MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A rectangular garden has side lengths, in meters, that are positive integers differing by 6, and its area is 2016 square meters. A square garden is fenced using exactly the same total length of fence as the perimeter of the rectangular garden. What is the area of the square garden, in square meters?

(A) 2025 **(B)** 2034 **(C)** 2052 **(D)** 2061 **(E)** 2116

2. A string of 10 digits, each 0 or 1, is written down (there are 1024 such strings). An *up* is a place where a 0 is immediately followed by a 1, and a *down* is a place where a 1 is immediately followed by a 0. For example, 0110100011 has 3 ups and 2 downs. How many of the 1024 strings have more ups than downs?

(A) 128 **(B)** 255 **(C)** 256 **(D)** 384 **(E)** 512

3. A surveyor marks out a triangular plot of land. She measures its three sides and finds that, in meters, they are $\sqrt{10}$, $\sqrt{13}$ and $\sqrt{17}$. What is the area of the plot, in square meters?

(A) 5 **(B)** $\frac{11}{2}$ **(C)** 6 **(D)** $\frac{13}{2}$ **(E)** 11

4. How many positive integers less than 1000 can be written in the form $2^a - 2^b$, where a and b are integers with $a > b \geq 0$?

(A) 50 **(B)** 51 **(C)** 54 **(D)** 55 **(E)** 65

5. The numbers $1, 2, 3, 4, 5, 6$ are written in some order in a row. Below each pair of neighbours their sum is written, giving a row of 5 numbers; the same is done to that row, giving 4 numbers, and so on until a single number remains at the bottom. Over all 720 orders, what is the largest possible bottom number minus the smallest possible bottom number?

(A) 60 **(B)** 64 **(C)** 66 **(D)** 70 **(E)** 72

6. Tickets are numbered 1 through 1000. A ticket numbered n wins a prize if $n^2 - 1$ is a multiple of 77; for example, ticket 76 wins because $76^2 - 1 = 75 \cdot 77$. How many tickets win a prize?

(A) 13 **(B)** 26 **(C)** 48 **(D)** 51 **(E)** 52

7. In triangle ABC, $AB = 10$, $AC = 24$, and the segment from A to the midpoint of $\overline{BC}$ has length 13. What is the area of triangle ABC?

(A) 60 **(B)** 65 **(C)** 78 **(D)** 104 **(E)** 120

8. A club has 10 members. It forms a committee of some of its members, then names one committee member chair and a different committee member secretary. The committee must contain at least one member who is neither chair nor secretary. Two outcomes are different if the committees differ or the chair or secretary differs. How many outcomes are possible?

(A) 11475 **(B)** 22950 **(C)** 23040 **(D)** 25200 **(E)** 46080

9. A frog starts at 0 on the number line. Each jump moves it either 7 units to the right or 5 units to the left. What is the smallest number of jumps after which the frog can be exactly at 2026?

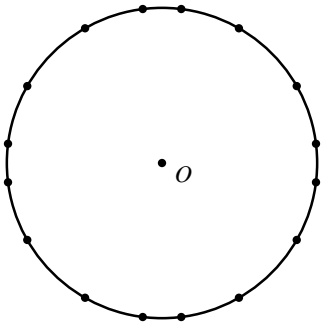
(A) 290 **(B)** 292 **(C)** 294 **(D)** 298 **(E)** 303

10. A list of numbers begins $0,\ 40$. Every later number is a weighted average of the two numbers just before

it: it is found by adding the earlier of the two to twice the later one and dividing by 3. (So the third number is $\frac{0+2\cdot 40}{3} = \frac{80}{3}$.) As the list goes on, its numbers get closer and closer to a single value. What is that value?

(A) 20 **(B)** 24 **(C)** $\frac{80}{3}$ **(D)** 30 **(E)** 32

11. Exactly 16 points with integer coordinates lie on the circle $x^2 + y^2 = 65$. Together with the centre $O = (0, 0)$ they make 17 points. How many triangles with all three vertices among these 17 points have a right angle?



(A) 112 **(B)** 120 **(C)** 128 **(D)** 144 **(E)** 240

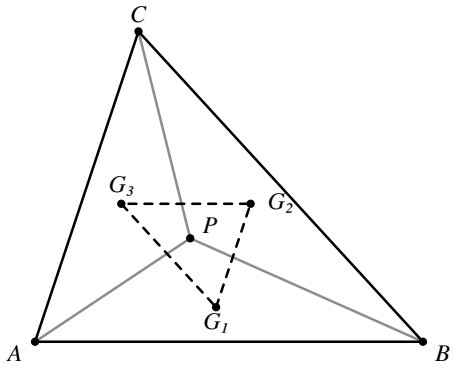
12. Sixteen dots are arranged in a 4×4 square array, at the points (x, y) with $x, y \in \{0, 1, 2, 3\}$. Of the $\binom{16}{2} = 120$ pairs of dots, for how many does the straight segment joining the two dots pass through no other dot of the array?

(A) 72 **(B)** 78 **(C)** 80 **(D)** 84 **(E)** 86

13. Real numbers x and y satisfy $x^2 + y^2 = 1$. What is the largest possible value of $x + y + xy$?

(A) $\frac{1}{2} + \sqrt{2}$ **(B)** 2 **(C)** $1 + \sqrt{2}$ **(D)** $\frac{3}{2} + \sqrt{2}$ **(E)** 3

14. Triangle ABC has area 90, and P is a point inside it. Let G_1, G_2 and G_3 be the centroids of triangles PAB, PBC and PCA, respectively. What is the area of triangle $G_1G_2G_3$?



