3} = 2\sqrt{6}$, so the distance is $\sqrt{6}$.

6. Answer: (B) $\frac{2\sqrt{3}}{3}$. The idea. In the complex plane, "turn 60° left and halve the length" is multiplication by $q = \frac{1}{2}e^{i\pi/3}$. So the jumps are $1, q, q^2, \ldots$ and P is the sum of a geometric series.

Work it out. With $O = 0$ and east the positive real direction, $P = 1 + q + q^2 + \cdots = \frac{1}{1-q}$, valid since $|q| = \frac{1}{2} < 1$. Now $q = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} + \frac{\sqrt{3}}{4}i$, so $1 - q = \frac{3}{4} - \frac{\sqrt{3}}{4}i$ and $|1 - q| = \sqrt{\frac{9}{16} + \frac{3}{16}} = \frac{\sqrt{3}}{2}$. Hence $OP = \frac{1}{|1-q|} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

Check. The six jumps of one full turn add to $(1 - q^6)/(1 - q)$ with $q^6 = \frac{1}{64}$, so after each full turn the whole picture repeats at scale $\frac{1}{64}$; the partial sums converge fast, and after two jumps the frog is already at $\left(\frac{5}{4}, \frac{\sqrt{3}}{4}\right)$, distance $\frac{\sqrt{28}}{4} \approx 1.32$ (the trap answer $\frac{\sqrt{7}}{2}$), and later jumps pull it back toward 1.155. The total path length 2 is only an upper bound.

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7. Answer: (E) 18**. The idea.** The centre cell splits the grid. The three cells above or to the left of the centre (top-left, top-middle, middle-left) must hold numbers smaller than 5, and the three cells below or to the right of it (middle-right, bottom-middle, bottom-right) must hold numbers larger than 5. That uses three of the four small numbers $1, 2, 3, 4$ and three of the four large numbers $6, 7, 8, 9$; the leftover small number and the leftover large number go in the two remaining corners, top-right and bottom-left.

Work it out. Suppose the fourth small number goes in the top-right corner. Then $1, 2, 3, 4$ fill the whole top row and the middle-left cell. The 1 must be top-left; the 4 is either top-right or middle-left (it cannot be top-middle, which is smaller than top-right). If the 4 is middle-left, the top row is $1, 2, 3$. If the 4 is top-right, the top row is $1, a, 4$ and the middle-left is the other of $2, 3$, with $a \in \{2, 3\}$ free: 2 ways. So the small part can be filled in 3 ways. The large numbers $6, 7, 8, 9$ then fill the whole bottom row and the middle-right cell, the mirror image of the small shape (turn the grid through 180° and replace each k by $10 - k$), so they too can be filled in 3 ways. Every choice is compatible, since each small number is below each large one. This gives $3 \cdot 3 = 9$ fillings, and by the symmetry across the main diagonal the case with the fourth small number in the bottom-left corner gives another 9. The total is 18.

Check. A computer check of all 9! arrangements finds 18 valid fillings with 5 in the centre (and 12 each with 4 or 6 in the centre, 42 in all). The trap 9 forgets that the leftover small number can take either corner, adding the two 3's instead of multiplying gives 6, and 12 is the count for the wrong centre value.

8. Answer: (D) $\frac{16\sqrt{3}}{9}$**. The idea.** Factor first: $z^3 - z^2 - z + 1 = z^2(z-1) - (z-1) = (z-1)^2(z+1)$. On the unit circle, the distances $|z - 1|$ and $|z + 1|$ are tied together (they are the legs of a right triangle on the diameter from -1 to 1), so everything reduces to a one-variable maximisation.

Work it out. Write $z = \cos\theta + i\sin\theta$ and $c = \cos\theta$. Then $|z-1|^2 = 2-2c$ and $|z+1|^2 = 2+2c$; in particular $|z-1|^2+|z+1|^2 = 4$. Let $u = |z+1|^2 \in [0, 4]$. The quantity is $|z-1|^2 \cdot |z+1| = (4-u)\sqrt{u}$. Put $u = v^2$ with $v \in [0, 2]$: maximise $g(v) = 4v - v^3$. Then $g'(v) = 4 - 3v^2 = 0$ at $v = \frac{2}{\sqrt{3}}$, giving $g = \frac{8}{\sqrt{3}} - \frac{8}{3\sqrt{3}} = \frac{16}{3\sqrt{3}} = \frac{16\sqrt{3}}{9}$. At the ends $g(0) = 0$ and $g(2) = 0$, so this is the maximum.

