MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

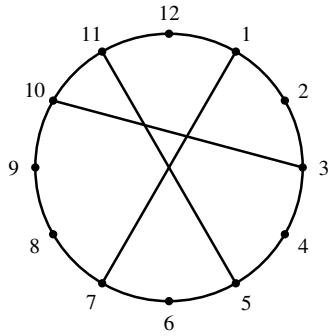
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A jar holds only red and blue marbles. If 4 red marbles were added to the jar, exactly half of the marbles in the jar would be red. If instead 4 red marbles were removed from the jar, exactly one third of the marbles in the jar would be red. Starting from the original jar, how many red marbles must be added so that exactly $\frac{3}{5}$ of the marbles in the jar are red?

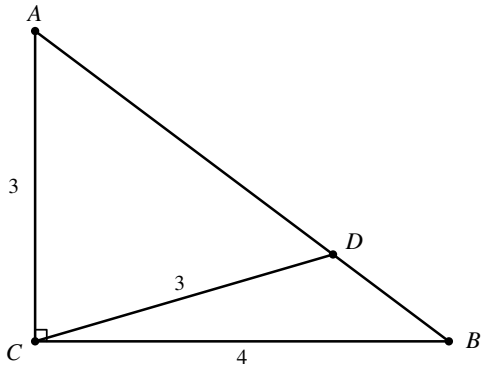
(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

2. The twelve hour marks of a clock face are equally spaced around a circle. Three straight chords are drawn: from the 1 o'clock mark to the 7 o'clock mark, from 3 to 10, and from 5 to 11. Each pair of these chords crosses inside the circle, and the three crossing points are the vertices of a small triangle. What is the measure, in degrees, of the smallest angle of this triangle?



(A) 30 **(B)** 45 **(C)** $\frac{105}{2}$ **(D)** 60 **(E)** 75

3. Right triangle ABC has its right angle at C, with $AC = 3$ and $BC = 4$. Point D is on the hypotenuse $\overline{AB}$, $D \neq A$, and $CD = 3$. What is the length BD?



(A) 1 **(B)** $\frac{7}{5}$ **(C)** $\frac{8}{5}$ **(D)** $\frac{9}{5}$ **(E)** 2

4. A sequence starts with $a_1 = 2$ and continues by the rule $a_{n+1} = \dfrac{1 + a_n}{1 - a_n}$ for $n \geq 1$. (No term turns out to equal 1, so every term is defined.) What is the product $a_1 \cdot a_2 \cdot a_3 \cdots a_{2026}$ of the first 2026 terms?

(A) -6 **(B)** -3 **(C)** 1 **(D)** 2 **(E)** 3

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5. The year 2024 has a curious property: it is divisible by 20 plus the number formed by its last two digits, since $20 + 24 = 44$ and $2024 = 44 \cdot 46$. For how many years from 2001 to 2099, inclusive, is the year divisible by 20 plus the number formed by its last two digits? (For example, for 2007 that number is $20 + 7 = 27$.)

(A) 8 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 14

6. A lock opens with a 4-digit code. Each digit is one of $0, 1, 2, \ldots, 9$, and a code may begin with 0. For this lock the four digits of the code add up to 9, and no digit of the code is larger than 4. How many codes are possible?

(A) 56 **(B)** 80 **(C)** 115 **(D)** 140 **(E)** 220

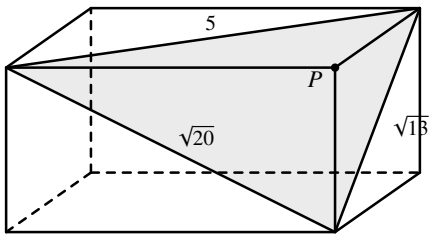
7. Find the sum of all real numbers x that satisfy $\big|\,\big||x| - 4\big| - 2\big| = \dfrac{x}{2}$.

(A) $\frac{40}{3}$ **(B)** 16 **(C)** $\frac{52}{3}$ **(D)** 20 **(E)** $\frac{64}{3}$

8. A basketball team scores only 2-point and 3-point baskets, and it ends a game with exactly 20 points. After each basket the scoreboard shows the team's running total. In how many different orders could the team's baskets have been scored if its scoreboard never showed exactly 10 points? (An order is a sequence of 2s and 3s adding to 20; for example 3, 3, 2, 2, 2, 3, 3, 2 is allowed.)