(A) 10 **(B)** 15 **(C)** 20 **(D)** 30 **(E)** 45

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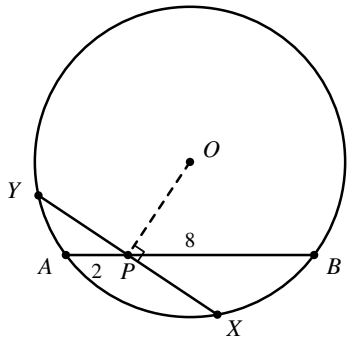
15. How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain exactly one pair of consecutive integers? For example, $\{2, 3, 7\}$ and $\{1, 2, 5, 9\}$ count, but $\{2, 3, 4\}$ does not (it contains two such pairs, 2, 3 and 3, 4), and neither does $\{1, 5\}$.

(A) 88 **(B)** 176 **(C)** 220 **(D)** 235 **(E)** 256

16. For how many integers n with $1 \le n \le 1050$ is $\gcd(n, 210) = \gcd(n + 2, 210)$? (Here $\gcd(a, b)$ denotes the greatest common divisor of a and b.)

(A) 75 **(B)** 150 **(C)** 240 **(D)** 300 **(E)** 480

17. In a circle with centre O, chord $\overline{AB}$ has length 8, and P is the point on $\overline{AB}$ with $AP = 2$. A second chord $\overline{XY}$ passes through P and is perpendicular to $\overline{OP}$. What is the length of $\overline{XY}$? (The radius of the circle is not given.)



(A) $4\sqrt{2}$ **(B)** 6 **(C)** $4\sqrt{3}$ **(D)** $2\sqrt{13}$ **(E)** 8

18. The numbers $1, 2, 3, \ldots, 10$ are split at random into two groups of five numbers each, with all 126 possible splits equally likely. What is the probability that the sums of the two groups differ by exactly 1?

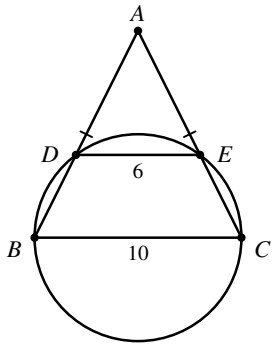
(A) $\frac{5}{63}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{10}{63}$ **(D)** $\frac{1}{5}$ **(E)** $\frac{20}{63}$

19. For how many positive integers n is $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor = n - 1$? (Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

(A) 20 **(B)** 21 **(C)** 22 **(D)** 41 **(E)** 42

20. Triangle ABC has $AB = AC$ and $BC = 10$. The circle with diameter $\overline{BC}$ crosses side $\overline{AB}$ again at a point D and side $\overline{AC}$ again at a point E, and $DE = 6$. What is the area of triangle ABC?

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(A) 25 **(B)** 30 **(C)** 40 **(D)** 48 **(E)** 50

21. The Fibonacci numbers are $F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$, so the list begins $1, 1, 2, 3, 5, 8, 13, \ldots$. For how many integers n with $1 \leq n \leq 2026$ is F_n divisible by 144?

(A) 84 **(B)** 126 **(C)** 168 **(D)** 169 **(E)** 337

22. How many different arrangements of the nine letters A, A, A, B, B, B, C, C, C in a row have no two identical letters next to each other?

(A) 174 **(B)** 180 **(C)** 186 **(D)** 210 **(E)** 216

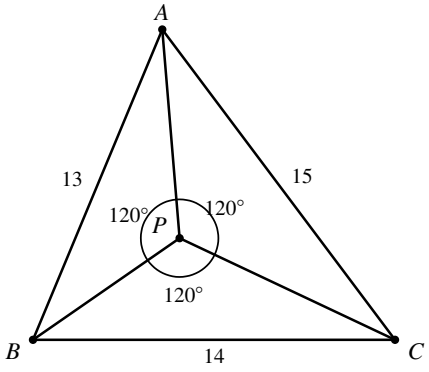
23. A loop on a cube is a closed path along the edges of the cube that starts and ends at the same vertex, uses no edge twice, and visits every other vertex on it only once (the starting vertex is revisited only at the end). Two loops are considered the same if they use the same set of edges. How many different loops are there on a cube?

(A) 16 **(B)** 22 **(C)** 25 **(D)** 28 **(E)** 31

24. How many ordered pairs (x, y) of real numbers satisfy both $(x^2+1)(y^2+1) = 25$ and $(x+y)(xy-1) = 12$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

25. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. Point P lies inside the triangle so that $\angle APB = \angle BPC = \angle CPA = 120^\circ$. What is $PA + PB + PC$?



(A) $\sqrt{295+56\sqrt{3}}$ **(B)** $\sqrt{295+84\sqrt{3}}$ **(C)** $\sqrt{295+112\sqrt{3}}$ **(D)** $\sqrt{295+168\sqrt{3}}$ **(E)** $\sqrt{590}$

Answer Key — AMC 10 Practice Contest 7

1. (A) 2025

2. (C) 256

3. (B) $\frac{11}{2}$

4. (A) 50

5. (E) 72

6. (E) 52

7. (E) 120

8. (B) 22950

9. (D) 298

10. (D) 30

11. (C) 128

12. (E) 86

13. (A) $\frac{1}{2} + \sqrt{2}$

14. (A) 10

15. (D) 235

16. (B) 150

17. (C) $4\sqrt{3}$

18. (C) $\frac{10}{63}$

19. (B) 21

20. (E) 50

21. (C) 168

22. (A) 174

23. (D) 28

24. (B) 5

25. (D) $\sqrt{295 + 168\sqrt{3}}$

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Solutions — AMC 10 Practice Contest 7

1. Answer: (A) 2025**. The idea.** If the square has side m, the rectangle, with the same perimeter and sides differing by 6, has sides $m-3$ and $m+3$. Its area is $(m-3)(m+3) = m^2 - 9$, so the square's area is always exactly 9 more than the rectangle's.

