MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

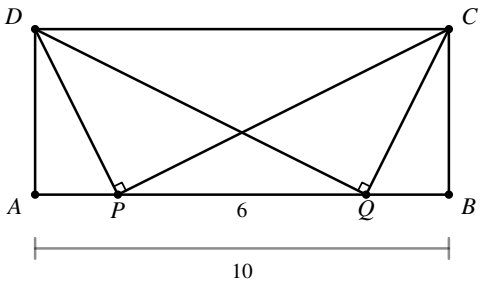
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Kai writes a list of numbers. He chooses the first two numbers himself: they are positive integers, and the first is smaller than the second. After that, every number he writes is the sum of all the numbers already in the list. For example, starting $2, 5$ he would continue $7, 14, 28, \ldots$.
Kai stops after writing ten numbers and finds that the sum of all ten is 3840. How many different values could the second number have?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 14 **(E)** 15

2. Rectangle $ABCD$ has $AB = 10$. Two different points P and Q on side $\overline{AB}$ satisfy $\angle DPC = \angle DQC = 90°$, and $PQ = 6$. What is the area of the rectangle?



(A) 24 **(B)** 30 **(C)** 32 **(D)** 40 **(E)** 48

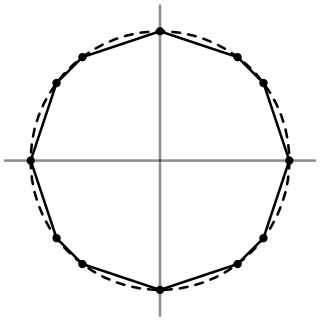
3. How many five-digit positive integers have digits that strictly increase from left to right and alternate between odd and even? (For example, 14567 qualifies, but 13468 and 12354 do not.)

(A) 15 **(B)** 21 **(C)** 25 **(D)** 26 **(E)** 27

4. How many ordered triples (x, y, z) of positive integers satisfy $x^2 + y^2 + z^2 = 2(x + y + z) + 6$?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 9

5. Exactly 12 points with integer coordinates lie on the circle $x^2 + y^2 = 25$. These 12 points are the vertices of a convex 12-sided polygon. What is the area of this polygon?



(A) 74 **(B)** 75 **(C)** 76 **(D)** 78 **(E)** 80

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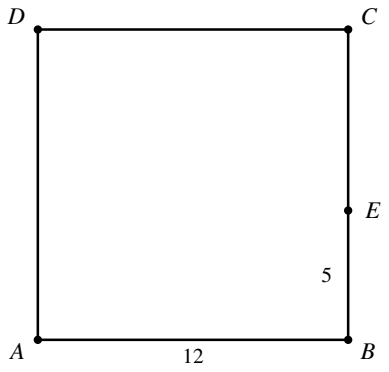
6. The eight numbers $1, 2, 3, 4, 5, 6, 8, 9$ are to be written in a row so that for every two numbers that are next to each other, one of them divides the other. In how many orders can this be done?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** 8

7. A geometric sequence of real numbers has first term 3. The sum of its first four terms is exactly five times the sum of its first two terms. What is the sum of all possible values of the fifth term of the sequence?

(A) 45 **(B)** 48 **(C)** 51 **(D)** 96 **(E)** 99

8. Square $ABCD$ has side length 12, and E is the point on side $\overline{BC}$ with $BE = 5$. Consider all points of the square (including its interior) that are closer to A than to E. What is the area of this region?



(A) $\frac{109}{2}$ **(B)** 60 **(C)** 65 **(D)** 72 **(E)** $\frac{179}{2}$

9. What is the product of all real numbers x that satisfy $x^2 + x = \dfrac{10}{x^2 + x - 3}$?

(A) -10 **(B)** -5 **(C)** -2 **(D)** 2 **(E)** 10

10. Three of the twelve vertices of a regular dodecagon are joined to make a triangle. Different choices can give congruent triangles. How many different triangles, up to congruence, can be made this way?

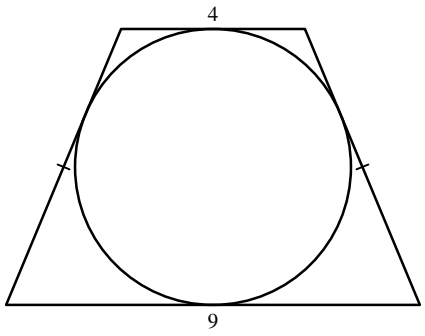
(A) 10 **(B)** 12 **(C)** 14 **(D)** 19 **(E)** 55

11. For how many positive integers $n \leq 1000$ do $2n$ and $3n$ have the same number of positive divisors?

(A) 333 **(B)** 342 **(C)** 388 **(D)** 397 **(E)** 398

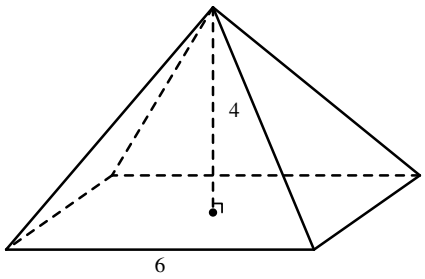
12. An isosceles trapezoid has parallel sides of lengths 4 and 9. A circle can be drawn inside the trapezoid that is tangent to all four of its sides. What is the area of the trapezoid?

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(A) 39 **(B)** 42 **(C)** 48 **(D)** 52 **(E)** 78

13. A pyramid has a square base with side length 6, and its apex is directly above the centre of the base at height 4. A cube sits inside the pyramid with its bottom face on the base of the pyramid. What is the largest possible edge length of the cube?



