MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

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1. A real number x satisfies $\sqrt{x+20} + \sqrt{x-4} = 12$. What is the value of the product $\sqrt{x+20} \cdot \sqrt{x-4}$?

(A) 24 **(B)** 29 **(C)** 32 **(D)** 35 **(E)** 36

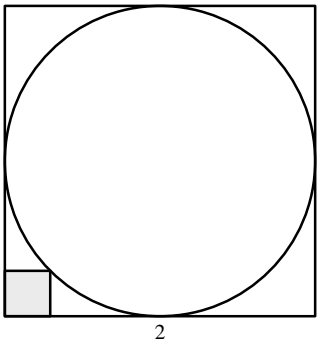
2. At a school supply stand, every pencil costs the same, every eraser costs the same, and every notebook costs the same. Three pencils, five erasers and one notebook cost \$7.20 in total. Four pencils, seven erasers and one notebook cost \$8.90 in total. How much do one pencil, one eraser and one notebook cost together?

(A) \$1.70 **(B)** \$3.40 **(C)** \$3.80 **(D)** \$4.10 **(E)** \$5.50

3. Seven different positive integers are written in increasing order. The mean of all seven is 15, the mean of the first four is 10, and the mean of the last four is 20. What is the largest possible value of the largest of the seven integers?

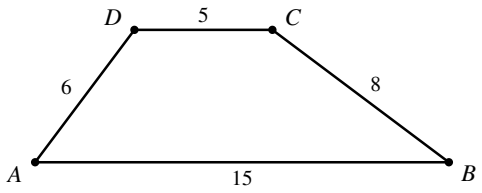
(A) 29 **(B)** 30 **(C)** 31 **(D)** 32 **(E)** 35

4. A circle is inscribed in a square of side length 2, touching all four sides. In one corner of the big square, in the region outside the circle, a small square is placed so that two of its sides lie along the two sides of the big square that meet at that corner, and the vertex of the small square farthest from that corner lies on the circle. What is the area of the small square?



(A) $17 - 12\sqrt{2}$ **(B)** $\frac{3}{2} - \sqrt{2}$ **(C)** $3 - 2\sqrt{2}$ **(D)** $1 - \frac{\sqrt{2}}{2}$ **(E)** $2 - \sqrt{2}$

5. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, with $AB = 15$ and $CD = 5$. Its legs have lengths $AD = 6$ and $BC = 8$. What is the area of the trapezoid?

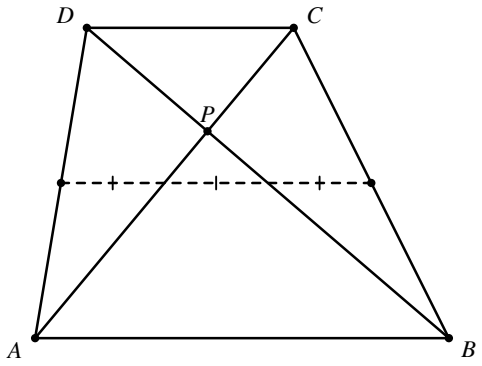


(A) 40 **(B)** 48 **(C)** 56 **(D)** 60 **(E)** 80

6. The equation $x^2 + x - 1 = 0$ has two real roots, r and s. Surprisingly, the two numbers $r^6 + 8r$ and $s^6 + 8s$ turn out to be equal, even though r and s are irrational and different. What is this common value?

(A) -3 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 13

7. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$ with $AB > CD$. The segment joining the midpoints of the legs $\overline{AD}$ and $\overline{BC}$ is cut by the two diagonals $\overline{AC}$ and $\overline{BD}$ into three pieces, and these three pieces all have the same length. The area of the trapezoid is 36. Let P be the point where the diagonals cross. What is the area of $\triangle ABP$?



(A) 4 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 16

8. A 3×4 rectangle of unit squares is to be covered completely, with no overlaps and nothing sticking out, using 1×1 tiles and 2×2 tiles (any number of each, including none). Tiles of the same size are identical, and two coverings are different if they look different (the rectangle is not turned or flipped). How many different coverings are there?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 13

9. How many subsets of $\{1, 2, 3, \ldots, 12\}$ have largest element at least twice their smallest element? (A subset with one element does not count, since its largest and smallest elements are equal.)

(A) 63 **(B)** 3906 **(C)** 3969 **(D)** 4032 **(E)** 4095

10. A calculator display starts at 1. It has three buttons: one doubles the number on the display, one adds 1 to it, and one subtracts 1 from it. What is the fewest number of button presses needed to make the display show 2026?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

11. Four friends hold a vote to choose a team captain. Each friend secretly votes for exactly one of the *other* three friends, each of the three choices being equally likely, and all four votes are independent. What is the probability that one friend receives strictly more votes than each of the other three?

(A) $\frac{4}{27}$ **(B)** $\frac{7}{27}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{20}{27}$ **(E)** $\frac{8}{9}$

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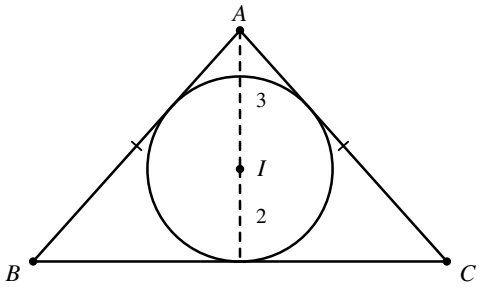
12. For a real number x, let $f(x) = |x-1| + 2|x-2| + 3|x-3| + \cdots + 10|x-10|$, where the kth term is $k|x-k|$. What is the smallest value that $f(x)$ takes?

(A) 100 **(B)** 112 **(C)** 113 **(D)** 125 **(E)** 136

13. For each $k = 1, 2, \ldots, 16$, let q_k be the whole-number part of $\dfrac{11k}{17}$, that is, $q_k = \left\lfloor \dfrac{11k}{17} \right\rfloor$. So $q_1 = 0$, $q_2 = 1$, $q_3 = 1$, and so on. What is $q_1 + q_2 + \cdots + q_{16}$?

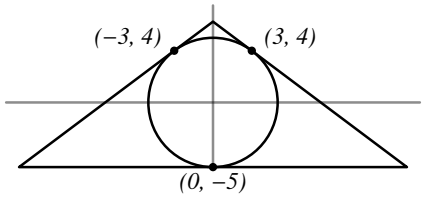
(A) 80 **(B)** 81 **(C)** 84 **(D)** 88 **(E)** 96

14. Triangle ABC has $AB = AC$. Its incircle has radius 2, and the center I of the incircle is at distance 3 from vertex A. What is the length of BC?