Check. It is attained at $\cos\theta = \frac{u}{2} - 1 = -\frac{1}{3}$: then $|z-1|^2 = \frac{8}{3}$ and $|z+1| = \frac{2}{\sqrt{3}}$, product $\frac{16}{3\sqrt{3}} \approx 3.079$. The triangle inequality only gives the bound 4, which is never reached; the natural guess $z = -1$ gives 0 and $z = \pm i$ gives $2 \cdot \sqrt{2} \approx 2.83$.

9. Answer: (A) $\frac{443}{1024}$**. The idea.** First count strings of H and T in a row (not a circle) with no HHH. Then handle the circle by looking at the run of heads that wraps around from the last seat to the first: cut the circle just after the tail that ends that run.

Work it out. Let t_n be the number of length-n rows with no three consecutive H's. Such a row is empty or ends in T, TH or THH after a shorter valid row, so $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ with $t_0 = 1$, $t_1 = 2$, $t_2 = 4$. This gives $t_3, \ldots, t_8 = 7, 13, 24, 44, 81, 149$.

Number the seats 1 to 10. At least one person flips tails (ten heads is bad). Suppose seats $1, \ldots, i$ are heads and seat $i + 1$ is tails, and seats $10, 10 - 1, \ldots, 11 - j$ are heads and seat $10 - j$ is tails. The run of heads through the seam has length $i + j$, so we need $i + j \le 2$; the seats strictly between the two tails form a row of $8 - i - j$ seats with no HHH, and anything allowed there works. The count is $t_8 + 2t_7 + 3t_6 = 149 + 162 + 132 = 443$, from $i + j = 0$ (one way), $i + j = 1$ (two ways) and $i + j = 2$ (three ways). The probability is $\frac{443}{1024}$.

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Check. Because $i + j \le 2 < 9$, the two tails are different seats, so every good circle is counted exactly once. Try the method on 4 seats: $t_2 + 2t_1 + 3t_0 = 4 + 4 + 3 = 11$; directly, of the 16 circles the bad ones are $HHHH$ and the four with a single tail, leaving 11. The trap $\frac{63}{128} = \frac{t_{10}}{1024} = \frac{504}{1024}$ counts the seats in a row and forgets the triples that wrap around.

10. Answer: (D) 15**. The idea.** A champion must be of the form $2^a 3^b 5^c 7^d \cdots$ with $a \ge b \ge c \ge d \ge \cdots$, using consecutive primes from 2. Otherwise swapping a larger prime's exponent onto a smaller prime gives a smaller number with the same number of divisors. That leaves few candidates, and we sweep through them tracking the record.

Work it out. Candidates up to 1000 with their divisor counts $d(n)$: 1(1), 2(2), 4(3), 6(4), 8(4), 12(6), 16(5), 24(8), 30(8), 32(6), 36(9), 48(10), 60(12), 64(7), 72(12), 96(12), 120(16), 128(8), 144(15), 180(18), 192(14), 210(16), 216(16), 240(20), 256(9), 288(18), 360(24), 384(16), 420(24), 432(20), 480(24), 512(10), 576(21), 720(30), 768(18), 840(32), 864(24), 900(27), 960(28).

Keep each one that beats every earlier count: 1(1), 2(2), 4(3), 6(4), 12(6), 24(8), 36(9), 48(10), 60(12), 120(16), 180(18), 240(20), 360(24), 720(30), 840(32). That is 15 champions.

Check. Near misses: 30 ties 24 with 8 divisors (ties do not count), 72 ties 60, 420 ties 360. A direct computer count of $d(n)$ for all $n \le 1000$ gives the same 15.

11. Answer: (A) 11**. The idea.** The distance from O to side BC is $R\cos A$ (drop the perpendicular to the midpoint of BC; the central angle $\angle BOC$ is $2A$, and half of it is A). So the sum is $R(\cos A + \cos B + \cos C)$, and a classical identity says $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$. The answer is simply $R + r$.

Work it out. Prove the identity. With $A + B + C = \pi$: $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 2\sin\frac{C}{2}\cos\frac{A-B}{2}$ and $\cos C = 1 - 2\sin^2\frac{C}{2} = 1 - 2\sin\frac{C}{2}\cos\frac{A+B}{2}$. Adding, $\cos A + \cos B + \cos C = 1 + 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

On the other hand $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$: the incentre I sees side BC with $BC = r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = r\frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$, while $BC = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$. Equating gives the formula. So $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$, and because the triangle is acute each distance is $R\cos A > 0$ etc., so the sum is $R + r = 8 + 3 = 11$.