(A) 2 **(B)** $\frac{12}{5}$ **(C)** $\frac{8}{3}$ **(D)** 3 **(E)** $\frac{24}{7}$

14. A function f defined on all real numbers satisfies $f(x) + f(x+1) = x^2$ for every real x, and $f(0) = 5$. What is $f(21)$?

(A) 200 **(B)** 205 **(C)** 210 **(D)** 215 **(E)** 231

15. Each of the 10 cells of a grid with 2 rows and 5 columns is coloured red, green or blue so that any two cells sharing a side have different colours. How many such colourings are there?

(A) 96 **(B)** 162 **(C)** 243 **(D)** 384 **(E)** 486

16. Consider every four-digit positive integer that is a multiple of 99. For each one, add up its four digits. What is the total of all these digit sums?

(A) 1656 **(B)** 1674 **(C)** 1692 **(D)** 1710 **(E)** 1818

17. How many subsets of $\{1, 2, 3, \ldots, 10\}$ with at least three elements have the property that the largest element equals the sum of all the other elements?

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(A) 24 **(B)** 30 **(C)** 32 **(D)** 35 **(E)** 40

18. For how many integers n with $1 \le n \le 100$ does the number n^n leave a remainder of 1 when it is divided by 5? (For example, $n = 4$ counts, since $4^4 = 256$.)

(A) 20 **(B)** 25 **(C)** 32 **(D)** 40 **(E)** 44

19. A point P is chosen uniformly at random from the interior of a square $ABCD$. From P, each side of the square is seen under an angle: $\angle APB$, $\angle BPC$, $\angle CPD$ and $\angle DPA$. What is the probability that exactly two of these four angles are obtuse?

(A) $1 - \frac{\pi}{4}$ **(B)** $\frac{\pi}{8}$ **(C)** $2 - \frac{\pi}{2}$ **(D)** $-1 + \frac{\pi}{2}$ **(E)** $\frac{\pi}{4}$

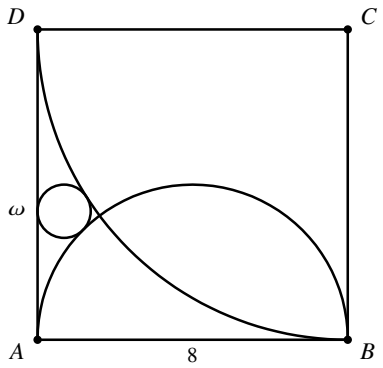
20. For each positive integer n, let $g(n) = n + \lfloor 2\sqrt{n} \rfloor$, where $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many integers from 1 to 2026, inclusive, are *not* equal to $g(n)$ for any positive integer n?

(A) 44 **(B)** 86 **(C)** 87 **(D)** 88 **(E)** 90

21. The number $\binom{2026}{1013} = \dfrac{2026!}{1013! \cdot 1013!}$ is written out in full in base ten. How many zeros does it end with?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

22. Square $ABCD$ has side length 8. Inside the square, a semicircle is drawn with diameter $\overline{AB}$, and a quarter-circle arc with centre C and radius 8 is drawn from B to D. A circle ω lies inside the square, outside the semicircle's disk and outside the quarter-circle's disk, is tangent to side $\overline{AD}$, and is tangent to both the semicircle and the quarter-circle arc. What is the radius of ω?



(A) $3 - 2\sqrt{2}$ **(B)** $-1 + \sqrt{2}$ **(C)** $12 - 8\sqrt{2}$ **(D)** $-2 + 2\sqrt{2}$ **(E)** $16 - 8\sqrt{3}$

23. Six boxes, labelled 1 to 6, stand in a circle, so box 6 is next to box 1. Ten identical balls are placed in the boxes (a box may hold any number of balls) so that no two neighbouring boxes are both empty. In how many ways can this be done?

(A) 882 **(B)** 1458 **(C)** 1638 **(D)** 1710 **(E)** 3003

24. The six vertices of a regular hexagon lie on a circle. Each of the 15 segments joining two of these vertices (the six sides and the nine diagonals) is drawn with probability $\frac{1}{2}$, independently of the others. The drawn segments cut the inside of the circle into regions. What is the expected number of regions?

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(A) $\frac{73}{8}$ **(B)** $\frac{89}{8}$ **(C)** $\frac{23}{2}$ **(D)** 12 **(E)** $\frac{97}{8}$

25. Parentheses may be inserted into the expression

$$1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10$$

in any way that produces a valid expression (for example, $1 - (2 - 3) - (4 - (5 - 6) - 7) - 8 - 9 - 10$), and the numbers and minus signs stay in place. Including the expression with no parentheses, how many different values can be obtained?

(A) 45 **(B)** 47 **(C)** 49 **(D)** 51 **(E)** 53

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Answer Key — AMC 10 Practice Contest 8

1. (B) 7

2. (D) 40

3. (E) 27

4. (C) 6

5. (A) 74

6. (E) 8

7. (C) 51

8. (A) $\frac{109}{2}$

9. (B) -5

10. (B) 12

11. (E) 398

12. (A) 39

13. (B) $\frac{12}{5}$

14. (B) 205

15. (E) 486

16. (A) 1656

17. (C) 32

18. (D) 40

19. (D) $-1+\frac{\pi}{2}$

20. (D) 88

21. (A) 3

22. (C) $12-8\sqrt{2}$

23. (D) 1710

24. (E) $\frac{97}{8}$

25. (C) 49

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Solutions — AMC 10 Practice Contest 8

1. Answer: (B) 7**. The idea.** The sum of the first ten numbers is exactly what the eleventh number would be, and from the third number on the list simply doubles.