(A) 49 **(B)** 64 **(C)** 65 **(D)** 72 **(E)** 114

9. A closed rectangular box has face diagonals of lengths $\sqrt{13}$, $\sqrt{20}$ and 5. Pick one corner P of the box. A flat saw cut passes through the three corners of the box that share an edge with P, slicing off a small pyramid containing P. How far is P from the plane of the cut?



(A) $\frac{8}{\sqrt{61}}$ **(B)** $\frac{12}{\sqrt{61}}$ **(C)** $\frac{12}{\sqrt{29}}$ **(D)** $\frac{\sqrt{29}}{2}$ **(E)** $\frac{24}{\sqrt{61}}$

10. A pile starts with n coins. Two players take turns; on each turn a player removes either exactly 1 coin or a prime number of coins (never more coins than remain). The player who takes the last coin wins. For how many values of n with $1 \le n \le 2026$ can the second player guarantee a win, no matter how the first player plays?

(A) 506 **(B)** 507 **(C)** 675 **(D)** 1013 **(E)** 1520

11. Some fractions have decimal expansions that repeat right from the decimal point: for example $\frac{1}{7}$ =

$0.\overline{142857}$, with a repeating block of 6 digits, and $\frac{1}{3} = 0.\overline{3}$, with a block of 1 digit. Others do not: $\frac{1}{6} = 0.1\overline{6}$ starts repeating only after one digit, and terminating decimals such as $\frac{1}{2} = 0.5$ do not count. For how many integers n with $2 \le n \le 50$ does $\frac{1}{n}$ repeat right from the decimal point with a repeating block of at most 6 digits?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

12. A positive integer N has exactly 8 positive divisors, and when they are listed in increasing order, the 4th and 5th divisors are consecutive integers. For example, $N = 30$ has this property: its divisors are $1, 2, 3, 5, 6, 10, 15, 30$. How many positive integers N less than 1000 have this property?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

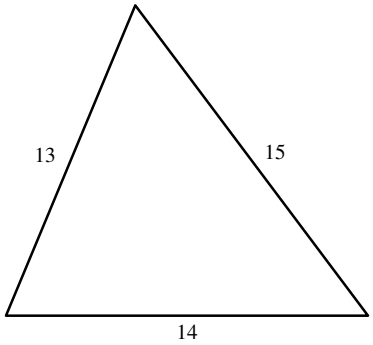
13. There is exactly one positive integer m for which the sum of the squares of the five consecutive integers $m, m+1, m+2, m+3, m+4$ equals the sum of the squares of the next four consecutive integers $m+5, m+6, m+7, m+8$. What is m?

(A) 32 **(B)** 36 **(C)** 40 **(D)** 41 **(E)** 44

14. How many 3-element subsets of $\{1, 2, 3, \ldots, 21\}$ contain one element that is the average of the other two? (For example, $\{4, 9, 14\}$ is such a subset, since 9 is the average of 4 and 14.)

(A) 100 **(B)** 110 **(C)** 121 **(D)** 200 **(E)** 210

15. A point is chosen uniformly at random from the interior of a triangle whose sides have lengths 13, 14 and 15. What is the probability that the point is at a distance of at least 1 from each of the three sides of the triangle?



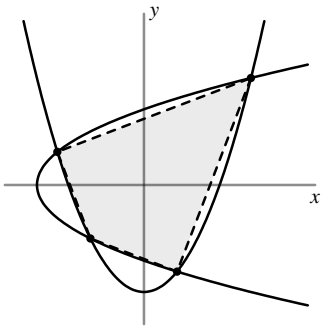
(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{25}{36}$ **(E)** $\frac{3}{4}$

16. A straight line crosses the graph of $y = x^3 - 3x^2 + 5x$ at three distinct points A, B and C, and B is the midpoint of segment $\overline{AC}$. The length of $\overline{AC}$ is $2\sqrt{10}$. What is the slope of the line?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 12

17. The parabolas $y = x^2 - 2$ and $x = y^2 - 2$ intersect in exactly four points. What is the area of the convex quadrilateral whose vertices are these four points?