(A) $2\sqrt{5}$ **(B)** $3\sqrt{5}$ **(C)** $4\sqrt{3}$ **(D)** $6\sqrt{2}$ **(E)** $4\sqrt{5}$

15. The circle $x^2 + y^2 = 25$ passes through the points $(3, 4)$, $(-3, 4)$ and $(0, -5)$. At each of these three points the tangent line to the circle is drawn. The three tangent lines enclose a triangle. What is its area?



(A) $\frac{675}{4}$ **(B)** 200 **(C)** 225 **(D)** $\frac{675}{2}$ **(E)** 675

16. For how many integers n with $1 \le n \le 2026$ is the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$ divisible by n?

(A) 337 **(B)** 338 **(C)** 673 **(D)** 674 **(E)** 675

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17. A sequence is defined by $a_1 = 1$ and $a_{n+1} = \dfrac{a_n}{1 + n\,a_n}$ for $n \geq 1$. For example, $a_2 = \frac{1}{2}$ and $a_3 = \dfrac{1/2}{1 + 2 \cdot \frac{1}{2}} = \frac{1}{4}$. For how many positive integers n is $a_n > \dfrac{1}{1000}$?

(A) 32 **(B)** 44 **(C)** 45 **(D)** 46 **(E)** 63

18. A 4×4 grid of unit squares contains nine 2×2 blocks, each made of four squares from two neighbouring rows and two neighbouring columns (blocks may overlap). In how many ways can exactly four of the 16 squares be shaded so that every one of the nine 2×2 blocks contains at least one shaded square?

(A) 64 **(B)** 72 **(C)** 79 **(D)** 81 **(E)** 256

19. The equation $x^4 = 8x + 7$ has exactly two real solutions (its other two solutions are not real). What is the product of the two real solutions?

(A) -7 **(B)** $-1 - 2\sqrt{2}$ **(C)** $-2\sqrt{2}$ **(D)** -2 **(E)** $1 - 2\sqrt{2}$

20. How many pairs of positive integers (a, b) with $a < b$ satisfy both $\gcd(a, b) = b - a$ and $\operatorname{lcm}(a, b) \leq 100$? Here gcd and lcm denote the greatest common divisor and the least common multiple.

(A) 87 **(B)** 88 **(C)** 90 **(D)** 94 **(E)** 96

21. The numbers $1, 2, \ldots, 9$ are placed in the cells of a 3×3 grid, one number per cell, with all 9! placements equally likely. What is the probability that some number in the grid is both the largest number in its row and the smallest number in its column?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{3}{5}$

22. A pyramid has a square base, and all eight of its edges have length 2. A ball inside the pyramid touches the base and all four triangular faces. A second, smaller ball sits above the first: it touches all four triangular faces and also touches the first ball. What is the radius of the smaller ball?

(A) $\dfrac{\sqrt{2} - 1}{4}$ **(B)** $\dfrac{3\sqrt{6} - 5\sqrt{2}}{2}$ **(C)** $\dfrac{\sqrt{3} - 1}{4}$ **(D)** $\dfrac{\sqrt{6} - \sqrt{2}}{4}$ **(E)** $2 - \sqrt{3}$

23. Call a positive integer n *sturdy* if every prime that divides n divides it at least twice. In other words, whenever a prime p divides n, so does p^2. For example, 1, 8 and $72 = 2^3 \cdot 3^2$ are sturdy, but $12 = 2^2 \cdot 3$ is not. How many sturdy numbers are there from 1 to 2026?

(A) 79 **(B)** 80 **(C)** 83 **(D)** 86 **(E)** 90

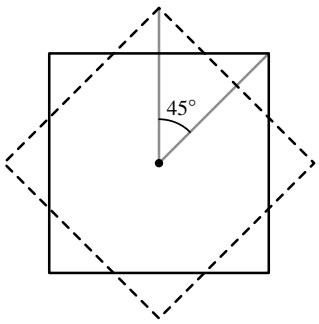
24. The nine points (x, y) with $x, y \in \{0, 1, 2\}$ form a 3×3 grid. A *quadrilateral* is drawn by choosing four of these points as vertices and joining them in a cycle by four straight sides. It must be a genuine quadrilateral: no two sides may cross or touch, except that consecutive sides share their common vertex, and no angle may be a straight angle (180°). Two quadrilaterals are different if their sets of sides are different (so the same four vertices, joined in a different order, can give different quadrilaterals). How many different quadrilaterals can be drawn?

(A) 70 **(B)** 78 **(C)** 86 **(D)** 94 **(E)** 102

25. A square of side 1 turns steadily about its centre through an angle of 45°, from its starting position to its final position. Some points of the plane are covered by the square at *some* moment during the turn,

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and some of those are covered at *every* moment of the turn (including the start and the end). What is the area of the set of points that are covered at some moment but not at every moment?



(A) $3-2\sqrt{2}$ **(B)** $1-\sqrt{2}+\frac{\pi}{4}$ **(C)** $\frac{\pi}{8}$ **(D)** $4-3\sqrt{2}+\frac{\pi}{4}$ **(E)** $3-2\sqrt{2}+\frac{\pi}{8}$

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Answer Key — AMC 10 Practice Contest 6

1. (D) 35

2. (C) \$3.80

3. (D) 32

4. (B) $\frac{3}{2} - \sqrt{2}$

5. (B) 48

6. (C) 5

7. (E) 16

8. (D) 11

9. (C) 3969

10. (B) 14

11. (D) $\frac{20}{27}$

12. (B) 112

13. (A) 80

14. (E) $4\sqrt{5}$

15. (A) $\frac{675}{4}$

16. (E) 675

17. (C) 45

18. (C) 79

19. (E) $1 - 2\sqrt{2}$

20. (A) 87

21. (A) $\frac{3}{10}$

22. (B) $\dfrac{3\sqrt{6} - 5\sqrt{2}}{2}$

23. (A) 79

24. (D) 94

25. (E) $3 - 2\sqrt{2} + \dfrac{\pi}{8}$

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Solutions — AMC 10 Practice Contest 6

1. Answer: (D) 35**. The idea.** Call the two roots $u = \sqrt{x+20}$ and $v = \sqrt{x-4}$. Their squares differ by a known amount, $u^2 - v^2 = (x+20) - (x-4) = 24$, no matter what x is. Since $u^2 - v^2 = (u+v)(u-v)$ and we know $u+v$, the difference $u-v$ comes for free, and then u and v themselves.

Work it out. From $(u+v)(u-v) = 24$ and $u+v = 12$ we get $u - v = 2$. Adding and subtracting, $u = 7$ and $v = 5$. The product is $uv = 35$.

Check. $u = 7$ gives $x + 20 = 49$, so $x = 29$, and then $\sqrt{29-4} = 5$ indeed; $7 + 5 = 12$. The tempting 29 is x itself, and $36 = 6 \cdot 6$ is what you get by assuming the two roots are equal.