Work it out. Call the first two numbers $a < b$. The third is $a + b$. Each later number equals (the number before it) + (everything before that), and everything before that is exactly the number before it again, so from the third term on each term is double the previous: $a+b$, $2(a+b)$, $4(a+b)$, ... The tenth number is $2^7(a + b)$, and the sum of all ten is what the eleventh would be, $2^8(a+b) = 256(a+b)$. So $256(a+b) = 3840$, giving $a+b = 15$. With $1 \le a < b$, the pairs are $(1, 14), (2, 13), \ldots, (7, 8)$, so b can be any of $8, 9, \ldots, 14$: 7 values.

Check. Start 7, 8: the list is $7, 8, 15, 30, 60, 120, 240, 480, 960, 1920$, whose sum is $15 + 15(1 + 2 + \cdots + 128) = 15 \cdot 256 = 3840$. Forgetting that the first number is smaller gives the tempting 14.

2. Answer: (D) 40**. The idea.** A point P sees segment DC at a right angle exactly when P lies on the circle with diameter $\overline{DC}$ (Thales). So P and Q are where that circle meets line AB.

Work it out. The circle has centre at the midpoint of $\overline{DC}$ and radius 5. Its centre is at height $h = AD$ above line AB, directly above the midpoint of $\overline{AB}$. It meets AB at two points symmetric about that midpoint, each at horizontal distance $\sqrt{25 - h^2}$, so $PQ = 2\sqrt{25 - h^2} = 6$. Hence $25 - h^2 = 9$, $h = 4$, and the area is $10 \cdot 4 = 40$.

Check. By symmetry $AP = 2$ and $PB = 8$. Triangles APD and BCP are similar (the right angle at P makes $\angle APD = \angle BCP$), so $AP \cdot PB = AD \cdot BC = AD^2$: $2 \cdot 8 = 16 = 4^2$. Confirmed.

3. Answer: (E) 27**. The idea.** Alternating parity means every step from one digit to the next is an odd number. Write each step as $2g+1$ with $g \ge 0$ and the problem becomes a small stars-and-bars count.

Work it out. The first digit d_1 is at least 1 (and then 0 can never appear). Write $d_{i+1} - d_i = 2g_i + 1$ with $g_1, g_2, g_3, g_4 \ge 0$. Then $d_5 = d_1 + 4 + 2G$ where $G = g_1 + g_2 + g_3 + g_4$, and we need $d_5 \le 9$, i.e. $d_1 + 2G \le 5$. The number of ways to write G as an ordered sum of four non-negative integers is $\binom{G+3}{3}$: 1, 4, 10 for $G = 0, 1, 2$.

$d_1 = 1$: $G \le 2$, giving $1 + 4 + 10 = 15$.

$d_1 = 2$ or 3: $G \le 1$, giving 5 each.

$d_1 = 4$ or 5: $G = 0$, giving 1 each.

Total $15 + 5 + 5 + 1 + 1 = 27$.

Check. Count by the first digit directly. If $d_1 = 5$ the only option is 56789 and if $d_1 = 4$ only 45678. If $d_1 = 3$: the row 34567 plus one step of size 3 somewhere, e.g. 36789, 34789, 34589, 34569: $1 + 4 = 5$, matching. Stopping after $d_1 = 1$ gives only 15, and forgetting the single rows 45678 and 56789 gives 25.

4. Answer: (C) 6**. The idea.** Complete the square in each variable: the equation says a point is at distance 3 from $(1, 1, 1)$, and positivity makes every shifted coordinate non-negative.

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Work it out. Rewrite as $(x-1)^2+(y-1)^2+(z-1)^2 = 6+3 = 9$. Let $a = x-1$, $b = y-1$, $c = z-1$; since $x, y, z \geq 1$, these are non-negative integers with $a^2+b^2+c^2 = 9$. The squares available are $0, 1, 4, 9$. Writing 9 as a sum of three of them: $9+0+0$ or $4+4+1$ (note $4+4+1 = 9$ and nothing else works: $4+1+1 = 6$, $1+1+1 = 3$, $4+4+0 = 8$). So (a, b, c) is a permutation of $(3, 0, 0)$ (3 ways) or of $(2, 2, 1)$ (3 ways): 6 triples, e.g. $(x, y, z) = (4, 1, 1)$ and $(3, 3, 2)$.

Check. $(3, 3, 2)$: $9+9+4 = 22$ and $2 \cdot 8+6 = 22$. Missing the $4+4+1$ family gives 3.

5. Answer: (A) 74**. The idea.** The polygon is inscribed in the circle, so split it into 12 triangles with apex at the centre O; each has area $\frac{1}{2}|x_1y_2 - x_2y_1|$ for consecutive vertices. By symmetry only one quadrant needs work.

Work it out. The lattice points are $(\pm5, 0)$, $(0, \pm5)$, $(\pm3, \pm4)$, $(\pm4, \pm3)$. In the first quadrant the vertices in order are $(5, 0)$, $(4, 3)$, $(3, 4)$, $(0, 5)$. Triangle areas: $\frac{1}{2}|5\cdot3-0| = \frac{15}{2}$; $\frac{1}{2}|4\cdot4-3\cdot3| = \frac{7}{2}$; $\frac{1}{2}|3\cdot5-0| = \frac{15}{2}$. The quadrant contributes $\frac{37}{2}$, and the four quadrants give 74.

Check. Pick's theorem: each edge (steps like $(1, 3)$ and $(1, 1)$) has no lattice points inside it, so there are $B = 12$ boundary points, and the interior lattice points are exactly those with $x^2+y^2 < 25$, of which there are 69. Then $A = I + \frac{B}{2} - 1 = 69+6-1 = 74$. (Forgetting the -1 in Pick's theorem gives 75, and using B in place of $\frac{B}{2}$ gives 80.)