Work it out. The perimeter of the rectangle is $2\big((m-3)+(m+3)\big) = 4m$, which is also the perimeter of a square of side m. Since $m^2 - 9 = 2016$, the square's area is $m^2 = 2025$.

Check. $m = 45$, so the rectangle is $42 \times 48 = 2016$ (true) with perimeter 180, and the square has side $180/4 = 45$ and area 2025. Adding 36 instead of 9 comes from using the full difference 6 in place of half of it, and $2116 = 46^2$ comes from an off-by-one side.

2. Answer: (C) 256**. The idea.** Reading left to right, every change of digit is an up or a down, and the changes must alternate: after an up you are on a 1, so the next change can only be a down, and vice versa. So the numbers of ups and downs differ by at most 1, and which is bigger is decided by the first and last digits.

Work it out. If the string starts and ends with the same digit, the changes pair off and ups = downs. If it starts with 1 and ends with 0, the changes run down, up, down, ..., down, so there is one more down. If it starts with 0 and ends with 1, there is exactly one more up. So we need first digit 0 and last digit 1; the middle 8 digits are free: $2^8 = 256$ strings.

Check. By symmetry (swap $0 \leftrightarrow 1$) exactly 256 strings have more downs, leaving $1024 - 512 = 512$ with a tie, which matches the $2 \cdot 2^8 = 512$ strings with equal end digits. The example 0110100011 starts with 0 and ends with 1: 3 ups, 2 downs.

3. Answer: (B) $\frac{11}{2}$**. The idea.** Each squared length is a sum of two squares: $10 = 1^2 + 3^2$, $17 = 4^2 + 1^2$, $13 = 3^2 + 2^2$. So the triangle can be drawn with its corners on grid points, and then its area is easy to read off.

Work it out. Take $O = (0,0)$, $P = (1,3)$, $Q = (4,1)$. Then $OP = \sqrt{1+9} = \sqrt{10}$, $OQ = \sqrt{16+1} = \sqrt{17}$ and $PQ = \sqrt{3^2+2^2} = \sqrt{13}$. Enclose it in the 4×3 box $[0,4] \times [0,3]$ of area 12 and remove three right triangles of areas $\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$, $\frac{1}{2} \cdot 4 \cdot 1 = 2$ and $\frac{1}{2} \cdot 3 \cdot 2 = 3$. The area is $12 - \frac{3}{2} - 2 - 3 = \frac{11}{2}$.

Check. With the shoelace formula, $\frac{1}{2}|1 \cdot 1 - 3 \cdot 4| = \frac{11}{2}$. Heron in squared form: $16K^2 = 4 \cdot 10 \cdot 13 - (10 + 13 - 17)^2 = 520 - 36 = 484$, so $K = \frac{22}{4} = \frac{11}{2}$.

4. Answer: (A) 50**. The idea.** $2^a - 2^b = 2^b(2^{a-b} - 1)$. In binary, $2^k - 1$ is a block of k ones, and multiplying by 2^b appends b zeros. So the numbers of this form are exactly those whose binary expansion is a block of ones followed by a block of zeros, and different pairs (a, b) give different binary strings, hence different numbers.

Work it out. Such a number has a binary digits ($a - b$ ones, then b zeros). Every number with at most 9 binary digits is below $512 < 1000$: for $a = 1, 2, \ldots, 9$ there are a choices of b, giving $1 + 2 + \cdots + 9 = 45$ numbers. With $a = 10$ digits, $2^{10} - 2^b = 1024 - 2^b < 1000$ needs $2^b > 24$, i.e. $b = 5, 6, 7, 8, 9$: 5 more. With $a \geq 11$ the number is at least $2^{11} - 2^{10} = 1024$. Total $45 + 5 = 50$.

Check. The five 10-digit ones are $1024 - 32 = 992, 960, 896, 768, 512$; $1024 - 16 = 1008$ is too big. Letting in 1008 as well gives 51; allowing all of $b = 0, \ldots, 9$ gives 55.

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5. Answer: (E) 72**. The idea.** Track how many times each starting number is counted in the bottom number. The counts follow Pascal's triangle: in a row of 6 the bottom number is $a_1 + 5a_2 + 10a_3 + 10a_4 + 5a_5 + a_6$.

Work it out. Each downward step adds neighbours, so the number of paths from position i down to the bottom is $\binom{5}{i-1}$, giving the weights $1, 5, 10, 10, 5, 1$. To make the total as large as possible, put the biggest numbers on the biggest weights (rearrangement): $6, 5$ in the middle, $4, 3$ next, $2, 1$ at the ends: $10(6 + 5) + 5(4 + 3) + (2 + 1) = 110 + 35 + 3 = 148$. For the smallest, reverse: $10(1 + 2) + 5(3 + 4) + (5 + 6) = 30 + 35 + 11 = 76$. The difference is $148 - 76 = 72$.

Check. Row $2, 4, 6, 5, 3, 1$: $6, 10, 11, 8, 4 \to 16, 21, 19, 12 \to 37, 40, 31 \to 77, 71 \to 148$. Row $5, 3, 1, 2, 4, 6$: $8, 4, 3, 6, 10 \to 12, 7, 9, 16 \to 19, 16, 25 \to 35, 41 \to 76$.

6. Answer: (E) 52**. The idea.** $77 = 7 \cdot 11$ with both prime, and a prime divides $n^2 - 1 = (n - 1)(n + 1)$ only if it divides one of the factors. So $n \equiv \pm 1 \pmod 7$ and $n \equiv \pm 1 \pmod{11}$: four combinations, each a single class mod 77 by the Chinese Remainder Theorem, not just the two obvious ones $n \equiv \pm 1 \pmod{77}$.