2. Answer: (C) \$3.80**. The idea.** Two equations cannot determine three prices, but they don't have to: we only need one particular combination, $p + e + n$. Look for multiples of the two equations whose difference (or sum) is exactly one of each item.

Work it out. Write the purchases as $3p + 5e + n = 7.20$ and $4p + 7e + n = 8.90$. Take α times the first plus β times the second and ask for coefficients 1, 1, 1: $3\alpha + 4\beta = 1$, $5\alpha + 7\beta = 1$, $\alpha + \beta = 1$. The last gives $\beta = 1 - \alpha$, and then the first gives $3\alpha + 4 - 4\alpha = 1$, so $\alpha = 3$, $\beta = -2$; the middle equation checks: $15 - 14 = 1$. So $p + e + n = 3(7.20) - 2(8.90) = 21.60 - 17.80 = 3.80$ dollars.

Check. The individual prices really are not determined, yet the total is. For example $p = 0.70$, $e = 0.50$, $n = 2.60$ gives $2.10 + 2.50 + 2.60 = 7.20$ and $2.80 + 3.50 + 2.60 = 8.90$; and $p = 0.50$, $e = 0.60$, $n = 2.70$ gives $1.50 + 3.00 + 2.70 = 7.20$ and $2.00 + 4.20 + 2.70 = 8.90$. Both have $p + e + n = 3.80$. The option \$1.70 is just the difference of the two totals, which is the price of one pencil and two erasers.

3. Answer: (D) 32**. The idea.** The "first four" and the "last four" overlap in exactly one number, the middle (fourth) one. Adding the two partial sums counts every number once except the middle one, which is counted twice, so comparing with the full sum pins down the middle number. After that, making the largest number big means making the fifth and sixth numbers as small as the ordering allows.

Work it out. Call the numbers $a_1 < a_2 < \cdots < a_7$. The sums are $a_1 + \cdots + a_7 = 7 \cdot 15 = 105$, $a_1 + a_2 + a_3 + a_4 = 4 \cdot 10 = 40$ and $a_4 + a_5 + a_6 + a_7 = 4 \cdot 20 = 80$. Adding the last two gives $105 + a_4 = 120$, so $a_4 = 15$. Then $a_5 + a_6 + a_7 = 80 - 15 = 65$. Since $a_5 > 15$ and $a_6 > a_5$, the smallest they can be is $a_5 = 16$ and $a_6 = 17$, leaving $a_7 = 65 - 33 = 32$. This is achievable: for example 1, 10, 14, 15, 16, 17, 32 (the first three sum to 25 and are less than 15).

Check. $1 + 10 + 14 + 15 = 40$, $15 + 16 + 17 + 32 = 80$ and the total is 105, with means 10, 20 and 15. The option 35 ignores that the numbers are different (taking $a_5 = a_6 = 15$).

4. Answer: (B) $\frac{3}{2} - \sqrt{2}$**. The idea.** By symmetry the far vertex of the small square lies on the diagonal of the big square through that corner. So measure along the diagonal: from the corner to the centre is half the diagonal, the circle takes up the last 1 unit of that, and what is left is the diagonal of the small square.

Work it out. Put the corner at the origin and the centre at $(1, 1)$. The distance from the corner to the centre is $\sqrt{2}$, and the circle (radius 1) meets that segment at distance $\sqrt{2} - 1$ from the

corner. That is the small square's diagonal, so its side is $t = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{\sqrt{2}}{2}$ and its area is $t^2 = 1 - \sqrt{2} + \frac{1}{2} = \frac{3}{2} - \sqrt{2}$.

Check. Area is half the diagonal squared: $\frac{(\sqrt{2}-1)^2}{2} = \frac{3-2\sqrt{2}}{2} = \frac{3}{2} - \sqrt{2} \approx 0.086$. The choice $3 - 2\sqrt{2}$ forgets to halve (it is the diagonal squared), and $1 - \frac{\sqrt{2}}{2}$ is the side, not the area. The small square lies outside the circle, since its nearest point to the centre is the far vertex, which is on the circle.

5. Answer: (B) 48**. The idea.** The height is the missing piece, and the two slanted legs are far apart. Slide one leg parallel to itself until it meets the other: the two legs and the difference of the bases then form a single triangle, whose height is the trapezoid's height.

Work it out. Draw the line through C parallel to $\overline{AD}$, meeting $\overline{AB}$ at E. Then $AECD$ is a parallelogram, so $CE = AD = 6$ and $AE = CD = 5$, which leaves $EB = 15 - 5 = 10$. Triangle EBC has sides 6, 8 and 10, and since $6^2 + 8^2 = 10^2$ it has a right angle at C. Its height to the side $\overline{EB}$ is therefore $\frac{6\cdot 8}{10} = \frac{24}{5}$, and this is the distance between the parallel sides. The area is $\frac{15+5}{2} \cdot \frac{24}{5} = 10 \cdot \frac{24}{5} = 48$.

Check. Coordinates: $A = (0,0)$, $B = (15,0)$, $D = (\frac{18}{5}, \frac{24}{5})$, $C = (\frac{43}{5}, \frac{24}{5})$ give $CD = 5$, $AD = \sqrt{\frac{324+576}{25}} = 6$ and $BC = \sqrt{\frac{1024+576}{25}} = 8$, so the height $\frac{24}{5}$ is right (swapping which leg is 6 gives the mirror image, with the same area). The choices 60 and 80 treat a leg of 6 or 8 as if it were the height.

6. Answer: (C) 5**. The idea.** There is no need to plug in $x = \frac{-1\pm\sqrt{5}}{2}$. The equation says $x^2 = 1 - x$, which lets us trade any power of x for a smaller one; repeating this reduces x^6 to a linear expression $ax + b$, valid for both roots at once.

Work it out. $x^2 = 1 - x$. Then $x^3 = x \cdot x^2 = x - x^2 = x - (1 - x) = 2x - 1$. Next $x^4 = x \cdot x^3 = 2x^2 - x = 2(1 - x) - x = 2 - 3x$. Then $x^5 = x \cdot x^4 = 2x - 3x^2 = 2x - 3(1 - x) = 5x - 3$, and finally $x^6 = x \cdot x^5 = 5x^2 - 3x = 5(1 - x) - 3x = 5 - 8x$. So $x^6 + 8x = 5$ for either root.

Check. The coefficients $1, 2, 3, 5, 8$ that appear are Fibonacci numbers, a sign that the reduction is right. Numerically, with $x = \frac{\sqrt{5}-1}{2} \approx 0.618$: $x^6 \approx 0.0557$ and $8x \approx 4.9443$, total 5. With the other root $x \approx -1.618$: $x^6 \approx 17.944$ and $8x \approx -12.944$, total 5 again. The option -3 is the value of $x^5 - 5x$, one reduction too early.