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(A) $2\sqrt{5}$ **(B)** 6 **(C)** $3\sqrt{5}$ **(D)** $5\sqrt{2}$ **(E)** $3\sqrt{10}$

18. A post office sells only 5-cent stamps and 7-cent stamps. A postage amount can be made up in several ways; for example, 35 cents can be made as seven 5-cent stamps or as five 7-cent stamps, and in no other way. For how many whole-number amounts from 1 cent to 100 cents, inclusive, can the postage be made up in exactly two ways? (Two ways are different if they use different numbers of 5-cent stamps.)

(A) 12 **(B)** 19 **(C)** 23 **(D)** 34 **(E)** 35

19. Call a set of vertices of a cube *scattered* if no two vertices in the set are the two endpoints of one edge of the cube. How many scattered sets are there, counting the empty set? (The cube has 8 vertices and 12 edges.)

(A) 29 **(B)** 33 **(C)** 35 **(D)** 37 **(E)** 41

20. For how many integers n with $1 \le n \le 2026$ does the base-4 representation of n end in exactly as many zeros as the base-6 representation of n? (For example, $n = 12$ is 30_4 and 20_6, one zero each, so 12 counts; $n = 8 = 20_4 = 12_6$ does not.)

(A) 1013 **(B)** 1351 **(C)** 1435 **(D)** 1444 **(E)** 1480

21. How many nonempty subsets of $\{1, 2, 3, \ldots, 100\}$ have an average (arithmetic mean) of exactly 4?

(A) 15 **(B)** 16 **(C)** 30 **(D)** 31 **(E)** 32

22. How many ordered triples (a, b, c) of positive integers satisfy $a + b + c = 30$ and $\gcd(a, b, c) = 1$? The three numbers need not be pairwise coprime: for example, $(2, 3, 25)$ counts, while $(6, 9, 15)$ does not, since all three are multiples of 3.

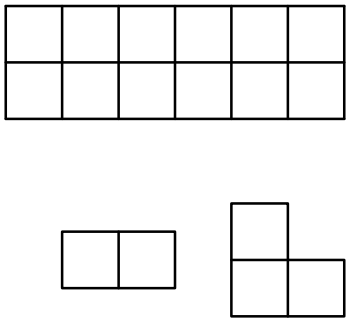
(A) 262 **(B)** 269 **(C)** 270 **(D)** 275 **(E)** 276

23. Let $f(x) = x^2 + bx + c$, where b and c are real numbers. The equation $f(f(x)) = 0$ has exactly four distinct real solutions, and three of them are -8, -10 and -12. What is c?

(A) 65 **(B)** 84 **(C)** 96 **(D)** 105 **(E)** 120

24. A 2×6 rectangle of unit squares is to be covered completely, with no overlaps and nothing sticking out, by tiles of two kinds: 1×2 dominoes, placed either way round, and L-shaped trominoes, each made of

three unit squares that form a 2×2 square with one corner square missing, placed in any of their four orientations. Any number of tiles of each kind may be used. How many coverings are possible?



(A) 13 **(B)** 24 **(C)** 45 **(D)** 53 **(E)** 117

25. How many ordered pairs (m, n) of integers with $1 \leq m \leq 30$ and $1 \leq n \leq 30$ have the property that $m + n$ divides $m^2 + n^2$? (Pairs with $m = n$ are allowed.)

(A) 36 **(B)** 46 **(C)** 48 **(D)** 54 **(E)** 66

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Answer Key — AMC 10 Practice Contest 5

1. (E) 12

2. (B) 45

3. (B) $\frac{7}{5}$

4. (A) -6

5. (D) 12

6. (B) 80

7. (C) $\frac{52}{3}$

8. (C) 65

9. (B) $\frac{12}{\sqrt{61}}$

10. (A) 506

11. (C) 11

12. (E) 6

13. (B) 36

14. (A) 100

15. (A) $\frac{9}{16}$

16. (A) 3

17. (C) $3\sqrt{5}$

18. (E) 35

19. (C) 35

20. (D) 1444

21. (D) 31

22. (E) 276

23. (D) 105

24. (D) 53

25. (E) 66

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Solutions — AMC 10 Practice Contest 5

1. Answer: (E) 12**. The idea.** Neither change touches the blue marbles, so compare everything with the fixed number b of blue marbles instead of with the changing total.