6. Answer: (E) 8**. The idea.** Look at the most constrained numbers first. The only number in the list that 5 divides or is divided by is 1, and 9 can only sit next to 1 or 3. These two facts almost build the row for us.

Work it out. Since 5 can only be next to 1, and 1 has at most two neighbours, 5 must be at an end with 1 beside it. By reversing, count rows starting $5, 1, \ldots$ and double at the end.

Now 9 is next to 1 or 3.

Case 1: 9 is right after 1: $5, 1, 9, 3, \ldots$ (9 needs a second neighbour, which can only be 3). The neighbours of 3 must be among $1, 6, 9$, so next is 6; 6's neighbours are among $1, 2, 3$, so next is 2; the last two places hold 4 and 8 in either order ($2 \mid 4, 2 \mid 8, 4 \mid 8$): 2 rows.

Case 2: 9 is not next to 1, so it is at the far end next to 3: $\ldots, 3, 9$. Then 3's other neighbour must be 6 (as 1 is taken), and 6's other neighbour must be 2: $5, 1, _, _, 2, 6, 3, 9$. The two gaps take 4 and 8 in either order, and both work: 2 rows.

That is 4 rows starting with 5, so 8 in all.

Check. The four rows starting with 5 are $5, 1, 9, 3, 6, 2, 4, 8$; $5, 1, 9, 3, 6, 2, 8, 4$; $5, 1, 4, 8, 2, 6, 3, 9$; $5, 1, 8, 4, 2, 6, 3, 9$; each checks pair by pair, and their reversals give the other four. Forgetting reversals gives 4.

7. Answer: (C) 51**. The idea.** The first four terms split as (first two) $+r^2\cdot$(first two), so the condition factors. The factorisation hides a ratio that makes both sums zero.

Work it out. With ratio r, the first two terms sum to $S = 3(1+r)$ and the next two to r^2S. The condition is $S + r^2S = 5S$, i.e. $S(r^2-4) = 0$. So $r = 2$, $r = -2$, or $S = 0$, which means $r = -1$. The fifth term $3r^4$ is $3 \cdot 16 = 48$ for $r = \pm2$ and $3 \cdot 1 = 3$ for $r = -1$. The possible values are 48 and 3, with sum 51.

Check. $r = -1$: the sequence $3, -3, 3, -3, \ldots$ has first-four sum $0 = 5 \cdot 0$, so it really qualifies. Dividing by S loses it and gives 48; counting 48 twice gives 96 or 99.

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8. Answer: (A) $\frac{109}{2}$**. The idea.** The points equally far from A and E form the perpendicular bisector of $\overline{AE}$, so the region is the part of the square on A's side of that line. The line crosses AB at P and CD at Q, and the region is the trapezoid $APQD$; each of AP and DQ comes from one equal-distance equation.

Work it out. P is on AB with $PA = PE$. Writing $AP = p$, we have $PB = 12 - p$ and $PE^2 = (12-p)^2 + 5^2$, so $p^2 = (12-p)^2 + 25$, i.e. $24p = 169$, $p = \frac{169}{24}$.

Q is on CD with $QA = QE$. Writing $DQ = q$: $QA^2 = 12^2 + q^2$ and $QE^2 = (12-q)^2 + 7^2$ (since $CQ = 12 - q$ and $CE = 12 - 5 = 7$). So $144 + q^2 = 144 - 24q + q^2 + 49$, giving $q = \frac{49}{24}$. Both lie inside their sides, so the region is the trapezoid $APQD$ with parallel sides AP and DQ and height $AD = 12$:

$$\text{area} = \tfrac{12}{2}\left(\tfrac{169}{24} + \tfrac{49}{24}\right) = 6 \cdot \tfrac{218}{24} = \tfrac{109}{2}.$$

Check. The trapezoid's area is its height times its width halfway up, at $y = 6$. With $A = (0,0)$ and $E = (12,5)$ the bisector is $12x + 5y = \frac{169}{2}$; at $y = 6$, $x = \frac{109}{24}$, and $12 \cdot \frac{109}{24} = \frac{109}{2}$. It is less than half the square (72), as it must be: the centre $(6,6)$ is closer to E (distance $\sqrt{37}$) than to A (distance $\sqrt{72}$), so the bisector passes on A's side of the centre.

9. Answer: (B) -5**. The idea.** The equation only involves $u = x^2 + x$, so solve for u first. But $x^2 + x$ can never be below $-\frac{1}{4}$, so one value of u gives no real x at all.

Work it out. With $u = x^2 + x$: $u(u-3) = 10$, so $u^2 - 3u - 10 = (u-5)(u+2) = 0$, giving $u = 5$ or $u = -2$ (neither makes the denominator $u - 3$ zero).

$u = -2$: $x^2 + x + 2 = 0$ has discriminant $1 - 8 < 0$: no real roots.

$u = 5$: $x^2 + x - 5 = 0$ has discriminant $21 > 0$: two real roots with product -5.

So the answer is -5.

Check. The roots are $\frac{-1 \pm \sqrt{21}}{2}$; their product is $\frac{1-21}{4} = -5$. Keeping the non-real pair as well multiplies in their product 2, giving the trap -10.

10. Answer: (B) 12**. The idea.** Three vertices cut the circumcircle into three arcs made of a, b, c edges of the 12-gon, with $a + b + c = 12$. Each side of the triangle is a chord determined by its arc, and the triangle is determined up to congruence by the unordered triple $\{a, b, c\}$. So count partitions of 12 into three positive parts.

Work it out. Why the triple determines the shape: rotating and reflecting the 12-gon can move any arrangement of arcs a, b, c (in any cyclic order) onto any other. Conversely different triples give different triangles, because the angles of the triangle are $15°a$, $15°b$, $15°c$ (inscribed angles). Now list $a \le b \le c$, $a + b + c = 12$:

$a = 1$: $(1,1,10)$, $(1,2,9)$, $(1,3,8)$, $(1,4,7)$, $(1,5,6)$ — five.