7. Answer: (E) 16**. The idea.** The midline of a trapezoid passes through the midpoints of both diagonals. So its outer pieces are midsegments of triangles standing on the short base, and its middle piece is what is left. Write $CD = a$ and $AB = b$.

Work it out. Let M be the midpoint of leg $\overline{AD}$. The midline is parallel to the bases, so in $\triangle ACD$ it is the line through the midpoint M of $\overline{AD}$ parallel to $\overline{CD}$, which meets $\overline{AC}$ at its midpoint; the piece from M to that point is a midsegment of $\triangle ACD$, of length $\frac{a}{2}$. In the same way (using $\triangle BCD$) the piece at the other end, from the midpoint of $\overline{BD}$ to the midpoint of $\overline{BC}$, also has length $\frac{a}{2}$. The whole midline has length $\frac{a+b}{2}$, so the middle piece is $\frac{a+b}{2} - a = \frac{b-a}{2}$. Equal pieces means $\frac{a}{2} = \frac{b-a}{2}$, so $b = 2a$.

Now triangles ABP and CDP are similar (alternate angles, since $AB \parallel CD$) with ratio $b : a = 2 : 1$, so P divides the height h in the ratio $2 : 1$: $\triangle ABP$ has height $\frac{2h}{3}$. The trapezoid's area is $\frac{a+2a}{2}h = \frac{3ah}{2} = 36$, so $ah = 24$. Then $[ABP] = \frac{1}{2} \cdot 2a \cdot \frac{2h}{3} = \frac{2ah}{3} = 16$.

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Check. The four triangles cut out by the diagonals have areas $[CDP] = \frac{1}{2}a \cdot \frac{h}{3} = 4$, $[ABP] = 16$, and the two side triangles 8 each (their areas are always equal and each is the geometric mean $\sqrt{4 \cdot 16}$). Total $4 + 16 + 8 + 8 = 36$. The choice 4 is $[CDP]$, the triangle on the short base; 12 comes from splitting the height evenly, and 9 assumes the diagonals cut the trapezoid into four equal parts.

8. Answer: (D) 11**. The idea.** Once the 2×2 tiles are placed, the 1×1 tiles must fill every remaining cell, in exactly one way. So we are really counting the ways to place non-overlapping 2×2 tiles in the rectangle.

Work it out. Number the rows $1, 2, 3$ and the columns $1, \ldots, 4$. A 2×2 tile uses rows $\{1, 2\}$ or $\{2, 3\}$ and columns $\{c, c + 1\}$ with $c \in \{1, 2, 3\}$: 6 positions.

No big tiles: 1 way. *One big tile:* 6 ways. *Two big tiles:* if both use the same pair of rows, they must use columns $\{1, 2\}$ and $\{3, 4\}$: 1 way for each row pair, so 2. If they use different row pairs, they share row 2, so their columns must be disjoint: starting columns 1 and 3, in either order (1 on top and 3 below, or 3 on top and 1 below): 2 ways. So 4 ways with two big tiles. *Three or more:* impossible, because every big tile covers exactly two cells of the middle row, which has only 4 cells.

Total: $1 + 6 + 4 = 11$.

Check. A direct computer enumeration (fill the first empty cell with either a small tile or the top-left of a big tile) also gives 11. Forgetting the staggered pairs gives 9; forgetting two-tile placements altogether gives 7.

9. Answer: (C) 3969**. The idea.** A subset is pinned down by its smallest element s, its largest element L, and a free choice of which of the numbers strictly between them to include. So fix the two ends and count the middle.

Work it out. Given s and L with $L \geq 2s$, the numbers $s + 1, \ldots, L - 1$ may each be in or out: 2^{L-s-1} subsets. We need $2s \leq 12$, so $s \leq 6$. For fixed s, summing 2^{L-s-1} over $L = 2s, \ldots, 12$ is a geometric series: $2^{s-1} + 2^s + \cdots + 2^{11-s} = 2^{12-s} - 2^{s-1}$. Now sum over $s = 1, \ldots, 6$: $(2^{11} + 2^{10} + \cdots + 2^6) - (2^0 + 2^1 + \cdots + 2^5) = 4032 - 63 = 3969$.

Check. Small case: for $\{1, 2, 3, 4\}$ the formula gives $s = 1$: $2^3 - 1 = 7$, $s = 2$: $2^2 - 2 = 2$, total 9; listing them, the subsets with $L \geq 2s$ are the 7 containing 1 and at least one other element, plus $\{2, 4\}$ and $\{2, 3, 4\}$. Nine. (And $3969 = 63^2$, a pleasant coincidence.) The option 4032 forgets to subtract, 4095 counts every nonempty subset.

10. Answer: (B) 14**. The idea.** Work backwards from 2026. Undoing a press means halving (when the number is even) or moving one step up or down. Halving is what makes progress, so at an odd number we should step to whichever neighbour is a multiple of 4: that neighbour can then be halved twice in a row.

Work it out. Backwards: $2026 \to 1013$ (halve), $1013 \to 1012$ (the neighbour divisible by 4), $1012 \to 506 \to 253$, $253 \to 252$, $252 \to 126 \to 63$, $63 \to 64$, then $64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$. That is 11 halvings and 3 unit steps: 14 presses. Read forwards: $1 \to 2 \to \cdots \to 64$, subtract 1 to get 63, double twice to 252, add 1, double twice to 1012, add 1, double to 2026.

Why is this greedy backward route the shortest? Two exchange arguments. *At an even number n, halving first is never worse:* a backward route that takes unit steps before halving must reach

some even $n + 2j$ (at least $2|j|$ unit steps) and then halve to $\frac{n}{2} + j$; halving first and then taking $|j|$ unit steps reaches the same number in fewer or equal presses. *At an odd number $n > 3$, step to the neighbour divisible by 4:* if $n - 1$ is that neighbour, three presses (−1, halve, halve) reach $m = \frac{n-1}{4}$, and four reach m or $m + 1$; the other choice (+1, halve, then an odd number forces a unit step, halve) needs four presses to reach m or $m + 1$ at all. The case where $n + 1$ is the multiple of 4 is symmetric. So the backward greedy route is optimal, and it takes 14 presses.

Check. Without the subtract button the answer would be 17: $2026 = 11111101010_2$ has 11 binary digits and eight 1s, so 10 doublings and 7 additions. The subtract button replaces the long run of 1s by a single borrow.