Work it out. Let the jar start with r red and b blue marbles. After adding 4 red, red is half of the jar, so red equals blue: $r + 4 = b$. After removing 4 red, red is one third, so blue is twice red: $b = 2(r - 4)$. Hence $r + 4 = 2r - 8$, giving $r = 12$ and $b = 16$, 28 marbles in all. Now add x red marbles so that red is $\frac{3}{5}$ of the jar: red : blue = 3 : 2, so $12 + x = \frac{3}{2} \cdot 16 = 24$ and $x = 12$.

Check. With 16 red and 16 blue, half are red; with 8 red and 16 blue, $8/24 = 1/3$ are red; with 24 red and 16 blue, $24/40 = 3/5$ are red.

2. Answer: (B) 45**. The idea.** Two chords crossing inside a circle make an angle equal to half the sum of the two arcs it and its vertical angle cut off. Adjacent hour marks are 30° apart, so every arc is a multiple of 30°.

Work it out. Chords 1–7 and 3–10: one pair of vertical angles cuts off arcs $1 \to 3$ (60°) and $7 \to 10$ (90°), so those angles are 75° and the other pair 105°. Chords 3–10 and 5–11: arcs $3 \to 5$ (60°) and $10 \to 11$ (30°) give 45° (and 135°). Chords 1–7 and 5–11: arcs $5 \to 7$ (60°) and $11 \to 1$ (60°) give 60° (and 120°). A triangle's angles must add to 180°, and from each pair of options the only choice that works is $75° + 45° + 60° = 180°$ (any obtuse option already leaves too little room). The smallest angle is 45°.

Check. Chords 1–7 and 5–11 are both diameters, so they meet at the centre, and the angle between them is the central angle from the 5 mark to the 7 mark, 60°, matching the value found above. The three angles $75°, 45°, 60°$ add to 180°.

3. Answer: (B) $\frac{7}{5}$**. The idea.** Since $CA = CD = 3$, triangle ACD is isosceles, so the foot H of the altitude from C to $\overline{AB}$ is the midpoint of $\overline{AD}$. And in a right triangle, the altitude to the hypotenuse cuts off $AH = AC^2/AB$.

Work it out. $AB = 5$. Triangles AHC and ACB are similar (they share angle A and each has a right angle), so $AH/AC = AC/AB$ and $AH = 9/5$. Then $AD = 2 \cdot AH = 18/5$ and $BD = 5 - 18/5 = 7/5$.

Check. Coordinates: $C = (0, 0)$, $A = (3, 0)$, $B = (0, 4)$, $D = (3 - 3t, 4t)$. $CD^2 = 9$ gives $(3 - 3t)^2 + 16t^2 = 9$, i.e. $25t^2 - 18t = 0$, so $t = 18/25$ and $BD = 5(1 - t) = 5 \cdot \frac{7}{25} = \frac{7}{5}$.

4. Answer: (A) -6**. The idea.** A rule like this is worth iterating a few times before doing anything clever: the sequence repeats with period 4, and the product of one full period is 1, so only the leftover terms matter.

Work it out. $a_2 = \frac{1+2}{1-2} = -3$, $a_3 = \frac{1-3}{1+3} = -\frac{1}{2}$, $a_4 = \frac{1/2}{3/2} = \frac{1}{3}$, $a_5 = \frac{4/3}{2/3} = 2 = a_1$. Since each term depends only on the one before, the sequence repeats $2, -3, -\frac{1}{2}, \frac{1}{3}, 2, -3, \ldots$ forever. One period multiplies to $2 \cdot (-3) \cdot (-\frac{1}{2}) \cdot \frac{1}{3} = 1$. Now $2026 = 4 \cdot 506 + 2$, so the product is $1^{506} \cdot a_1 a_2 = 2 \cdot (-3) = -6$.