$a = 2$: $(2,2,8)$, $(2,3,7)$, $(2,4,6)$, $(2,5,5)$ — four.

$a = 3$: $(3,3,6)$, $(3,4,5)$ — two.

$a = 4$: $(4,4,4)$ — one.

Total 12.

Check. The angle view: the triangle's angles are multiples of $15°$ summing to $180°$, i.e. partitions of 12 into 3 parts, again 12. Counting ordered arc triples instead gives $\binom{11}{2} = 55$.

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11. Answer: (E) 398**. The idea.** Multiplying by 2 or by 3 only changes the exponents of 2 and 3, so split off those two primes: $n = 2^a 3^b m$ with m coprime to 6. Comparing the two divisor counts forces $a = b$, and then it is a counting job.

Work it out. If m has t divisors, then $2n = 2^{a+1}3^b m$ has $(a+2)(b+1)t$ divisors and $3n = 2^a 3^{b+1} m$ has $(a+1)(b+2)t$. These are equal exactly when $(a+2)(b+1) = (a+1)(b+2)$, i.e. $ab + a + 2b + 2 = ab + 2a + b + 2$, i.e. $a = b$. So $n = 6^k m$ with m coprime to 6.

The number of $m \le X$ coprime to 6 is $X - \lfloor X/2 \rfloor - \lfloor X/3 \rfloor + \lfloor X/6 \rfloor$ (inclusion–exclusion).

$k = 0$, $m \le 1000$: $1000 - 500 - 333 + 166 = 333$.

$k = 1$, $m \le 166$: $166 - 83 - 55 + 27 = 55$.

$k = 2$, $m \le 27$: $27 - 13 - 9 + 4 = 9$.

$k = 3$, $m \le 4$: only $m = 1$, so 1 (that is $n = 216$).

$k = 4$: $6^4 = 1296 > 1000$, none.

Total $333 + 55 + 9 + 1 = 398$.

Check. $n = 6$: 12 and 18 both have 6 divisors. $n = 2$: 4 has 3 divisors but 6 has 4. $n = 216$: $432 = 2^4 3^3$ and $648 = 2^3 3^4$ both have 20. Stopping after $k = 0$ gives 333, after $k = 1$ gives 388, and missing 216 gives 397.

12. Answer: (A) 39**. The idea.** The two tangent segments from a point to a circle are equal. Going around the trapezoid, each side is made of two tangent segments, one from each end, and adding up alternate sides uses every tangent segment once. So in a quadrilateral with an inscribed circle the two pairs of opposite sides have equal sums: the two legs together equal $4 + 9 = 13$.

Work it out. Each leg is $\frac{13}{2}$. Dropping perpendiculars from the ends of the short base, the long base sticks out by $\frac{9-4}{2} = \frac{5}{2}$ on each side, so the height is $\sqrt{\left(\frac{13}{2}\right)^2 - \left(\frac{5}{2}\right)^2} = \sqrt{\frac{144}{4}} = 6$. The area is $\frac{4+9}{2} \cdot 6 = 39$.

Check. The height equals the diameter of the circle, so its radius is 3, and the area also equals the radius times the semiperimeter: $3 \cdot \frac{4+9+13}{2} = 39$.

13. Answer: (B) $\frac{12}{5}$**. The idea.** Only the top face of the cube is at risk: the pyramid narrows as it rises, so the cube fits exactly when its top square fits inside the pyramid's horizontal cross-section at the cube's height.

Work it out. The cross-section of the pyramid at height h is a square similar to the base, shrunk by the factor $1 - \frac{h}{4}$, so its side is $6\left(1 - \frac{h}{4}\right)$. Centre the cube under the apex. A cube of edge s has its top face at height s, a square of side s, which fits in the cross-section (and then the whole cube fits, since lower sections are bigger) exactly when $s \le 6 - \frac{3s}{2}$, that is, $\frac{5s}{2} \le 6$, so $s \le \frac{12}{5}$.

Check. At $s = \frac{12}{5}$ the section at height $\frac{12}{5}$ has side $6 \cdot \frac{2}{5} = \frac{12}{5}$: the top edges of the cube lie exactly on the four slanted faces. Working in the triangle through the apex and a base diagonal gives the same equation, $\frac{s\sqrt{2}}{2} = 3\sqrt{2}\left(1 - \frac{s}{4}\right)$.

14. Answer: (B) 205**. The idea.** The rule walks f along the integers, but each step flips the sign of the starting value. Split $f(n)$ into a part that satisfies the rule with $f(0) = 0$ and a part that just alternates.

Work it out. From $f(n+1) = n^2 - f(n)$: $f(1) = -5$, $f(2) = 1+5 = 6$, $f(3) = 4-6 = -2$, $f(4) = 9+2 = 11$, $f(5) = 16-11 = 5$. Compare with the triangular numbers $T_n = \frac{n(n-1)}{2} = 0, 0, 1, 3, 6, 10$: the values are $T_n + 5$ for even n and $T_n - 5$ for odd n. Indeed $g(n) = \frac{n(n-1)}{2}$ satisfies $g(n) + g(n+1) = \frac{n(n-1)+(n+1)n}{2} = n^2$, and $5(-1)^n$ satisfies $h(n) + h(n+1) = 0$, so $f(n) = \frac{n(n-1)}{2} + 5(-1)^n$ fits the rule and $f(0) = 5$. Therefore $f(21) = 210 - 5 = 205$.

Check. $f(20) + f(21) = (190+5) + 205 = 400 = 20^2$. Forgetting the alternating part gives 210; flipping its sign gives 215.