Check. Equilateral case: $R = 2r$ and each distance is r, sum $3r = R + r$. For the given data, a computer builds an isosceles and a scalene acute triangle with $R = 8$, $r = 3$ and measures the three distances directly: both sums are 11. (If the triangle were obtuse, one term $R\cos A$ would be negative and the actual distances would not add to 11, which is why acuteness is stated.)

12. Answer: (D) $\frac{1+\sqrt{5}}{4}$**. The idea.** Guess the shape of the optimum from symmetry ($a \leftrightarrow d$, $b \leftrightarrow c$), then prove the bound by writing $M(a^2 + b^2 + c^2 + d^2) - (ab + bc + cd)$ as a sum of squares. The right constant turns out to be $M = \frac{\varphi}{2}$ where $\varphi = \frac{1+\sqrt{5}}{2}$, the golden ratio, which satisfies $\varphi^2 = \varphi + 1$.

Work it out. Try $a = d = p$, $b = c = q$ with $2p^2 + 2q^2 = 1$: the value is $2pq + q^2$. With $t = \frac{q}{p}$ this is $\frac{t^2+2t}{2(1+t^2)}$, whose derivative vanishes when $t^2 - t - 1 = 0$, that is, at $t = \varphi$. Taking $q = \varphi p$ gives $2p^2(1 + \varphi^2) = 1$ and value $p^2(2\varphi + \varphi^2) = p^2(3\varphi + 1)$; since $1 + \varphi^2 = \varphi + 2$, this is $\frac{3\varphi+1}{2(\varphi+2)}$. Now $\frac{\varphi}{2}(\varphi + 2)\cdot 2 = \varphi^2 + 2\varphi = 3\varphi + 1$, so the value equals $\frac{\varphi}{2} = \frac{1+\sqrt{5}}{4}$.

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Upper bound: let $m = \frac{\varphi}{2}$, so $4m^2 = \varphi^2 = \varphi + 1 = 2m + 1$. Then $m(a^2+b^2+c^2+d^2) - (ab+bc+cd) = m\left(a - \frac{b}{2m}\right)^2 + m\left(d - \frac{c}{2m}\right)^2 + \left(m - \frac{1}{4m}\right)(b^2+c^2) - bc$. Since $4m^2 - 1 = 2m$, we get $m - \frac{1}{4m} = \frac{4m^2-1}{4m} = \frac{1}{2}$, and the last part is $\frac{1}{2}(b^2+c^2) - bc = \frac{1}{2}(b-c)^2 \geq 0$. So $ab + bc + cd \leq m = \frac{1+\sqrt{5}}{4}$, with equality in the case found above.

Check. Numerically $\frac{1+\sqrt{5}}{4} \approx 0.809 = \cos 36°$, the largest eigenvalue of the symmetric matrix of the form. The trap $\frac{\sqrt{2}}{2}$ is the answer for $ab + bc$ with three variables; the crude bound $xy \leq \frac{x^2+y^2}{2}$ on each term gives only 1, which is not attained.

13. Answer: (E) $\frac{336}{25}$. The idea. If the incircle touches BC at D, then $\overline{ID}$ is a radius perpendicular to BC, so the mirror image of I in BC is the point on ray ID at distance $2r$ from I. In other words, P, Q, R are the three touch points D, E, F pushed out from I by a factor of 2: triangle PQR is triangle DEF enlarged by 2 about I, so its area is 4 $[DEF]$. And $[DEF]$ is easy from I, because I is the circumcenter of DEF (all three are at distance r) and the angle $\angle EIF$ is $180° - A$ (quadrilateral $AEIF$ has right angles at E and F).

Work it out. Here $s = 21$ and by Heron $K = \sqrt{21 \cdot 1 \cdot 6 \cdot 14} = 42$, so $r = K/s = 2$ and $R = \frac{abc}{4K} = \frac{20 \cdot 15 \cdot 7}{168} = \frac{25}{2}$. Triangle DEF is acute (its angles are $90° - \frac{A}{2}$ and so on), so its circumcenter I lies inside it and $[DEF] = [IEF] + [IFD] + [IDE] = \frac{1}{2}r^2\big(\sin(180°-A) + \sin(180°-B) + \sin(180°-C)\big) = \frac{1}{2}r^2(\sin A + \sin B + \sin C)$. By the law of sines $\sin A + \sin B + \sin C = \frac{a+b+c}{2R} = \frac{2s}{2R} = \frac{s}{R}$, so $[DEF] = \frac{r^2 s}{2R} = \frac{4 \cdot 21}{25} = \frac{84}{25}$. Therefore $[PQR] = 4 \cdot \frac{84}{25} = \frac{336}{25}$. (Note that angle A is obtuse here, $\cos A = -\frac{3}{5}$, and nothing in the argument cares.)