Work it out. The four classes mod 77: $(1, 1) \to 1$; $(-1, -1) \to 76$; $n \equiv 1 \pmod 7$, $n \equiv -1 \pmod{11} \to 43$ (since $43 = 6 \cdot 7 + 1 = 4 \cdot 11 - 1$); $n \equiv -1 \pmod 7$, $n \equiv 1 \pmod{11} \to 34$ ($34 = 5 \cdot 7 - 1 = 3 \cdot 11 + 1$). Since $1000 = 12 \cdot 77 + 76$, the numbers 1 to 924 contain 12 full blocks with 4 solutions each (48), and 925 to 1000 covers residues 1 to 76, which include all four of $1, 34, 43, 76$. Total $48 + 4 = 52$.

Check. $34^2 - 1 = 1155 = 15 \cdot 77$ and $43^2 - 1 = 1848 = 24 \cdot 77$. Counting only $n \equiv \pm 1 \pmod{77}$ gives the trap answer 26.

7. Answer: (E) 120**. The idea.** Double the median: reflect A through the midpoint M of BC to get A'. Then $ABA'C$ is a parallelogram, and triangle ABA' has sides $AB = 10$, $BA' = AC = 24$ and $AA' = 26$, a 10–24–26 right triangle with the same area as ABC.

Work it out. The diagonal BC splits the parallelogram $ABA'C$ into two triangles congruent to ABC, and so does the diagonal AA'; hence $[ABA'] = [ABC]$. Since $10^2 + 24^2 = 676 = 26^2$, triangle ABA' is right-angled at B, with area $\frac{1}{2} \cdot 10 \cdot 24 = 120$.

Check. Apollonius: $AB^2 + AC^2 = 2AM^2 + \frac{1}{2}BC^2$ gives $676 = 338 + \frac{1}{2}BC^2$, so $BC = 26$. Then $AM = 13 = \frac{1}{2}BC$, so A is on the circle with diameter BC and $\angle A = 90°$; the area is $\frac{1}{2} \cdot 10 \cdot 24 = 120$ again.

8. Answer: (B) 22950**. The idea.** Adding up over committee sizes, $\sum k(k-1)\binom{10}{k}$, is slow. Instead choose the officers first, then the rest of the committee.

Work it out. Choose the chair (10 ways) and then the secretary (9 ways). Each of the other 8 club members is either on the committee or not, $2^8 = 256$ ways, but the choice with none of them is forbidden, leaving 255. Total $90 \cdot 255 = 22950$.

Check. $\sum_{k=2}^{10} k(k-1)\binom{10}{k} = 90 \cdot 2^8 = 23040$ counts all committees with officers; removing the 90 two-person committees leaves 22950. Forgetting that chair and secretary are different roles halves the count to 11475.

9. Answer: (D) 298**. The idea.** Only the numbers of each kind of jump matter: with a right jumps and b left jumps the frog is at $7a - 5b$. The total $a + b = a + \frac{7a-2026}{5}$ grows with a, so we want the smallest usable a, and usability is a divisibility condition.

Work it out. We need $7a - 5b = 2026$ with $a, b \geq 0$. Then $7a \geq 2026$, so $a \geq 290$. Mod 5: $7a \equiv 2a \equiv 2026 \equiv 1$, so $a \equiv 3 \pmod 5$. The least such $a \geq 290$ is 293, giving $5b = 7 \cdot 293 - 2026 = 2051 - 2026 = 25$, $b = 5$. Total $293 + 5 = 298$ jumps.

Check. $a = 290$ gives 2030, 4 too far, not a multiple of 5; $a = 291, 292$ give excesses 11, 18, also not multiples of 5. The answer 290 ignores that the frog must land exactly on 2026.

10. Answer: (D) 30**. The idea.** Look for a combination of two neighbouring terms that never changes. If x, y are consecutive and $z = \frac{x+2y}{3}$ comes next, then $z + \frac{1}{3}y = \frac{1}{3}x + y$: the quantity (later term) $+\frac{1}{3}$(earlier term) is the same for every pair of neighbours.

Work it out. For the first pair it is $40 + \frac{1}{3} \cdot 0 = 40$. If the terms approach L, the invariant approaches $L + \frac{1}{3}L = \frac{4}{3}L$, so $\frac{4}{3}L = 40$ and $L = 30$.

Check. The terms are $0, 40, \frac{80}{3}, \frac{280}{9}, \frac{800}{27}, \ldots$; their distances from 30 are $-30, 10, -\frac{10}{3}, \frac{10}{9}, -\frac{10}{27}$, each $-\frac{1}{3}$ times the previous, so they shrink to 0. The plain average of 0 and 40, 20, ignores the weighting.

11. Answer: (C) 128**. The idea.** Split by whether O is a vertex. If all three vertices are on the circle, the triangle has a right angle exactly when one side is a diameter (Thales). If O is a vertex, the other two are points P, Q on the circle, and the right angle can only be at O: a right angle at P would put Q on the tangent line at P, which meets the circle only at P.

Work it out. The points are $(\pm1, \pm8), (\pm8, \pm1), (\pm4, \pm7), (\pm7, \pm4)$, forming 8 diametrically opposite pairs. *Three points on the circle:* pick the diameter that is the hypotenuse (8 ways) and any one of the other 14 points as the right-angle vertex; a triangle has only one right angle, so nothing is counted twice: $8 \cdot 14 = 112$. *With O:* we need $OP \perp OQ$, so Q is P turned by 90° about O; turning (x, y) gives $(-y, x)$ or $(y, -x)$, again lattice points on the circle. Each of the 16 choices of P has 2 partners, and each pair is counted from both ends: $16 \cdot 2/2 = 16$. (An opposite pair $P, -P$ with O is not a triangle at all, since O is the midpoint.) Total $112 + 16 = 128$.

Check. The 16 triangles at O are halves of the 4 squares inscribed in the circle (4 squares, 4 right-angled corners at O each). Forgetting O gives 112; counting ordered perpendicular pairs gives $112 + 32 = 144$.