Check. Why period 4: applying the rule twice gives $a_{n+2} = \dfrac{1 + \frac{1+a}{1-a}}{1 - \frac{1+a}{1-a}} = \dfrac{2}{-2a} = -\dfrac{1}{a_n}$, so $a_{n+4} = -1/a_{n+2} = a_n$, and also $a_n a_{n+2} = -1$, so any four consecutive terms multiply to $(-1)(-1) = 1$.

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5. Answer: (D) 12**. The idea.** Write the year as $2000 + y$ with $1 \le y \le 99$. Subtract the divisor from the year: $2000 + y = (20 + y) + 1980$. So $20 + y$ divides the year exactly when it divides 1980.

Work it out. $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$, and we need its divisors $d = 20+y$ with $21 \le d \le 119$. Listing divisors of 1980 in that range: 22, 30, 33, 36, 44, 45, 55, 60, 66, 90, 99, 110. (The next ones down, 20 and 18, and up, 132, fall outside.) That is 12 divisors, so 12 years: 2002, 2010, 2013, 2016, 2024, 2025, 2035, 2040,

Check. $2079 = 99 \cdot 21$ and $2090 = 110 \cdot 19$; a non-example: 2020 has $20+20 = 40$, and $1980/40$ is not whole, so $40 \nmid 2020$ indeed ($2020 = 40 \cdot 50.5$).

6. Answer: (B) 80**. The idea.** Without the cap on the digits this is a stars-and-bars count. The cap is easy to enforce afterwards: a digit of 5 or more can happen in at most one place, because two such digits would already add to $10 > 9$. So subtract the bad codes once, with nothing to add back.

Work it out. Codes whose four digits add to 9 (no cap): place 9 stars and 3 bars, $\binom{12}{3} = 220$. Codes with some digit at least 5: choose that position (4 ways) and take 5 away from its digit; the four digits now add to 4, which can be done in $\binom{7}{3} = 35$ ways. That is $4 \cdot 35 = 140$ bad codes, and no code is counted twice. So $220 - 140 = 80$ codes remain.

Check. List the digit multisets with sum 9 and every digit at most 4, with their numbers of orders: $\{4,4,1,0\}$: 12; $\{4,3,2,0\}$: 24; $\{4,3,1,1\}$: 12; $\{4,2,2,1\}$: 12; $\{3,3,3,0\}$: 4; $\{3,3,2,1\}$: 12; $\{3,2,2,2\}$: 4. Total $12 + 24 + 12 + 12 + 4 + 12 + 4 = 80$.

7. Answer: (C) $\frac{52}{3}$**. The idea.** The left side is never negative, so any solution has $x \ge 0$. For $x \ge 0$ the graph of the left side is a zigzag: it falls from 2 to 0 on $[0, 2]$, rises to 2 on $[2, 4]$, falls to 0 on $[4, 6]$, and rises with slope 1 after 6. Solve on each straight piece, keeping only solutions that lie on that piece, and watch the corner at $x = 4$.

Work it out. On $[0, 2]$: $2 - x = x/2$ gives $x = 4/3$. On $[2, 4]$: $x - 2 = x/2$ gives $x = 4$. On $[4, 6]$: $6 - x = x/2$ gives $x = 4$ again: the line $y = x/2$ passes exactly through the peak $(4, 2)$, so this is the same solution, not a new one. On $[6, \infty)$: $x - 6 = x/2$ gives $x = 12$. The solutions are $4/3$, 4, 12, with sum $\frac{4}{3} + 4 + 12 = \frac{52}{3}$.

Check. $x = 12$: $||12 - 4| - 2| = 6 = 12/2$. $x = 4/3$: $||4/3 - 4| - 2| = |8/3 - 2| = 2/3 = (4/3)/2$. Counting 4 twice would give the trap $64/3$.

8. Answer: (C) 65**. The idea.** Count all orders reaching 20, then subtract those that pass through 10. An order through 10 splits into an order reaching 10 followed by an order for the remaining 10 points, so those number $f(10)^2$, where $f(n)$ counts orders with total n.