Check. Coordinates: $B = (0,0)$, $C = (20,0)$, $A = \left(\frac{28}{5}, \frac{21}{5}\right)$. The incircle touches BC at distance $s - b = 6$ from B, so $I = (6,2)$ and $P = (6,-2)$. Reflecting across CA and AB gives $Q = \left(\frac{178}{25}, \frac{146}{25}\right)$ and $R = \left(\frac{18}{5}, \frac{26}{5}\right)$, and the shoelace formula gives area $\frac{336}{25}$ again. Forgetting the factor 4 gives $\frac{84}{25}$, the area of the contact triangle.

14. Answer: (C) 675. The idea. Draw a graph on $\{1, \ldots, 2026\}$ with an edge from n to $2n$ and from n to $3n$. A valid colouring is a proper 2-colouring of this graph. Every number is $m \cdot 2^i 3^j$ with m coprime to 6, and the edges never change m; so the graph splits into one piece for each m. Each piece is a staircase-shaped patch of the (i, j) grid, which is connected and has no odd cycle, so it has exactly 2 proper colourings.

Work it out. Fix m coprime to 6 with $m \leq 2026$. The numbers of the form $m \cdot 2^i 3^j$ up to 2026 correspond to lattice points (i, j) with $i, j \geq 0$ and $2^i 3^j \leq 2026/m$. This set is closed under decreasing i or j, so from any point you can walk down to $(0,0)$ step by step along edges: the piece is connected. Each edge changes $i + j$ by exactly 1, so colouring by the parity of $i + j$ works, and once m itself has a colour, connectivity forces every other colour in the piece. Hence each piece has exactly 2 colourings, the pieces are independent, and the total is 2^k where k is the number of $m \leq 2026$ coprime to 6. Since $2026 = 6 \cdot 337 + 4$, each of the 337 full blocks of six contributes the two residues 1 and 5, and the leftover 2023, 2024, 2025, 2026 contributes only 2023. So $k = 2 \cdot 337 + 1 = 675$.

Check. The piece containing 1 is $\{1, 2, 3, 4, 6, 8, 9, 12, \ldots\}$, and the parity rule gives 1 red, 2, 3 blue, 4, 6, 9 red, consistent around every little square $n, 2n, 3n, 6n$. Using only the doubling rule would count the 1013 odd numbers; using only the tripling rule would count the 1351 numbers

not divisible by 3; dropping the leftover 2023 gives 674.

15. Answer: (C) $\frac{3}{8}$. The idea. The denominator factors by completing a square: $n^4+4 = (n^2+2)^2 - (2n)^2 = (n^2-2n+2)(n^2+2n+2)$. The two factors differ by exactly $4n$, a multiple of the numerator, so each term splits into a difference of two reciprocals. Writing $f(n) = n^2-2n+2 = (n-1)^2+1$, the larger factor is $n^2+2n+2 = (n+1)^2+1 = f(n+2)$, so the sum telescopes, but with a shift of two instead of one.

Work it out. Since $f(n+2) - f(n) = 4n$, $\frac{n}{n^4+4} = \frac{n}{f(n)f(n+2)} = \frac{1}{4}\left(\frac{1}{f(n)} - \frac{1}{f(n+2)}\right)$. Summing from $n = 1$ to N, every $\frac{1}{f(j)}$ with $3 \le j \le N$ appears once with a plus sign and once with a minus sign, leaving $\frac{1}{4}\left(\frac{1}{f(1)} + \frac{1}{f(2)} - \frac{1}{f(N+1)} - \frac{1}{f(N+2)}\right)$. As $N \to \infty$ the last two terms go to 0, and $f(1) = 1$, $f(2) = 2$, so the sum is $\frac{1}{4}\left(1 + \frac{1}{2}\right) = \frac{3}{8}$.

Check. The first terms are $\frac{1}{5} + \frac{2}{20} + \frac{3}{85} + \frac{4}{260} + \frac{5}{629} = 0.2+0.1+0.0353+0.0154+0.0079 \approx 0.3586$, and the tail beyond is about $\frac{1}{4}\left(\frac{1}{26} + \frac{1}{37}\right) \approx 0.0164$, giving $0.375 = \frac{3}{8}$. Telescoping with a shift of one (keeping only $\frac{1}{f(1)}$) gives $\frac{1}{4}$; forgetting the factor $\frac{1}{4}$ gives $\frac{3}{2}$.

16. Answer: (B) $\frac{225}{4}$. The idea. Rewrite $\cos^6 x$ as a constant plus cosines of multiples of x. Each of those cosine sums runs over equally spaced angles that wrap around the circle a whole number of times, so it vanishes; only the constant survives, 180 times.