11. Answer: (D) $\frac{20}{27}$**. The idea.** There are $3^4 = 81$ equally likely ballots. It is easier to count the ballots with *no* clear winner. The four votes can be spread as 3+1, 2+1+1, 2+2 or 1+1+1+1, and only the last two are ties at the top.

Work it out. *Everyone gets one vote.* Then the votes form a rearrangement of the four friends with nobody voting for themselves: a derangement of 4 objects, of which there are 9.

Two friends get two votes each. Choose the two leaders A and B in $\binom{4}{2} = 6$ ways. The other two get no votes, so all four votes go to A or B. A cannot vote for A, so A votes B; likewise B votes A. Now each leader has one vote and needs exactly one more from the remaining two friends, who must split: 2 ways. That gives $6 \cdot 2 = 12$ ballots.

So $9 + 12 = 21$ ballots have no clear winner, and the probability of a clear winner is $\frac{81-21}{81} = \frac{60}{81} = \frac{20}{27}$.

Check. Count directly. A 3+1 split: the winner gets all three other votes (4 choices of winner) and votes for anyone (3 ways): 12. A 2+1+1+0 split: the remaining $81 - 12 - 12 - 9 = 48$. Clear winners: $12 + 48 = 60$. Forgetting the 2+2 ties gives $\frac{72}{81} = \frac{8}{9}$; $\frac{4}{27}$ is the chance of a three-vote winner only.

12. Answer: (B) 112**. The idea.** Think of k people standing at position k on a number line (for $k = 1, \ldots, 10$), and $f(x)$ as the total distance everyone walks to meet at x. Moving x to the right by a little brings everyone to the right of x closer and everyone to the left farther away, so f decreases as long as more people are to the right of x than to the left. The minimum is at the (weighted) median person.

Work it out. There are $1 + 2 + \cdots + 10 = 55$ people. The slope of f just to the right of x equals (people at or left of x) minus (people right of x). At positions 1 through 6 there are $1+2+\cdots+6 = 21$ people, fewer than half of 55; through 7 there are 28, more than half. So f decreases until $x = 7$ (slope $21 - 34 = -13$ just left of 7) and increases after (slope $28 - 27 = 1$ just right of 7). The minimum is $f(7) = 1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1 + 7 \cdot 0 + 8 \cdot 1 + 9 \cdot 2 + 10 \cdot 3 = 56 + 56 = 112$.

Check. The neighbours are larger: $f(8) = 7 + 12 + 15 + 16 + 15 + 12 + 7 + 0 + 9 + 20 = 113$ and $f(6) = 5 + 8 + 9 + 8 + 5 + 0 + 7 + 16 + 27 + 40 = 125$. The trap 113 is $f(8)$, from misplacing the weighted median by one; 125 is $f(6)$.

13. Answer: (A) 80**. The idea.** Adding sixteen floors one by one is slow. Pair k with $17 - k$ instead: the two fractions add up to a whole number, and that pins down the sum of their floors.

Work it out. For $1 \le k \le 16$, $\dfrac{11k}{17} + \dfrac{11(17-k)}{17} = 11$. Neither fraction is a whole number, because 17 is prime and divides neither 11 nor k. If two non-integers add up to the integer 11,

their fractional parts add up to exactly 1. So their floors add up to $11 - 1 = 10$.

The sixteen values of k form the eight pairs $\{1, 16\}, \{2, 15\}, \ldots, \{8, 9\}$. The total is $8 \cdot 10 = 80$.

Check. Without the floors, the sum would be $\frac{11}{17}(1 + 2 + \cdots + 16) = \frac{11 \cdot 136}{17} = 88$. Each floor removes a fractional part, and the sixteen fractional parts form eight pairs that each sum to 1. That removes 8, and $88 - 8 = 80$. The first few floors are $0, 1, 1, 2, 3, 3, 4, 5$, which sum to 19. The last eight are $5, 6, 7, 7, 8, 9, 9, 10$, which sum to 61. Together they give 80.

14. Answer: (E) $4\sqrt{5}$**. The idea.** By symmetry, the incircle touches BC at its midpoint M, and A, I, M lie on the axis of symmetry. So the height is $AM = AI + IM = 3 + 2 = 5$. The tangent point T on AB gives a small right triangle ATI that is similar to the big right triangle AMB.

Work it out. In right triangle ATI (right angle at T, since a radius is perpendicular to the tangent), $IT = 2$ and $AI = 3$, so $AT = \sqrt{9 - 4} = \sqrt{5}$.

Triangles ATI and AMB share the angle at A and both have a right angle (at T and at M). So $\dfrac{MB}{AM} = \dfrac{TI}{AT} = \dfrac{2}{\sqrt{5}}$, which gives $MB = \dfrac{5 \cdot 2}{\sqrt{5}} = 2\sqrt{5}$.

Hence $BC = 2 \cdot MB = 4\sqrt{5}$.

Check. $AB = \sqrt{AM^2 + MB^2} = \sqrt{25 + 20} = 3\sqrt{5}$. The area is $\frac{1}{2} \cdot 4\sqrt{5} \cdot 5 = 10\sqrt{5}$, and the semiperimeter is $\frac{1}{2}(3\sqrt{5} + 3\sqrt{5} + 4\sqrt{5}) = 5\sqrt{5}$. So the inradius is $10\sqrt{5}/5\sqrt{5} = 2$, as required.

15. Answer: (A) $\frac{675}{4}$**. The idea.** A tangent is perpendicular to the radius at its point of contact. For the circle $x^2 + y^2 = 25$, the tangent at (a, b) is therefore the line $ax + by = 25$. Its normal direction is (a, b), and the point (a, b) satisfies it because $a^2 + b^2 = 25$.

Work it out. The three tangents are $3x + 4y = 25$, $\quad -3x + 4y = 25$, $\quad y = -5$ (the last one is $0 \cdot x - 5y = 25$). The first two lines meet where $x = 0$, at $y = 25/4$. The first line meets $y = -5$ where $3x = 45$, so at $(15, -5)$. By symmetry the second line meets it at $(-15, -5)$.

So the triangle has base 30 on the line $y = -5$ and apex $(0, 25/4)$. Its height is $25/4 + 5 = 45/4$, and its area is $\frac{1}{2} \cdot 30 \cdot \frac{45}{4} = \frac{675}{4}$.

Check. The circle is the incircle of this triangle, so the area equals $r \cdot s$ with $r = 5$. The apex is $\sqrt{15^2 + (45/4)^2} = 75/4$ from each base vertex, so $s = \frac{1}{2}\left(30 + \frac{75}{2}\right) = \frac{135}{4}$. Then $rs = 5 \cdot \frac{135}{4} = \frac{675}{4}$.

16. Answer: (E) 675**. The idea.** Use the closed form $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$. Dividing by n leaves $\dfrac{(n+1)(2n+1)}{6}$, so the only question is when this is a whole number.