12. Answer: (E) 86**. The idea.** The segment from (x_1, y_1) to (x_2, y_2) with $\Delta x = |x_1 - x_2|$, $\Delta y = |y_1 - y_2|$ contains $\gcd(\Delta x, \Delta y) - 1$ lattice points strictly inside it, and inside a convex square array those points are dots. So a pair is clear exactly when $\gcd(\Delta x, \Delta y) = 1$.

Work it out. Count the blocked pairs, those with gcd > 1: the displacements with entries in $\{0, 1, 2, 3\}$ are $(2, 0), (0, 2), (3, 0), (0, 3), (2, 2), (3, 3)$. Along one row, $(2, 0)$ occurs 2 times and $(3, 0)$ once, so rows give $4 \cdot 3 = 12$ and columns 12. The diagonal step $(2, 2)$ in either diagonal direction fits in $2 \cdot 2$ positions each, 8 in all, and $(3, 3)$ gives the 2 main diagonals. Blocked: $12 + 12 + 8 + 2 = 34$, so clear: $120 - 34 = 86$.

Check. Pairs with $(\Delta x, \Delta y) = (2, 1)$ etc. have gcd = 1 and are clear even though they are long. Counting each $(2, 2)$ diagonal step twice gives $120 - 42 = 78$.

13. Answer: (A) $\frac{1}{2} + \sqrt{2}$**. The idea.** The expression is symmetric in x and y, so write it in terms of the sum $s = x + y$. The constraint turns the product into a function of s too, and the only thing left to track is which values of s are possible.

Work it out. From $s^2 = x^2+2xy+y^2 = 1+2xy$ we get $xy = \frac{s^2-1}{2}$, so $x+y+xy = s+\frac{s^2-1}{2} = \frac{(s+1)^2}{2}-1$. Which s can occur? Since $(x-y)^2 \geq 0$, we have $s^2 = 2(x^2+y^2)-(x-y)^2 \leq 2$, so $-\sqrt{2} \leq s \leq \sqrt{2}$, and both ends occur (at $x = y = \pm\frac{\sqrt{2}}{2}$). On this interval $(s+1)^2$ is largest at $s = \sqrt{2}$ (the point farthest from -1), giving $\frac{(1+\sqrt{2})^2}{2} - 1 = \frac{3+2\sqrt{2}}{2} - 1 = \frac{1}{2} + \sqrt{2}$.

Check. At $x = y = \frac{\sqrt{2}}{2}$: $x + y = \sqrt{2}$ and $xy = \frac{1}{2}$, total $\frac{1}{2} + \sqrt{2} \approx 1.914$. The tempting $1 + \sqrt{2}$ would need $x + y$ and xy to be largest separately with $xy = 1$, which is impossible on the unit circle.

14. Answer: (A) 10**. The idea.** A centroid is the average of its three vertices. Subtracting two of these averages makes P cancel, so the small triangle is a scaled copy of ABC no matter where P is.

Work it out. Treat points as vectors: $G_1 = \frac{P+A+B}{3}$, $G_2 = \frac{P+B+C}{3}$, $G_3 = \frac{P+C+A}{3}$. Then $G_1 - G_2 = \frac{A-C}{3}$, $\quad G_2 - G_3 = \frac{B-A}{3}$, $\quad G_3 - G_1 = \frac{C-B}{3}$. So each side of $\triangle G_1G_2G_3$ is parallel to a side of $\triangle ABC$ and exactly one third as long. The triangles are similar with ratio $\frac{1}{3}$, so the area ratio is $\frac{1}{9}$: the area is $\frac{90}{9} = 10$.

Check. Coordinates: $A = (0,0)$, $B = (15,0)$, $C = (0,12)$ (area 90) and $P = (3,3)$ give $G_1 = (6,1)$, $G_2 = (6,5)$, $G_3 = (1,5)$, a right triangle with legs 4 and 5: area 10. (The tempting 30 comes from using the ratio $\frac{1}{3}$ for area instead of length.)

15. Answer: (D) 235**. The idea.** Such a set is made of blocks of consecutive chosen numbers with gaps between them: exactly one block is a pair and all the others are single numbers. Count the sets of each size k by placing $k-1$ blocks into the gaps left by the unchosen numbers, then choosing which block is the pair.

Work it out. A set with k elements has $k-1$ blocks. The $10-k$ unchosen numbers leave $11-k$ slots (before, between and after them), and the blocks must go into different slots, in order: $\binom{11-k}{k-1}$ ways. Then choose which block is the pair: $k-1$ ways. $k = 2$: $\binom{9}{1} \cdot 1 = 9$; $k = 3$: $\binom{8}{2} \cdot 2 = 56$; $k = 4$: $\binom{7}{3} \cdot 3 = 105$; $k = 5$: $\binom{6}{4} \cdot 4 = 60$; $k = 6$: $\binom{5}{5} \cdot 5 = 5$; for $k \geq 7$ there are too few slots. Total $9 + 56 + 105 + 60 + 5 = 235$.

Check. For $k = 2$ the sets are the 9 pairs $\{i, i+1\}$, as the formula says. For $k = 6$, the four unchosen numbers must each separate two blocks, e.g. $\{1,2,4,6,8,10\}$, and the pair can be any of the 5 blocks. Forgetting the factor $k-1$ would give only $9 + 28 + 35 + 15 + 1 = 88$.

16. Answer: (B) 150**. The idea.** Since $210 = 2 \cdot 3 \cdot 5 \cdot 7$ is a product of distinct primes, $\gcd(n, 210)$ is just the product of those primes among $2, 3, 5, 7$ that divide n. So the two gcds are equal exactly when each of the four primes divides both n and $n+2$ or neither. Handle the primes one at a time.

Work it out. *The prime* 2. The numbers n and $n+2$ have the same parity, so 2 divides both or neither: no condition.