Work it out. With $c = \cos x = \frac{e^{ix}+e^{-ix}}{2}$, the binomial theorem gives $\cos^6 x = \frac{1}{64}\left(e^{6ix} + 6e^{4ix} + 15e^{2ix} + 20 + 15e^{-}\right.$ $\frac{10+15\cos 2x+6\cos 4x+\cos 6x}{32}$. Now sum over $x = 0°, 1°, \ldots, 179°$. The angles $2x$ are $0°, 2°, \ldots, 358°$: 180 equally spaced points once round the circle, and the cosines (real parts of the 180th roots of unity) add to 0. Likewise $4x$ runs twice round in steps of $4°$ and $6x$ three times round in steps of $6°$, and those sums are 0 too. What remains is $180 \cdot \frac{10}{32} = \frac{225}{4}$.

Check. The average of $\cos^6$ over a full period is $\frac{10}{32} = \frac{5}{16}$ (compare $\frac{1}{2}$ for $\cos^2$ and $\frac{3}{8}$ for $\cos^4$), and $180 \cdot \frac{5}{16} = 56.25$; adding the terms numerically gives the same. Using the $\cos^4$ average by mistake gives $\frac{135}{2}$, and the $\cos^2$ average gives 90.

17. Answer: (B) $4\sqrt{2}$. The idea. Two facts about a corner with three right angles. First, the three-dimensional Pythagorean theorem (de Gua's theorem): the square of the area of the slanted face equals the sum of the squares of the areas of the three right faces. That gives the missing face OCA. Second, if $OA = a$, $OB = b$, $OC = c$, the three right-face areas are $\frac{ab}{2}$, $\frac{bc}{2}$, $\frac{ca}{2}$, and their product is $\frac{(abc)^2}{8}$; the volume is $\frac{abc}{6}$, so the volume comes straight from the product of the areas, with no need to find the edges one by one.

Work it out. Put O at the origin and $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$. Then $\overrightarrow{AB} \times \overrightarrow{AC} = (bc,\ ca,\ ab)$, so $[ABC]^2 = \frac{(bc)^2+(ca)^2+(ab)^2}{4} = [OBC]^2+[OCA]^2+[OAB]^2$. Hence $[OCA]^2 = 169-9-16 = 144$ and $[OCA] = 12$. So $ab = 6$, $bc = 8$, $ca = 24$, and $(abc)^2 = 6 \cdot 8 \cdot 24 = 1152$, giving $abc = 24\sqrt{2}$. The tetrahedron is a right-angled corner cut from an $a \times b \times c$ box, with volume $\frac{1}{3} \cdot \frac{ab}{2} \cdot c = \frac{abc}{6} = \frac{24\sqrt{2}}{6} = 4\sqrt{2}$.

Check. The edges are $a = \frac{abc}{bc} = 3\sqrt{2}$, $b = \frac{abc}{ca} = \sqrt{2}$, $c = \frac{abc}{ab} = 4\sqrt{2}$: indeed $ab = 6$, $bc = 8$, $ca = 24$, and $[ABC] = \frac{1}{2}\sqrt{8^2 + 24^2 + 6^2} = \frac{1}{2}\sqrt{676} = 13$. Treating areas as if they added (third

face $13-3-4=6$) gives volume 4; using $\frac{abc}{3}$ instead of $\frac{abc}{6}$ gives $8\sqrt{2}$, and forgetting the division altogether gives $24\sqrt{2}$.

18. Answer: (E) 5. **The idea.** Write $X = I_1 + I_2 + \cdots + I_{10}$, where $I_j = 1$ if guest j gets their own coat. Then X^3 expands into a sum of products $I_a I_b I_c$ over ordered triples (a, b, c), and such a product only depends on the *set* of distinct guests in the triple: it is 1 exactly when all of them get their own coats. A set of k specified guests all gets their own coats with probability $\frac{(10-k)!}{10!}$.

Work it out. Group the 10^3 ordered triples by the number k of distinct guests they involve. $k=1$: the triples (a,a,a); there are 10, each with probability $\frac{1}{10}$, contributing 1. $k=2$: choose the two guests ($\binom{10}{2}=45$ ways) and fill three positions using both of them ($2^3-2=6$ ways): 270 triples, each with probability $\frac{1}{10\cdot 9}$, contributing 3. $k=3$: $10\cdot 9\cdot 8$ triples, each with probability $\frac{1}{10\cdot 9\cdot 8}$, contributing 1. So $E[X^3] = 1+3+1 = 5$.