Work it out. We need $6 \mid (n+1)(2n+1)$.

Factor 2. $2n + 1$ is odd, so $n + 1$ must be even, that is, n is odd.

Factor 3. Check n mod 3. If $n \equiv 0$, then $(n+1)(2n+1) \equiv 1 \cdot 1$, which is not divisible by 3. If $n \equiv 1$, then $2n + 1 \equiv 0$. If $n \equiv 2$, then $n + 1 \equiv 0$. So n must not be a multiple of 3.

Together, n is odd and not divisible by 3, so $n \equiv 1$ or $5 \pmod 6$. Since $2026 = 6 \cdot 337 + 4$, the 337 complete blocks of six give $2 \cdot 337 = 674$ values. The leftover numbers $2023, \ldots, 2026$ are $\equiv 1, 2, 3, 4$, and they add 2023 only. The total is 675.

Check. For $n = 5$ the sum is $55 = 5 \cdot 11$, which works. For $n = 3$ the sum is 14, which fails. For $n = 7$ the sum is $140 = 7 \cdot 20$, which works. For $n = 9$ the sum is 285, and $285/9$ is not whole, which fails as predicted.

17. Answer: (C) 45**. The idea.** The rule looks awkward, but its reciprocal is simple: $\dfrac{1}{a_{n+1}} = \dfrac{1 + na_n}{a_n} = \dfrac{1}{a_n} + n$. So the reciprocals just grow by 1, then 2, then 3, and so on.

Work it out. Starting from $\frac{1}{a_1} = 1$, we get $\dfrac{1}{a_n} = 1 + (1 + 2 + \cdots + (n-1)) = 1 + \dfrac{n(n-1)}{2}$. The terms decrease, and $a_n > \frac{1}{1000}$ exactly when $1 + \frac{n(n-1)}{2} < 1000$, that is, $n(n-1) < 1998$. Since $45 \cdot 44 = 1980 < 1998$ but $46 \cdot 45 = 2070 > 1998$, this holds for $n = 1, 2, \ldots, 45$: 45 values.

Check. The formula gives $a_3 = \frac{1}{1+3} = \frac{1}{4}$, as in the statement, and $a_{45} = \frac{1}{991}$, $a_{46} = \frac{1}{1036}$, on either side of $\frac{1}{1000}$.

18. Answer: (C) 79**. The idea.** The four corner blocks do not overlap, so each of them holds exactly one of the four shaded squares. Record, for each corner block, whether its shaded square lies in the block's column nearer the middle of the grid, and whether it lies in the row nearer the middle; the other five blocks then turn into simple yes/no conditions.

Work it out. Call the corner blocks TL, TR, BL, BR, and for each let $h = 1$ if its shaded square is in the inner column (column 2 or 3) and $v = 1$ if it is in the inner row; the pair (h, v) pins down the square, so there are $4^4 = 256$ candidates. The top-middle block (rows 1–2, columns 2–3) is hit exactly when $h_{TL} = 1$ or $h_{TR} = 1$; likewise the bottom-middle block needs h_{BL} or h_{BR}, the left-middle block v_{TL} or v_{BL}, and the right-middle block v_{TR} or v_{BR}. These leave $3 \cdot 3 = 9$ choices for the h's and 9 for the v's, so 81 shadings. The centre block needs some corner with $h = v = 1$. If no corner has this, then $h_{TL} = 1$ forces $v_{TL} = 0$, so $v_{BL} = 1$, so $h_{BL} = 0$, so $h_{BR} = 1$, so $v_{BR} = 0$, so $v_{TR} = 1$, so $h_{TR} = 0$: one shading; and $h_{TL} = 0$ forces $h_{TR} = 1$ and, around the same cycle, one more. Removing these 2 leaves $81 - 2 = 79$.

Check. The two excluded shadings are the pinwheels such as the squares in positions (row, column) $(1, 2)$, $(2, 4)$, $(3, 1)$, $(4, 3)$: each of the eight outer blocks is hit but the centre block is empty. Shading the four central squares is allowed, and three squares can never suffice because the corner blocks are disjoint.

19. Answer: (E) $1 - 2\sqrt{2}$**. The idea.** Vieta's formulas give the product of *all four* roots, -7, but not the product of the two real ones. So we need to split the quartic into two quadratics. The trick is to add the same expression to both sides so that *both* sides become perfect squares.

Work it out. Add $2tx^2 + t^2$ to both sides for a number t still to be chosen: $(x^2 + t)^2 = 2tx^2 + 8x + (t^2 + 7)$. The right side is a perfect square in x when its discriminant vanishes: $64 = 4 \cdot 2t \cdot (t^2 + 7)$, that is, $t(t^2 + 7) = 8$, and $t = 1$ works. With $t = 1$, $(x^2 + 1)^2 = 2x^2 + 8x + 8 = 2(x + 2)^2$, so $x^2 + 1 = \sqrt{2}\,(x + 2)$ or $x^2 + 1 = -\sqrt{2}\,(x + 2)$.

The first equation is $x^2 - \sqrt{2}\,x + (1 - 2\sqrt{2}) = 0$. Its discriminant is $2 - 4(1 - 2\sqrt{2}) = 8\sqrt{2} - 2 > 0$, so it has two real roots, with product $1 - 2\sqrt{2}$.

The second equation is $x^2 + \sqrt{2}\,x + (1 + 2\sqrt{2}) = 0$. Its discriminant is $2 - 4(1 + 2\sqrt{2}) < 0$, so it has no real roots.

So the product of the real solutions is $1 - 2\sqrt{2}$.

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Check. The product of the two quadratics is $(x^2+1)^2-2(x+2)^2 = x^4-8x-7$, as it should be. Their constant terms multiply to $(1-2\sqrt{2})(1+2\sqrt{2}) = -7$, which is Vieta's product of all four roots. Numerically the real roots are about 2.233 and -0.819, with product about $-1.828 = 1-2\sqrt{2}$.

20. Answer: (A) 87**. The idea.** Let $d = b - a$. The condition says d divides a, so $a = kd$ and $b = (k+1)d$ for a positive integer k. Conversely every such pair works, since k and $k+1$ share no factor: $\gcd(kd, (k+1)d) = d$. And then $\operatorname{lcm}(a,b) = \frac{ab}{d} = k(k+1)d$.

Work it out. We count pairs (k, d) of positive integers with $k(k+1)d \le 100$. For each k there are $\left\lfloor \frac{100}{k(k+1)} \right\rfloor$ choices of d: $k = 1$: 50; $k = 2$: 16; $k = 3$: 8; $k = 4$: 5; $k = 5$: 3; $k = 6$: 2; $k = 7, 8, 9$: 1 each; and none for $k \ge 10$, since $10 \cdot 11 > 100$. Total $50 + 16 + 8 + 5 + 3 + 2 + 1 + 1 + 1 = 87$.