15. Answer: (E) 486**. The idea.** Build the grid one column at a time. A column is an ordered pair of different colours, and the key fact is that the number of ways to colour the next column is the same whatever the current column is.

Work it out. The first column: 3 choices for the top cell and 2 for the bottom, so 6 ways. Now suppose a column is (top a, bottom b) with $a \neq b$, and let c be the third colour. The next column (top x, bottom y) needs $x \neq a$, $y \neq b$, $x \neq y$. The options for x are b and c; if $x = b$ then $y \in \{a, c\}$ (2 ways), and if $x = c$ then y must avoid b and c, so $y = a$ (1 way). That is 3 ways, whatever a, b were. With 4 more columns the total is $6 \cdot 3^4 = 486$.

Check. For a 2×2 grid the rule gives $6 \cdot 3 = 18$; listing next columns after (R, G): (G, R), (G, B), (B, R) — three, as claimed. Guessing "2 ways per cell" gives $6 \cdot 2^4 = 96$, and stopping one column short gives 162.

16. Answer: (A) 1656**. The idea.** Split a four-digit number $\overline{abcd}$ into its two-digit halves: $\overline{abcd} = 100 \cdot \overline{ab} + \overline{cd}$, and $100 \equiv 1 \pmod{99}$. So divisibility by 99 only depends on $\overline{ab} + \overline{cd}$.

Work it out. $99 \mid \overline{abcd}$ exactly when $99 \mid \overline{ab} + \overline{cd}$. Here $10 \leq \overline{ab} \leq 99$ and $0 \leq \overline{cd} \leq 99$, so the sum is between 10 and 198: it is 99 or 198. If $\overline{ab} + \overline{cd} = 99$, no carrying happens (units digits add to 9 and tens digits add to 9), so $a + c = 9$ and $b + d = 9$: digit sum 18. The sum 198 happens only for $99 + 99$, the number 9999, with digit sum 36. The four-digit multiples of 99 are $99 \cdot 11 = 1089$ through $99 \cdot 101 = 9999$: 91 numbers. Total: $90 \cdot 18 + 36 = 1656$.

Check. 1089: $1+0+8+9 = 18$; 5049: 18; 9900: 18. Counting 9999 among the 18s and then again as 36 gives 1674, the tempting wrong answer.

17. Answer: (C) 32**. The idea.** Fix the largest element m. The rest of the set is a collection of at least two distinct positive integers adding up to m — automatically all smaller than m. So we are counting partitions of m into distinct parts, minus the one-part partition "m" itself.

Work it out. Partitions of m into at least two distinct parts: $m = 3$: $2+1$ (1). $m = 4$: $3+1$ (1). $m = 5$: $4+1$, $3+2$ (2). $m = 6$: $5+1$, $4+2$, $3+2+1$ (3). $m = 7$: $6+1$, $5+2$, $4+3$, $4+2+1$ (4). $m = 8$: $7+1$, $6+2$, $5+3$, $5+2+1$, $4+3+1$ (5). $m = 9$: $8+1$, $7+2$, $6+3$, $5+4$, $6+2+1$, $5+3+1$, $4+3+2$ (7). $m = 10$: $9+1$, $8+2$, $7+3$, $6+4$, $7+2+1$, $6+3+1$, $5+4+1$, $5+3+2$, $4+3+2+1$ (9). Total $1+1+2+3+4+5+7+9 = 32$.

Check. The number of partitions of m into distinct parts is $2, 2, 3, 4, 5, 6, 8, 10$ for $m = 3, \ldots, 10$ (a known table); subtracting 1 each time gives the same list. Forgetting the four-part $4+3+2+1$ or three-part cases gives the smaller distractors.

18. Answer: (D) 40**. The idea.** The remainder of n^n depends on the base n modulo 5 and on the exponent n. For a base not divisible by 5, the powers repeat every 4 steps modulo 5 (for instance

$2, 4, 3, 1, 2, \ldots$), so only n modulo 4 matters in the exponent. Both are fixed by n modulo 20, so count in one block of 20 and multiply by 5.

Work it out. Among $n = 1, \ldots, 20$: if $n \equiv 0 \pmod 5$, never. If $n \equiv 1 \pmod 5$, always: $n = 1, 6, 11, 16$. If $n \equiv 4 \equiv -1 \pmod 5$, we need n even: $n = 4, 14$. If $n \equiv 2 \pmod 5$, $2^n \equiv 1$ needs $4 \mid n$: only $n = 12$. If $n \equiv 3 \pmod 5$, $3^n \equiv 1$ needs $4 \mid n$: only $n = 8$. That is $4+2+1+1 = 8$ per block of 20, so $5 \cdot 8 = 40$ values up to 100.

Check. $12^{12} \equiv 2^{12} = 4096 \equiv 1$ and $8^8 \equiv 3^8 = 6561 \equiv 1 \pmod 5$, while $7^7 \equiv 2^7 = 128 \equiv 3$ and $9^9 \equiv (-1)^9 \equiv 4$ do not count.

19. Answer: (D) $-1 + \frac{\pi}{2}$**. The idea.** By Thales, $\angle APB$ is obtuse exactly when P lies inside the circle with diameter $\overline{AB}$, that is, in the half-disc on $\overline{AB}$ inside the square. The four angles add up to 360°, so they cannot all be acute, and the question is how the four half-discs overlap.