An odd prime $p \in \{3, 5, 7\}$. If p divided both n and $n+2$, it would divide their difference 2, which is impossible. So "both or neither" means neither: $n \not\equiv 0$ and $n \not\equiv -2 \pmod{p}$. That rules out 2 of the p residues, leaving 1 residue mod 3, 3 residues mod 5 and 5 residues mod 7.

By the Chinese Remainder Theorem these choices combine into $1 \cdot 3 \cdot 5 = 15$ allowed residues modulo 105. The range $1 \leq n \leq 1050$ is exactly 10 full blocks of 105 consecutive integers, each containing every residue once, so the count is $10 \cdot 15 = 150$.

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Check. $n = 2$: $\gcd(2, 210) = 2 = \gcd(4, 210)$, and indeed 2 avoids 0 and -2 modulo 3, 5 and 7. $n = 1$ fails because $3 \mid n + 2$: the gcds are 1 and 3. Forgetting that the common value may be 2 (keeping only odd n) gives the trap 75; excluding only $n \equiv 0$ for each odd prime gives $2 \cdot 4 \cdot 6 = 48$ residues and the trap 480; counting n coprime to 210 gives 240.

17. Answer: (C) $4\sqrt{3}$**. The idea.** The perpendicular from the centre to a chord meets the chord at its midpoint. For chord $\overline{XY}$ that perpendicular is $\overline{OP}$ itself, so P is the midpoint of $\overline{XY}$. Then the intersecting-chords theorem finishes the job without ever needing the radius.

Work it out. Chords $\overline{AB}$ and $\overline{XY}$ cross at P, so $PX \cdot PY = PA \cdot PB = 2 \cdot 6 = 12$. Since $PX = PY$, $PX = \sqrt{12} = 2\sqrt{3}$ and $XY = 4\sqrt{3}$.

Check. Try radius 5: put O at the origin and $\overline{AB}$ on the line $y = -3$ from $(-4, -3)$ to $(4, -3)$. Then $P = (-2, -3)$, $OP^2 = 13$, and half of $\overline{XY}$ is $\sqrt{25 - 13} = \sqrt{12}$. With radius 4 instead ($\overline{AB}$ a diameter), $OP = 2$ and half of $\overline{XY}$ is $\sqrt{16 - 4} = \sqrt{12}$ again.

18. Answer: (C) $\frac{10}{63}$**. The idea.** The total is $1 + 2 + \cdots + 10 = 55$, so sums differing by 1 must be 27 and 28. Each good split has exactly one group summing to 27, so count the five-element subsets with sum 27.

Work it out. Subtract $1, 2, 3, 4, 5$ from the five chosen numbers in increasing order: if $a_1 < a_2 < \cdots < a_5$, set $c_i = a_i - i$. Then $0 \le c_1 \le c_2 \le \cdots \le c_5 \le 5$ and $\sum c_i = 27 - 15 = 12$. So we count partitions of 12 into at most 5 parts, each at most 5. List by the largest part:

Largest part 5: the rest is 7 in at most 4 parts each ≤ 5: 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1
9 ways.

Largest part 4: the rest is 8 in at most 4 parts each ≤ 4: 4+4, 4+3+1, 4+2+2, 4+2+1+1, 3+3+2, 3+3+1+1, 3+2+2+1, 2
8 ways.

Largest part 3: the rest is 9 in at most 4 parts each ≤ 3: 3+3+3, 3+3+2+1, 3+2+2+2: 3 ways.

Largest part at most 2 gives at most $10 < 12$. Total $9 + 8 + 3 = 20$ subsets, so 20 good splits and the probability is $\frac{20}{126} = \frac{10}{63}$.

Check. By symmetry ($a \mapsto 11 - a$ turns a sum of 27 into 28) there are also 20 subsets with sum 28; counting both as separate splits would double-count and give the trap $\frac{20}{63}$.

19. Answer: (B) 21**. The idea.** Since $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$, the three floors add up to $\frac{41n}{42}$ minus the fractional parts, which depend only on the remainders of n. Writing the shortfall exactly turns the equation into a formula for n in terms of its own remainders.

Work it out. Let n leave remainders a, b, c on division by $2, 3, 7$. Then $\lfloor n/2 \rfloor = \frac{n-a}{2}$, and so on, and the equation becomes $\frac{n-a}{2} + \frac{n-b}{3} + \frac{n-c}{7} = n - 1 \iff n = 42 - 21a - 14b - 6c$. Conversely, pick any $a \in \{0, 1\}$, $b \in \{0, 1, 2\}$, $c \in \{0, \ldots, 6\}$ and put $n = 42 - 21a - 14b - 6c$. Then $n \equiv a \pmod 2$ (as 21 is odd), $n \equiv -14b \equiv b \pmod 3$, and $n \equiv -6c \equiv c \pmod 7$: the remainders of n really are a, b, c, so n solves the equation. Different triples give different n (their remainders differ), so the solutions correspond exactly to triples with $21a + 14b + 6c \le 41$ (so that $n \ge 1$).

Count them. $a = 0$: $b = 0$ allows $c \le 6$ (7 ways), $b = 1$ allows $6c \le 27$ (5 ways), $b = 2$ allows $6c \le 13$ (3 ways). $a = 1$: $b = 0$ allows $6c \le 20$ (4 ways), $b = 1$ allows $6c \le 6$ (2 ways), $b = 2$ is impossible. Total $7 + 5 + 3 + 4 + 2 = 21$.

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Check. Replacing (a, b, c) by $(1-a, 2-b, 6-c)$ sends $m = 21a + 14b + 6c$ to $85 - m$, and no triple has $m = 42$ or 43; so exactly half of the 42 triples have $m \le 41$, confirming 21. For example $n = 42$ (triple $0, 0, 0$): $21 + 14 + 6 = 41$.