Check. Extreme cases: $(50, 100)$ has gcd $= 50 = 100 - 50$ and lcm $= 100$; $(9, 10)$ has gcd $= 1$ and lcm $= 90$. A non-example: $(4, 6)$ has $\gcd 2 = 6 - 4$ and lcm 12, which is $k = 2$, $d = 2$ and is counted.

21. Answer: (A) $\frac{3}{10}$**. The idea.** Two observations make this easy. First, *at most one* cell can have the property, so the nine events (one per cell) cannot overlap and their probabilities simply add. Second, whether a given cell has the property depends only on the five numbers in its row and column.

Work it out. *At most one.* Suppose x (row i, column j) and y (row k, column l) both have the property. They cannot share a row, because both would be the row maximum. They cannot share a column either, because both would be the column minimum. So $i \ne k$ and $j \ne l$. Let z be the number in row i, column l. Then $x \ge z$ (x is the largest in row i) and $z \ge y$ (y is the smallest in column l). So $x > y$, since the numbers are distinct. The same argument with the roles swapped gives $y > x$, a contradiction.

One cell. Fix a cell. Its row and column together contain 5 cells, and the 5 numbers in them are placed in those cells uniformly at random. The cell has the property exactly when the two smaller of the other numbers are its row-mates and the two larger are its column-mates. So the cell holds the middle number of the five, its row-mates hold the two smallest, and its column-mates hold the two largest. That is $1 \cdot 2! \cdot 2! = 4$ of the $5! = 120$ placements, a probability of $\frac{1}{30}$.

Adding over the 9 disjoint events: $9 \cdot \frac{1}{30} = \frac{3}{10}$.

Check. Test the method on a single row of 3 cells. Each column has just one cell, so a cell has the property exactly when it holds the row maximum, and some cell always does. The method agrees: each cell holds the largest of its 3 row numbers with probability $\frac{1}{3}$, the 3 events are disjoint, and $3 \cdot \frac{1}{3} = 1$. In general the answer for an $m \times n$ grid is $\dfrac{m!\,n!}{(m+n-1)!}$, which is $\dfrac{3!\,3!}{5!} = \dfrac{3}{10}$ here.

22. Answer: (B) $\dfrac{3\sqrt{6}-5\sqrt{2}}{2}$**. The idea.** Everything happens in the vertical cross-section through the apex V and the midpoints of two opposite base edges. By symmetry both balls are centred on the pyramid's axis. A ball touches all four slanted faces exactly when its circle in this cross-section touches the two slanted sides of the triangle, and the base cross-section behaves the same way. The two balls are then related by a scaling about V.

Work it out. *The cross-section.* Each triangular face is equilateral with side 2, so its height (the slant height) is $\sqrt{3}$. The cross-section is an isosceles triangle with base 2 and equal sides $\sqrt{3}$. Its height is $\sqrt{3-1} = \sqrt{2}$, which is the pyramid's height.

First ball. Its radius is the inradius of this triangle: area/semiperimeter $= \dfrac{\frac{1}{2}\cdot 2\cdot\sqrt{2}}{(2+2\sqrt{3})/2} = \dfrac{\sqrt{2}}{1+\sqrt{3}} = \dfrac{\sqrt{6}-\sqrt{2}}{2} = r_1$.

Distance to the apex. Let α be the angle between the axis and a slanted side at V. Then $\sin\alpha = \frac{1}{\sqrt{3}}$ (opposite side 1, hypotenuse $\sqrt{3}$). A circle with centre on the axis at distance d from V that touches both slanted sides has radius $d\sin\alpha = d/\sqrt{3}$. So the first centre is at $d_1 = \sqrt{3}\,r_1$.

Second ball. Its centre is at $d_2 = \sqrt{3}\,r_2$, and tangency of the balls gives $d_1 - d_2 = r_1 + r_2$. So $\sqrt{3}r_1 - \sqrt{3}r_2 = r_1 + r_2$, and $r_2 = r_1\cdot\dfrac{\sqrt{3}-1}{\sqrt{3}+1} = r_1(2-\sqrt{3}) = \dfrac{(\sqrt{6}-\sqrt{2})(2-\sqrt{3})}{2} = \dfrac{3\sqrt{6}-5\sqrt{2}}{2}$.

Check. Numerically $r_1 \approx 0.518$ and $d_1 = \sqrt{3}r_1 \approx 0.897$. Adding r_1 puts the bottom of the first ball 1.414 below V, which is exactly on the base ($\sqrt{2} \approx 1.414$). Then $r_2 \approx 0.518\cdot 0.268 \approx 0.139$. The top of the second ball is at distance $d_2 - r_2 = (\sqrt{3}-1)r_2 \approx 0.10$ from V, which is still inside the pyramid.

23. Answer: (A) 79**. The idea.** Every sturdy number can be written in exactly one way as $n = a^2b^3$, where a is any positive integer and b is *squarefree* (a product of distinct primes). Look at each prime: an exponent $e \geq 2$ is either even ($e = 2k$, put p^k into a) or odd and at least 3 ($e = 2k+3$, put p^k into a and one p into b). So b is exactly the product of the primes with odd exponent, and it is determined by n. Conversely every a^2b^3 is sturdy. So we count pairs (a, b) instead of numbers.

Work it out. For a fixed squarefree b, the number of a with $a^2b^3 \leq 2026$ is $\left\lfloor\sqrt{2026/b^3}\right\rfloor$. We need $b^3 \leq 2026$, so $b \leq 12$, and the squarefree choices are $b = 1, 2, 3, 5, 6, 7, 10, 11$ (4, 8, 9 and 12 are not squarefree):

$b = 1$: $\lfloor\sqrt{2026}\rfloor = 45$. $b = 2$: $\lfloor\sqrt{253.25}\rfloor = 15$. $b = 3$: $\lfloor\sqrt{75.03\ldots}\rfloor = 8$. $b = 5$: $\lfloor\sqrt{16.2\ldots}\rfloor = 4$.

$b = 6$: $\lfloor\sqrt{9.37\ldots}\rfloor = 3$. $b = 7$: $\lfloor\sqrt{5.90\ldots}\rfloor = 2$. $b = 10$: $\lfloor\sqrt{2.026}\rfloor = 1$. $b = 11$: $\lfloor\sqrt{1.52\ldots}\rfloor = 1$.

Total: $45 + 15 + 8 + 4 + 3 + 2 + 1 + 1 = 79$.