Check. The same bookkeeping gives $E[X] = 1$ and $E[X^2] = 1+1 = 2$ (the familiar mean 1 and variance 1). The pattern $1, 2, 5$ is the Bell numbers, counting the ways to split the positions $\{1,2,3\}$ into blocks of equal guests, and it holds whenever there are at least 3 guests. Directly: $P(X=k) = \binom{10}{k} D_{10-k}/10!$ with D_m the derangement numbers, and $\sum k^3 P(X=k)$ is exactly 5.

19. Answer: (B) $\frac{1168}{21}$. **The idea.** The tangent line parallel to BC lies at distance $2r$ from BC (the incircle is squeezed between the two parallel lines), so it cuts off a triangle at A similar to ABC with ratio $\frac{h_a-2r}{h_a}$. Using $h_a = \frac{2K}{a}$ and $r = \frac{K}{s}$, this ratio is $1-\frac{a}{s} = \frac{s-a}{s}$: a clean expression in tangent lengths.

Work it out. Here $s=21$ and $K = \sqrt{21\cdot 7\cdot 6\cdot 8} = 84$. The three ratios are $\frac{s-a}{s} = \frac{7}{21}$, $\frac{s-b}{s} = \frac{6}{21}$, $\frac{s-c}{s} = \frac{8}{21}$ (with $a=14$, $b=15$, $c=13$). The small triangles have areas 84 times the squares of these ratios, in total $84\cdot\frac{49+36+64}{441} = \frac{84\cdot 149}{441} = \frac{596}{21}$. The three cut-off triangles do not overlap (their ratios add up to $\frac{21}{21}=1$, so along each side the two corners meet nowhere: along BC the cuts at B and C take $\frac{8}{21}$ and $\frac{7}{21}$ of it, leaving $\frac{6}{21}$ as a side of the hexagon). The hexagon's area is $84-\frac{596}{21} = \frac{1764-596}{21} = \frac{1168}{21}$.

Check. Concretely, with $B=(0,0)$, $C=(14,0)$, $A=(5,12)$: $r=4$, $h_a=12$, and the tangent line $y=8$ cuts off a triangle at A of height 4, ratio $\frac{4}{12}=\frac{7}{21}$ as claimed, area $\frac{84}{9}=\frac{196}{21}$. Reporting the total area of the three corners gives $\frac{596}{21}$ instead of the hexagon; using the tangent at distance r rather than $2r$ from each side would make the corners overlap, a sign that something is off.

20. Answer: (C) $\frac{112}{15}$. **The idea.** Put the triangle in coordinates: everything we need (the inradius, the incenter, the excircle touch point) comes from tangent lengths. Then the chord length is $2\sqrt{r^2-d^2}$, where d is the distance from the incenter to line AD.

Work it out. The semiperimeter is $s=27$ and Heron gives $K = \sqrt{27\cdot 6\cdot 7\cdot 14} = 126$, so $r = \frac{126}{27} = \frac{14}{3}$. Take $B=(0,0)$, $C=(21,0)$; then $A=(5,12)$ (check: $5^2+12^2=13^2$ and $16^2+12^2=20^2$). The incircle touches BC at distance $s-b=7$ from B, so the incenter is $I = \left(7, \frac{14}{3}\right)$. The excircle opposite A touches BC at distance $s-c=14$ from B, so $D=(14,0)$. Line AD has slope $\frac{-12}{9} = -\frac{4}{3}$, i.e. $4x+3y=56$. Its distance from I is $d = \frac{|28+14-56|}{5} = \frac{14}{5}$. Hence $XY = 2\sqrt{\frac{196}{9}-\frac{196}{25}} = 2\cdot 14\sqrt{\frac{16}{225}} = \frac{112}{15}$.

Check. A classical lemma says line AD passes through the point of the incircle diametrically opposite its touch point with BC (the homothety at A taking the incircle to the excircle sends that

antipode to D). Here the touch point is $(7, 0)$, its antipode is $\left(7, \frac{28}{3}\right)$, and indeed $4 \cdot 7 + 3 \cdot \frac{28}{3} = 56$. So $\overline{XY}$ is a chord through one end of a diameter, which is why it is shorter than the diameter $\frac{28}{3}$.

21. Answer: (A) $\frac{1512}{1513}$. The idea. Find a change of variable that turns $\star$ into ordinary multiplication. Put $T(a) = \frac{1+a}{1-a}$. Then $T(a \star b) = \frac{1+ab+a+b}{1+ab-a-b} = \frac{(1+a)(1+b)}{(1-a)(1-b)} = T(a)\,T(b)$. (This is the addition law for tanh, the same rule as adding velocities in special relativity.)