20. Answer: (E) 50**. The idea.** Two facts about the circle on $\overline{BC}$. First, D sees the diameter at a right angle, so $\overline{CD} \perp \overline{AB}$ and $AD = AC\cos A$. Second, B, C, D, E lie on one circle, so $\angle ADE = \angle ACB$ and triangle ADE is similar to triangle ACB, with ratio $\frac{AD}{AC} = \cos A$.

Work it out. The similarity gives $\frac{DE}{BC} = \cos A$, so $\cos A = \frac{6}{10} = \frac{3}{5}$ and $\sin A = \frac{4}{5}$. With $AB = AC = b$, the law of cosines gives $100 = 2b^2 - 2b^2 \cdot \frac{3}{5} = \frac{4}{5}b^2$, so $b^2 = 125$. The area is $\frac{1}{2}b^2 \sin A = \frac{1}{2} \cdot 125 \cdot \frac{4}{5} = 50$.

Check. Coordinates: $B = (-5, 0)$, $C = (5, 0)$ and $A = (0, 10)$ give $AB^2 = 125$. The foot of the perpendicular from C to line AB is $D = (-3, 4)$, which is on the circle $x^2 + y^2 = 25$; by symmetry $E = (3, 4)$ and $DE = 6$. The area is $\frac{1}{2} \cdot 10 \cdot 10 = 50$.

21. Answer: (C) 168**. The idea.** Listing the first twelve terms shows $F_{12} = 144$. The addition rule $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ then says that stepping 12 places multiplies the term by a unit modulo 144, so the multiples of 144 recur exactly every 12 places.

Work it out. The terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$, so $F_{12} = 144$ and none of $F_1, \ldots, F_{11}$ is a multiple of 144.

The addition rule (proved by induction on m from the defining recursion) with $m = 12$ reads $F_{n+12} = F_{13}F_n + F_{12}F_{n-1} = 233F_n + 144F_{n-1}$ for $n \ge 2$, and it also holds for $n = 1$ with $F_0 = 0$. Hence $F_{n+12} \equiv 233F_n \equiv 89F_n \pmod{144}$. Since 89 and 144 share no factor, $144 \mid F_{n+12}$ exactly when $144 \mid F_n$. So the pattern of which terms are multiples of 144 repeats every 12 steps, and in each block of 12 only the last term qualifies: $144 \mid F_n \iff 12 \mid n$.

The multiples of 12 up to 2026 number $\lfloor 2026/12 \rfloor = 168$.

Check. $F_{24} = 46368 = 144 \cdot 322$, while $F_6 = 8$ and $F_{18} = 2584 = 144 \cdot 17 + 136$ are not multiples of 144, even though $F_6 \mid F_{18}$. The trap 84 assumes the full repeating period of the sequence modulo 144, which is 24, contains only one zero; it contains two.

22. Answer: (A) 174**. The idea.** The usual "place one letter, drop the rest into gaps" trick fails because the other letters can themselves be adjacent. Instead, first write the six letters A and B in a row, then insert the three C's: every place where two equal letters touch must receive a C, and the remaining C's go into any other gaps.

Work it out. Remove the C's from a good arrangement: what remains is a word of three A's and three B's, and the C's sit in distinct gaps of that word (no two C's adjacent), at most one per gap, among the 7 gaps (including both ends). Conversely, a word plus a choice of gaps gives a good arrangement exactly when every adjacent equal pair in the word gets a C.

If the word has r runs (maximal blocks of equal letters), it has $6 - r$ adjacent equal pairs; call this j. We need $j \le 3$, then the remaining $3 - j$ C's go into any of the other $7 - j$ gaps: $\binom{7-j}{3-j}$ ways. Count words by runs (a run structure is fixed by the first letter and how each letter's three copies are split into runs, which alternate):

$r = 6$, $j = 0$: ABABAB, BABABA: 2 words, $\binom{7}{3} = 35$ each: 70.

$r = 5$, $j = 1$: one letter in 3 runs and the other in 2 runs (the 3-run letter comes first); 2 ways to split the 2-run letter, 2 choices of which letter: 4 words, $\binom{6}{2} = 15$ each: 60.

$r = 4$, $j = 2$: both letters in 2 runs: $2 \cdot 2$ splits times 2 starting letters: 8 words, $\binom{5}{1} = 5$ each: 40.

$r = 3$, $j = 3$: one letter in 2 runs, other in 1, the 2-run letter first: $2 \cdot 2 = 4$ words, $\binom{4}{0} = 1$ each: 4.

$r = 2$ (AAABBB, BBBAAA) would need 4 C's: impossible. The word counts $2 + 4 + 8 + 4 + 2 = 20 = \binom{6}{3}$ check out. Total $70 + 60 + 40 + 4 = 174$.

Check. The same method on the smaller word AABBCC: ABAB, BABA take the two C's in $\binom{5}{2} = 10$ ways each, ABBA, BAAB in $\binom{4}{1} = 4$ ways each, AABB, BBAA in 1 way each: $20+8+2 = 30$, which matches a direct listing. Using $\binom{4}{1}$ in place of $\binom{4}{0}$ for the four $j = 3$ words gives the trap 186.

23. Answer: (D) 28**. The idea.** Colour the cube's vertices black and white alternately (neighbours get different colours). Every loop alternates colours, so it has even length 4, 6 or 8. Count each length with its own idea: faces for 4, the two missing vertices for 6, and the one missing edge at each vertex for 8.

Work it out. *Length 4.* A 4-cycle on the cube is a face: 6 loops.