Work it out. Take side length 2. Half-discs on opposite sides have radius 1 and centres 2 apart, so they meet only at the centre: two opposite angles are never both obtuse, and three obtuse angles are impossible. Half-discs on neighbouring sides, say on $\overline{AB}$ and $\overline{BC}$, have centres $\sqrt{2}$ apart and cross at B and at the centre O. Their overlap is a lens split by the chord $\overline{BO}$ into two equal circular segments; each is a quarter disc minus a right triangle, $\frac{\pi}{4} - \frac{1}{2}$. So each lens has area $\frac{\pi}{2} - 1$, the four lenses do not overlap, and the region where exactly two angles are obtuse has area $4\left(\frac{\pi}{2} - 1\right) = 2\pi - 4$. The probability is $\frac{2\pi-4}{4} = \frac{\pi}{2} - 1 \approx 0.57$.

Check. A point near vertex B sees both $\overline{AB}$ and $\overline{BC}$ at angles close to 180°, so it lies in a lens, as expected. Also, each half-disc has area $\frac{\pi}{2}$ and together they cover the whole square, with the lenses covered twice: $4 \cdot \frac{\pi}{2} - (2\pi - 4) = 4$, the area of the square.

20. Answer: (D) 88**. The idea.** As n increases by 1, $g(n)$ increases by 1 except when $\lfloor 2\sqrt{n} \rfloor$ also ticks up, when it jumps by 2 and skips exactly one value. So count the jumps: each one is a missed number. The jumps happen twice between consecutive squares, not once.

Work it out. Fix $k \geq 1$ and let $k^2 \leq n < (k+1)^2$. Then $2k \leq 2\sqrt{n} < 2k + 2$, and $2\sqrt{n} \geq 2k + 1$ exactly when $4n \geq 4k^2+4k+1$, i.e. $n \geq k^2+k+1$. So: for $n = k^2, \ldots, k^2+k$, $g(n) = n+2k$ covers $k^2+2k, \ldots, k^2+3k$; for $n = k^2+k+1, \ldots, k^2+2k$, $g(n) = n+2k+1$ covers $k^2+3k+2, \ldots, k^2+4k+1$. The next block $(k+1)$ starts at $(k+1)^2 + 2(k+1) = k^2 + 4k + 3$. So for each k exactly two values are skipped: $k^2 + 3k + 1$ and $k^2 + 4k + 2 = (k+2)^2 - 2$, and before everything, $g(1) = 3$, so 1 and 2 are skipped. Now count up to 2026. $k^2 + 3k + 1 \leq 2026$: $k = 43$ gives 1979, $k = 44$ gives 2069, so 43 values. $(k+2)^2 - 2 \leq 2026$: $(k+2)^2 \leq 2028$, $k + 2 \leq 45$, $k \leq 43$, so 43 values. Total $2 + 43 + 43 = 88$.

Check. The values of g start $3, 4, 6, 8, 9, 10, 12, 13, 15, 16, 17, 18, 20, \ldots$, missing $1, 2, 5, 7, 11, 14, 19, \ldots$ — matching $k^2 + 3k + 1 = 5, 11, 19$ and $(k+2)^2 - 2 = 7, 14$. With $\lfloor \sqrt{n} \rfloor$ instead there is only one jump per k and the answer would be 44.

21. Answer: (A) 3**. The idea.** Trailing zeros count factors of 10, so we need the exact powers of 5 and of 2 in the binomial coefficient, and take the smaller. Legendre's formula gives each power in a factorial; subtracting cancels most of it.

Work it out. The power of 5 in 2026! is $\lfloor \frac{2026}{5} \rfloor + \lfloor \frac{2026}{25} \rfloor + \lfloor \frac{2026}{125} \rfloor + \lfloor \frac{2026}{625} \rfloor = 405+81+16+3 = 505$. In 1013! it is $202 + 40 + 8 + 1 = 251$. So the binomial coefficient has $5^{505-502} = 5^3$ exactly. For 2: in 2026! it is $1013 + 506 + 253 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2018$, and in 1013! it is

$506 + 253 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 1005$, so the power of 2 is $2018 - 2010 = 8$. The number of trailing zeros is $\min(3, 8) = 3$.

Check. Kummer's theorem: the power of a prime p in $\binom{2n}{n}$ is the number of carries when adding $n + n$ in base p. In base 5, $1013 = 13023_5$, and $13023_5 + 13023_5$ carries in the units, fives and 125s places: 3 carries. In base 2, $1013 = 1111110101_2$ has eight 1s, so doubling it carries 8 times. Both agree.

22. Answer: (C) $12 - 8\sqrt{2}$**. The idea.** Put coordinates on the square and translate each tangency into a distance between centres: tangent to a line means distance r, external tangency means the centres are r plus the other radius apart. The first two conditions pin the centre in terms of r; the third gives one equation, which becomes a quadratic in $\sqrt{r}$.

Work it out. Let $A = (0, 0)$, $B = (8, 0)$, $C = (8, 8)$, $D = (0, 8)$. The semicircle has centre $M = (4, 0)$ and radius 4; the arc has centre C and radius 8; ω sits near A, between them. Tangency to $\overline{AD}$ (the line $x = 0$) puts the centre at (r, y). External tangency to the semicircle: $(r-4)^2 + y^2 = (r+4)^2$, so $y^2 = 16r$ and $y = 4\sqrt{r}$. External tangency to the arc: $(8 - r)^2 + (8 - 4\sqrt{r})^2 = (8 + r)^2$. Expand: $64 - 16r + r^2 + 64 - 64\sqrt{r} + 16r = 64 + 16r + r^2$, so $64 - 64\sqrt{r} = 16r$, i.e. $r + 4\sqrt{r} - 4 = 0$. As a quadratic in $\sqrt{r}$: $\sqrt{r} = -2 + 2\sqrt{2}$, so $r = (2\sqrt{2} - 2)^2 = 12 - 8\sqrt{2}$.