Work it out. For $x_k = \frac{2k}{k^2+1}$, $T(x_k) = \frac{k^2+2k+1}{k^2-2k+1} = \left(\frac{k+1}{k-1}\right)^2$. The product of $\frac{k+1}{k-1}$ over $k = 2, \ldots, 10$ telescopes: $\frac{3 \cdot 4 \cdot 5 \cdots 11}{1 \cdot 2 \cdot 3 \cdots 9} = \frac{10 \cdot 11}{1 \cdot 2} = 55$. So the whole expression y has $T(y) = 55^2 = 3025$. Inverting $\frac{1+y}{1-y} = 3025$ gives $y = \frac{3025-1}{3025+1} = \frac{3024}{3026} = \frac{1512}{1513}$.

Check. Start small: $x_2 \star x_3 = \frac{4/5+3/5}{1+12/25} = \frac{7/5}{37/25} = \frac{35}{37}$, and the formula predicts $T = \left(\frac{3 \cdot 4}{1 \cdot 2}\right)^2 = 36$, i.e. $\frac{35}{37}$. Telescoping to $10 \cdot 11 = 110$ without the $1 \cdot 2$ gives $T = 110^2$ and $\frac{12099}{12101}$; inverting T carelessly as $\frac{T-1}{T}$ or $\frac{T}{T+1}$ gives $\frac{3024}{3025}$ or $\frac{3025}{3026}$.

22. Answer: (B) 671. The idea. Modulo a prime p, the nonzero residues are all powers of a single *primitive root* g. The number $m(g^j)$, the order of g^j, is $\frac{p-1}{\gcd(j, p-1)}$, so for each divisor d of $p-1$ exactly $\varphi(d)$ residues have order d. The sum we want is therefore $\sum_{d \mid 36} d\,\varphi(d)$.

Work it out. Here $p - 1 = 36$, and 2 is a primitive root mod 37 (its order divides 36, and $2^{12} \equiv 26$, $2^{18} \equiv 36 \equiv -1$, so the order is not 12 or 18 or any divisor of them). The residues with order d are the g^j with $\gcd(j, 36) = 36/d$, and there are $\varphi(d)$ of them. So $\sum_{a=1}^{36} m(a) = \sum_{d \mid 36} d\,\varphi(d) = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 2 + 6 \cdot 2 + 9 \cdot 6 + 12 \cdot 4 + 18 \cdot 6 + 36 \cdot 12$, which is $1+2+6+8+12+54+48+108+432 = 671$.

Check. The function $n \mapsto \sum_{d \mid n} d\varphi(d)$ is multiplicative, so the answer is also $(1 + 2 + 8)(1 + 6 + 54) = 11 \cdot 61 = 671$. The counts are consistent: $\sum_{d \mid 36} \varphi(d) = 36$ residues. The sum $1 + 2 + \cdots + 36 = 666$ is a tempting but unrelated number, and $36 \cdot 36 = 1296$ assumes every a has order 36.

23. Answer: (A) $\frac{7}{24}$. The idea. The loops are the cycles of a uniformly random permutation. Build such a permutation one element at a time: element k either starts a new cycle or is inserted into an existing cycle immediately after one of the $k-1$ earlier elements, and each of these k options has probability $\frac{1}{k}$. Every permutation of $\{1, \ldots, n\}$ arises from exactly one sequence of choices, so this produces the uniform distribution; and inserting later elements never separates two elements already in a common cycle. So we only have to follow elements $1, 2, 3, 4$.

Work it out. Element 2 joins the cycle of 1 with probability $\frac{1}{2}$; otherwise we fail. Given that, element 3 either joins the cycle $\{1, 2\}$ (probability $\frac{2}{3}$) or starts its own (probability $\frac{1}{3}$). Case 1, $\{1, 2, 3\}$ together (probability $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$): element 4 must follow one of $1, 2, 3$, probability $\frac{3}{4}$. Contribution $\frac{1}{4}$. Case 2, 3 alone (probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$): element 4 must be inserted right after 3, probability $\frac{1}{4}$. Contribution $\frac{1}{24}$. Total: $\frac{1}{4} + \frac{1}{24} = \frac{7}{24}$.

Check. The same method says two given boxes share a loop with probability $\frac{1}{2}$, so a naive "independent" guess would be $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$; the answer is a bit larger because the two events help each other (all four boxes can be in one loop, which happens with probability $\frac{1}{4}$). Counting by computer over all permutations of $4, 5, 6$ or 7 boxes gives $\frac{7}{24}$ every time, as the insertion argument predicts: the answer does not depend on the number of boxes.