Check. Up to 100 the method gives 10 ($b = 1$) + 3 ($b = 2$: $8, 32, 72$) + 1 ($b = 3$: 27) = 14. Listing them directly: $1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 72, 81, 100$, which is 14. Letting b also be 4, 8 or 9 would count numbers such as $64 = 1^2\cdot 4^3 = 8^2\cdot 1^3$ twice. That mistake gives 86.

24. Answer: (D) 94**. The idea.** Four points can be joined in a cycle in 3 ways (choose which point is opposite a fixed one). Everything depends on the shape of the four points:

(a) If three of them are collinear, no cycle works. Either the middle point is a vertex with a 180° angle, or two sides overlap, or a side runs through the middle point.

(b) If they are in convex position, exactly one cycle works (the other two have crossing sides).

(c) If one point lies inside the triangle of the other three, all three cycles work. Each one is the triangle with one side pushed in to the inner point, and these are three different quadrilaterals.

So the answer is (number of convex 4-sets) + 3× (number of 4-sets with a point inside the triangle of the others).

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Work it out. *Collinear triples.* The grid has 8 lines of three points (3 rows, 3 columns, 2 diagonals). A 4-set containing one of them is that line plus one of the other 6 points: $8 \cdot 6 = 48$ sets. None is counted twice, because two different lines of three share at most one point and so need 5 points. That leaves $\binom{9}{4} - 48 = 126 - 48 = 78$ sets with no three collinear.

Which have a point inside? Every point except the centre $O = (1, 1)$ lies on the edge of the big square, so it cannot be strictly inside a triangle of grid points. So we need triangles of the 8 outer points that contain O strictly inside, where no vertex, O and another vertex are collinear.

Seen from O, the 8 outer points lie at directions $0°, 45°, \ldots, 315°$, one every $45°$. Number them $0, \ldots, 7$ around. Three of them surround O strictly exactly when every gap between consecutive chosen points is less than $180°$, that is, at most 3 steps. A gap of exactly 4 steps is an opposite pair, collinear with O, which is also not allowed. Three gaps of at most 3 that sum to 8 must be $3, 3, 2$ in some order. There are 8 such triangles: the gap of 2 can start at any of the 8 points, and the rest is then forced. The outer three are never collinear, since their gaps are not $1, 1, 6$. So 8 sets are of type (c).

Total. $78 - 8 = 70$ convex sets give 70 quadrilaterals, and the 8 other sets give 3 each. The answer is $70 + 24 = 94$.

Check. Example of type (c): $(2, 1), (0, 2), (1, 0)$ around the centre $(1, 1)$. The cycle order $(2, 1) \to (1, 1) \to (0, 2) \to (1, 0)$ gives an arrowhead, and so do the two other orders. Example of type (b): the four corners give only the square itself, because the other two orders produce crossing "bow-ties".

25. Answer: (E) $3 - 2\sqrt{2} + \dfrac{\pi}{8}$**. The idea.** Look from the centre O. In direction φ, a square covers the segment from O out to its boundary, at distance ρ (the square's "radius" in that direction). This distance is smallest, $\frac{1}{2}$, towards an edge midpoint, and largest, $\frac{\sqrt{2}}{2}$, towards a corner. It grows as the direction moves from an edge midpoint towards a corner. During the turn, the direction φ meets the turning square at every relative angle in a $45°$ window. So the *swept* set reaches the *largest* radius in the window, and the *always-covered* set only the *smallest*. By symmetry we can work in one $90°$ quarter and multiply by 4. Measure angles from O, with the starting square's edges at directions $0°$ and $90°$ (its corners at $45°$). The square turns counterclockwise through $45°$, so its final corners are at $0°, 90°, \ldots$ and its final edge midpoints at $45°, 135°, \ldots$.

Swept region (one quarter, $0°$ to $90°$). A corner passes through direction φ during the turn exactly for $45° \le \varphi \le 90°$. That gives a circular sector of radius $\frac{\sqrt{2}}{2}$ and angle $45°$, with area $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{16}$. For $0° \le \varphi \le 45°$ no corner passes, and the farthest reach is at the start or at the end of the turn. The starting square wins for $22.5° \le \varphi \le 45°$ and the final square wins for $0° \le \varphi \le 22.5°$. The starting piece is the triangle between the rays at $22.5°$ and $45°$ and the edge $x = \frac{1}{2}$. Its side on that edge has length $\frac{1}{2}(\tan 45° - \tan 22.5°) = \frac{1}{2}(2 - \sqrt{2})$, and its height from O is $\frac{1}{2}$, so its area is $\frac{2-\sqrt{2}}{8}$. The final piece is congruent to it. Quarter total: $\frac{\pi}{16} + \frac{2-\sqrt{2}}{4}$, so the swept area is $\frac{\pi}{4} + 2 - \sqrt{2}$.

Always-covered region (same quarter). An edge midpoint (radius $\frac{1}{2}$) passes through direction φ exactly for $0° \le \varphi \le 45°$. At those moments only the disc of radius $\frac{1}{2}$ is covered in that direction, and it is covered at all other moments too. That gives a sector of radius $\frac{1}{2}$ and angle $45°$, with area $\frac{\pi}{32}$. For $45° \le \varphi \le 90°$ the smallest reach occurs at the start or at the end. The final square's edge (whose midpoint is at $45°$) is nearer for $45° \le \varphi \le 67.5°$, and the starting square's top edge is nearer for $67.5° \le \varphi \le 90°$. Each piece is a right triangle with legs $\frac{1}{2}$ and $\frac{1}{2}\tan 22.5° = \frac{1}{2}(\sqrt{2}-1)$, so area $\frac{\sqrt{2}-1}{8}$. Quarter total: $\frac{\pi}{32} + \frac{\sqrt{2}-1}{4}$, so the always-covered area is $\frac{\pi}{8} + \sqrt{2} - 1$.

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Work it out. The always-covered set lies inside the swept set, so the answer is $\left(2 - \sqrt{2} + \frac{\pi}{4}\right) - \left(\sqrt{2} - 1 + \frac{\pi}{8}\right) = 3 - 2\sqrt{2} + \frac{\pi}{8} \approx 0.564$.

Check. The always-covered set is *not* the overlap of the starting and final squares, which is a regular octagon of area $2(\sqrt{2} - 1) \approx 0.828$. Halfway through the turn, the square's edge midpoints point at $22.5°, 112.5°, \ldots$, where the octagon's corners are, and they cut those corners off. Consistently, $\frac{\pi}{8} + \sqrt{2} - 1 \approx 0.807$, which lies between the disc of radius $\frac{1}{2}$ (≈ 0.785) and the octagon. Similarly, the swept area ≈ 1.371 lies between 1 and the disc of radius $\frac{\sqrt{2}}{2}$ (≈ 1.571).

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