Check. $r \approx 0.686$ and the centre is $(0.686, 8\sqrt{2} - 8) \approx (0.686, 3.31)$, comfortably inside the square. Distances: to M, $\sqrt{3.314^2 + 3.314^2} \approx 4.686 = 4 + r$; to C, $\sqrt{7.314^2 + 4.686^2} \approx 8.686 = 8 + r$. Stopping at $\sqrt{r}$ gives $2\sqrt{2} - 2$, a distractor.

23. Answer: (D) 1710**. The idea.** First decide which boxes are empty; the empty boxes must form a set with no two neighbours on the 6-cycle. Then count distributions where every other box gets at least one ball, by stars and bars.

Work it out. Sets of pairwise non-neighbouring boxes on a circle of 6: the empty set (1); single boxes (6); pairs: $\binom{6}{2} - 6 = 9$; triples: only $\{1, 3, 5\}$ and $\{2, 4, 6\}$ (2); four or more is impossible. If exactly k boxes are non-empty, the 10 balls go into them with each getting at least one in $\binom{9}{k-1}$ ways. Total: $\binom{9}{5} + 6\binom{9}{4} + 9\binom{9}{3} + 2\binom{9}{2} = 126 + 756 + 756 + 72 = 1710$.

Check. Without the restriction there are $\binom{15}{5} = 3003$ placements, so the rule removes 1293 — plausible, since two specific neighbours are both empty in $\binom{11}{3} = 165$ placements and there are 6 neighbouring pairs, with overlaps. Forgetting the alternating triples gives 1638; stopping after one empty box gives 882.

24. Answer: (E) $\frac{97}{8}$**. The idea.** Count regions by adding the segments one at a time. A new segment that meets the earlier ones at q *distinct* interior points is cut into $q + 1$ pieces, and each piece splits one region into two. Adding this up over the segments: every drawn segment contributes 1, and an interior point where $k \geq 2$ drawn segments meet contributes $k - 1$ (the first segment through it adds nothing there, each later one adds 1). So

$$\text{regions} = 1 + (\text{segments drawn}) + \sum_{\text{interior points}} (k - 1).$$

For points in general position the last sum just counts crossing pairs; the regular hexagon is *not* in general position, because its three long diagonals all pass through the centre.

Work it out. Expected number of segments: $\frac{15}{2}$. Crossing pairs: two segments cross inside exactly when their four endpoints alternate around the circle, which gives one crossing pair for

each of the $\binom{6}{4} = 15$ sets of four vertices. Three of these pairs are pairs of long diagonals, all meeting at the centre O. The other 12 crossings are ordinary points where exactly two segments meet (the only point of the hexagon where three segments meet is O), each contributing 1 when both segments are drawn, with probability $\frac{1}{4}$: expected contribution $12 \cdot \frac{1}{4} = 3$. At O, let k be the number of drawn long diagonals, so $k \sim \text{Binomial}(3, \frac{1}{2})$. The contribution is $k - 1$ when $k \geq 1$ and 0 when $k = 0$, that is, $k - 1 + [k = 0]$, with expectation $\frac{3}{2} - 1 + \frac{1}{8} = \frac{5}{8}$. Total: $1 + \frac{15}{2} + 3 + \frac{5}{8} = \frac{8+60+24+5}{8} = \frac{97}{8}$.

Check. With all 15 segments drawn the formula gives $1 + 15 + 12 + 2 = 30$: the 6 circular caps outside the sides plus the 24 regions into which the diagonals cut a regular hexagon (the known count). Leaving out the starting region (the leading 1) gives $\frac{89}{8}$; ignoring the centre altogether gives $\frac{23}{2}$; using $\frac{1}{2}$ for the centre (forgetting the $[k = 0]$ correction) gives 12.

25. Answer: (C) 49**. The idea.** Removing all parentheses turns any such expression into $1 \pm 2 \pm 3 \pm \cdots \pm 10$. First decide which sign patterns can occur; then count the distinct sums, which is a subset-sum question.

Work it out. *Which signs are possible.* The 1 is never inside a subtraction's right side, so it is always +. The 2 is always the first number of whatever group contains it, and that group is subtracted from something starting with 1; so 2 is always −. Every sign pattern for $3, \ldots, 10$ is achievable: to make a block of consecutive numbers +, open a parenthesis just before the − number immediately preceding the block and close it after the block; e.g. signs $-2, +3, +4, -5, +6$ come from $1 - (2 - 3 - 4) - (5 - 6)$. Each maximal run of + numbers is preceded by a − number (since 2 is −), and these groups never overlap. *Which values occur.* If S is the sum of the numbers from 3 to 10 that get a minus sign, the value is $1 - 2 + (3 + 4 + \cdots + 10) - 2S = 51 - 2S$. So the number of values equals the number of possible subset sums S of $\{3, 4, \ldots, 10\}$, which lie between 0 and 52. The sums 1 and 2 are impossible, and so are 51 and 50 (their complements); every other value from 0 to 52 occurs: 0 (empty), 3 to 10 (single numbers), $11 = 3 + 8$, $12 = 3 + 9$, 13 to 19 as $10 + 3$ through $10 + 9$, 20 to 22 as $10 + 3 + 7$, $10 + 3 + 8$, $10 + 3 + 9$, and 23 to 26 as $10 + 9 + 4$ through $10 + 9 + 7$; the complement of a set with sum S has sum $52 - S$, which covers 26 to 49 and 52. That is $53 - 4 = 49$ values.

Check. Extremes: $S = 0$ gives $1 - (2 - 3 - 4 - \cdots - 10) = 51$, the largest value, and $S = 52$ gives -53, the plain expression $1 - 2 - \cdots - 10 = -53$. Answering 53 forgets the four impossible subset sums; answering 51 forgets their complements.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.