Length 6. Six vertices use three of each colour, so the two missing vertices have opposite colours: they are adjacent (12 edges) or antipodal (4 long diagonals). If two adjacent vertices are removed, the other six vertices and the 7 edges among them form two squares sharing an edge (a 2×3 array of points): for example, removing 000 and 001 leaves 010, 110, 100 in one row and 011, 111, 101 in the other, with three rungs. Its only loop through all six vertices is the outer rectangle: 1 loop each. If two antipodal vertices are removed, the other six vertices with their edges form a single hexagon ("equator"): 1 loop each. So $12 + 4 = 16$ loops.

Length 8. Such a loop uses 2 of the 3 edges at every vertex, so the unused edges form a perfect matching (4 edges, one at every vertex). Conversely, removing a perfect matching leaves every vertex with 2 edges: either one 8-loop or two 4-loops. Count perfect matchings by viewing the cube as a top square and a bottom square joined by 4 vertical edges. Using all 4 vertical edges: 1 matching. Using none: match the top square (2 ways) and the bottom square (2 ways): 4. Using exactly 2: the two unmatched top vertices must be adjacent, so the vertical edges sit at adjacent corners (4 ways) and the rest is forced: 4. Using 1 or 3 leaves an odd number of top vertices: impossible. That is 9 matchings, of which 3 have all edges parallel (all vertical, or top and bottom matched in the same direction). Removing a parallel matching leaves two opposite faces (two 4-loops), so the 8-loops come from the other 6 matchings: 6 loops.

Total $6 + 16 + 6 = 28$.

Check. Label vertices by coordinates $xyz \in \{0, 1\}^3$. Removing the mixed matching {000-010, 100-110, 001-101, leaves the single loop $000, 100, 101, 111, 110, 010, 011, 001$, as claimed; the other five mixed matchings are its images under the cube's symmetries. Stopping at lengths 4 and 6 gives the trap 22; counting all 9 matchings as 8-loops gives 31.

24. Answer: (B) 5**. The idea.** Expand the first product: $(x^2 + 1)(y^2 + 1) = x^2y^2 + x^2 + y^2 + 1 = (xy - 1)^2 + (x + y)^2$. So with $u = x + y$ and $v = xy - 1$ the system is just $u^2 + v^2 = 25$, $uv = 12$. The subtle part is going back from (u, v) to real (x, y): some choices give no real pair, and one gives a repeated one.

Work it out. From $u^2 + v^2 = 25$ and $2uv = 24$: $(u + v)^2 = 49$ and $(u - v)^2 = 1$, so (u, v) is one of $(4, 3)$, $(3, 4)$, $(-4, -3)$, $(-3, -4)$.

For each, x and y are the roots of $t^2 - ut + (v + 1) = 0$, which are real only when $u^2 \geq 4(v + 1)$.

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$(4,3)$: $t^2-4t+4=(t-2)^2$: a double root, so $x=y=2$: one ordered pair.

$(3,4)$: $9<20$: no real pair.

$(-4,-3)$: $t^2+4t-2=0$, roots $-2\pm\sqrt{6}$, distinct: two ordered pairs.

$(-3,-4)$: $t^2+3t-3=0$, roots $\frac{-3\pm\sqrt{21}}{2}$, distinct: two ordered pairs.

Total $1+0+2+2=5$.

Check. $(2,2)$: $5\cdot 5=25$ and $4\cdot 3=12$. For $x,y=-2\pm\sqrt{6}$: $x+y=-4$, $xy=4-6=-2$, so $(x+y)(xy-1)=12$ and $(xy-1)^2+(x+y)^2=9+16=25$. Counting the double root as two pairs gives the trap 6; forgetting to test real roots gives 8 (or 7).

25. Answer: (D) $\sqrt{295+168\sqrt{3}}$**. The idea.** Call the three distances $x=PA$, $y=PB$, $z=PC$. The three small triangles at P each have a 120° angle, which gives one equation per side (law of cosines) and one equation for the total area. We never need x,y,z individually: the side equations give $x^2+y^2+z^2$ in terms of $xy+yz+zx$, and the area gives $xy+yz+zx$, which is exactly what $(x+y+z)^2$ needs.

Work it out. In a triangle with sides x,y around a 120° angle, the third side w satisfies $w^2=x^2+y^2+xy$ (drop a perpendicular: the 60° outside angle makes a 30-60-90 triangle). So $x^2+xy+y^2=169$, $y^2+yz+z^2=196$, $z^2+zx+x^2=225$. Adding: $2(x^2+y^2+z^2)+(xy+yz+zx)=590$.

The area of such a triangle is $\frac{1}{2}xy\sin 120^\circ=\frac{\sqrt{3}}{4}xy$. The three pieces fill $\triangle ABC$, whose area by Heron's formula is $\sqrt{21\cdot 8\cdot 7\cdot 6}=84$. So $\frac{\sqrt{3}}{4}(xy+yz+zx)=84$, i.e. $xy+yz+zx=112\sqrt{3}$.

Then $x^2+y^2+z^2=\frac{590-112\sqrt{3}}{2}=295-56\sqrt{3}$ and $(x+y+z)^2=(x^2+y^2+z^2)+2(xy+yz+zx)=295-56\sqrt{3}+224\sqrt{3}=295+168\sqrt{3}$. So $PA+PB+PC=\sqrt{295+168\sqrt{3}}\approx 24.21$.

Check. Rotate $\triangle APB$ by 60° about B away from C: P goes to P' with $\triangle BPP'$ equilateral and A goes to A' with $\triangle ABA'$ equilateral. The 120° angles make C,P,P',A' collinear, so $PA+PB+PC=CA'$. With $B=(0,0)$, $C=(14,0)$, $A=(5,12)$, the apex A' of the outward equilateral triangle on AB is $\left(\frac{5}{2}-6\sqrt{3},\ 6+\frac{5\sqrt{3}}{2}\right)$, and $CA'^2=\left(\frac{23}{2}+6\sqrt{3}\right)^2+\left(6+\frac{5\sqrt{3}}{2}\right)^2=295+168\sqrt{3}$.

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