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24. Answer: (C) $\frac{1}{2}$**. The idea.** The shadow is the convex hull of the shadows of the four vertices, so it is either a triangle or a quadrilateral. A triangular shadow is the projection of one face (area at most the face's area $\frac{\sqrt{3}}{4}$). A quadrilateral shadow has two diagonals, and since every pair of vertices of a tetrahedron is joined by an edge, those diagonals are the shadows of two *opposite* edges; that caps the area.

Work it out. Triangle case: the shadow is the projection of a face, whose area is the face's area times $|\cos\theta|$, so at most $\frac{\sqrt{3}}{4} \approx 0.433$. Quadrilateral case: its diagonals are shadows of two opposite edges, so each has length at most 1 (a projection never lengthens a segment). A convex quadrilateral with diagonals d_1, d_2 meeting at angle φ has area $\frac{1}{2}d_1 d_2 \sin\varphi \le \frac{1}{2}$. The bound $\frac{1}{2}$ is reached: in a regular tetrahedron opposite edges are perpendicular, and their common perpendicular joins their midpoints. Shine the light along that common perpendicular. Both edges are then parallel to the screen, so their shadows have full length 1, and they stay perpendicular, crossing at their midpoints. The shadow is a square with diagonals 1, of area $\frac{1}{2}$. Since $\frac{1}{2} > \frac{\sqrt{3}}{4}$, the maximum is $\frac{1}{2}$.

Check. Use vertices (t,t,t), $(t,-t,-t)$, $(-t,t,-t)$, $(-t,-t,t)$ with $t = \frac{\sqrt{2}}{4}$ (edge $2\sqrt{2}t = 1$). Looking along the x-axis, the shadows are $(\pm t, \pm t)$, a square of side $2t = \frac{\sqrt{2}}{2}$ and area $\frac{1}{2}$. A computer search over thousands of light directions never finds a larger shadow. Looking straight down onto a face gives only $\frac{\sqrt{3}}{4}$, the most common wrong answer.

25. Answer: (E) -11427840**. The idea.** The signs split over the binary digits: if $n = \sum_{j=0}^{4} b_j 2^j$ then $(-1)^{s(n)} = \prod_j (-1)^{b_j}$. So the exponential generating function factors: $G(t) = \sum_{n=0}^{31} (-1)^{s(n)} e^{nt} = \prod_{j=0}^{4} \left(1 - e^{2^j t}\right)$, and since $e^{nt} = \sum_k \frac{n^k t^k}{k!}$, the sum we want, $\sum (-1)^{s(n)} n^6$, is 6! times the coefficient of t^6 in $G(t)$. Each factor vanishes at $t = 0$, so G starts at t^5, and we need just one term past the start.

Work it out. Expanding the product chooses, for each binary digit, either 1 (digit 0) or $-e^{2^j t}$ (digit 1), which gives exactly $(-1)^{s(n)} e^{nt}$ for each n. Since $1 - e^x = -x\left(1 + \frac{x}{2} + O(x^2)\right)$, each factor is $1 - e^{2^j t} = -2^j t\left(1 + 2^{j-1}t + O(t^2)\right)$. Multiplying the five factors, $G(t) = (-1)^5 2^{0+1+2+3+4} t^5 \left(1 + \frac{1+2+4+8+16}{2} t + O(t$ $-1024\,t^5 - 1024 \cdot \frac{31}{2} t^6 + O(t^7)$. The coefficient of t^6 is -15872, so the sum is $6! \cdot (-15872) = 720 \cdot (-15872) = -11427840$. (The coefficients of $t^0, \ldots, t^4$ vanish: the power sums of degree 0 to 4 cancel exactly, which is the Prouhet splitting of $0, \ldots, 31$ into two classes. The cancellation stops at degree 5, and degree 6 needs the second term of each factor.)

Check. A small case of the same law: over $n = 0, 1, 2, 3$ (two binary digits), $G(t) = (1 - e^t)(1 - e^{2t}) = 2t^2\left(1 + \frac{t}{2} + \cdots\right)(1 + t + \cdots) = 2t^2 + 3t^3 + \cdots$, predicting $\sum (-1)^{s(n)} n^3 = 3! \cdot 3 = 18$; directly, $0 - 1 - 8 + 27 = 18$. Adding all 32 terms of the original sum by computer also gives -11427840. Forgetting the $\frac{1}{2}$ in $e^x = 1 + x + \frac{x^2}{2} + \cdots$ doubles the answer to -22855680; using 32 instead of $1 + 2 + 4 + 8 + 16 = 31$ gives -11796480, or -23592960 with the $\frac{1}{2}$ also forgotten; multiplying by 7! instead of 6! gives -79994880.

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