Work it out. The last basket is a 2 or a 3, so $f(n) = f(n-2) + f(n-3)$ with $f(0) = 1$, $f(1) = 0$, $f(2) = 1$. This gives, for $n = 3, 4, \ldots, 20$: 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114. So $f(10) = 7$ and $f(20) = 114$. Orders passing through 10: $7 \cdot 7 = 49$. Orders avoiding 10: $114 - 49 = 65$.

Check. Count $f(20)$ another way: with a twos and b threes, $2a+3b = 20$ allows $(10, 0)$, $(7, 2)$, $(4, 4)$, $(1, 6)$, giving $1 + \binom{9}{2} + \binom{8}{4} + \binom{7}{1} = 1 + 36 + 70 + 7 = 114$. Similarly $f(10)$: $(5, 0)$, $(2, 2)$ give $1 + 6 = 7$.

9. Answer: (B) $\frac{12}{\sqrt{61}}$**. The idea.** First find the edges: adding the three face-diagonal equations gives $a^2 + b^2 + c^2$ at once. Then compute the volume of the cut-off pyramid in two ways: with a

right-angled corner at P as the base, and with the cut triangle as the base.

Work it out. Let the edges be a, b, c with $a^2 + b^2 = 13$, $b^2 + c^2 = 20$, $c^2 + a^2 = 25$. Adding, $2(a^2 + b^2 + c^2) = 58$, so $a^2 + b^2 + c^2 = 29$ and $c^2 = 16$, $a^2 = 9$, $b^2 = 4$: the edges are $3, 2, 4$. The pyramid has three perpendicular edges $3, 2, 4$ at P, so its volume is $\frac{1}{3} \cdot \frac{3\cdot 2}{2} \cdot 4 = 4$. Its other face is the triangle with sides $\sqrt{13}, \sqrt{20}, 5$ (the face diagonals). By Heron in squared form, $16K^2 = 4 \cdot 13 \cdot 20 - (13 + 20 - 25)^2 = 1040 - 64 = 976$, so $K^2 = 61$ and $K = \sqrt{61}$. With the cut as base, $4 = \frac{1}{3}\sqrt{61}\, h$, so $h = \frac{12}{\sqrt{61}}$.

Check. Put P at the origin and the edges along the axes: the cut plane is $\frac{x}{3} + \frac{y}{2} + \frac{z}{4} = 1$, whose distance from the origin is $1 \Big/ \sqrt{\frac{1}{9} + \frac{1}{4} + \frac{1}{16}} = 1 \Big/ \sqrt{\frac{61}{144}} = \frac{12}{\sqrt{61}}$.

10. Answer: (A) 506**. The idea.** Look for the losing piles (where the player to move loses with best play). Small cases suggest $4, 8, 12, \ldots$. The reason is that no legal move removes a multiple of 4: 1 is not, 2 and 3 are not, and every larger prime is odd.

Work it out. Claim: a pile is losing exactly when n is a multiple of 4. From $n \equiv 1, 2, 3 \pmod 4$ you can remove 1, 2 or 3 coins (all legal, since 2 and 3 are prime) and leave a multiple of 4 (possibly 0, which means you just won). From a multiple of 4, whatever you remove is not a multiple of 4, so the pile you leave is not a multiple of 4 and in particular not 0; your opponent then uses the strategy above. So the second player wins exactly when $4 \mid n$: $\lfloor 2026/4 \rfloor = 506$ values.

Check. $n = 4$: removing $1, 2, 3$ leaves $3, 2, 1$, and the opponent takes them all. $n = 5$: removing 5 (prime) wins at once, but removing 1 to leave 4 also wins. The period is 4, not 2 or 3, because 2 and 3 are both moves.

11. Answer: (C) 11**. The idea.** If $\frac{1}{n} = 0.\overline{d_1 d_2 \ldots d_k}$, then multiplying by 10^k shifts one whole block in front of the point, so $\frac{10^k}{n} - \frac{1}{n} = \overline{d_1 \ldots d_k}$ is a whole number: n divides $10^k - 1$. Conversely, if $n \mid 10^k - 1$, then $\frac{1}{n} = \frac{m}{10^k - 1}$ repeats the k-digit block m from the start. So we need $n \mid 10^k - 1$ for some $k \le 6$.

Work it out. Since 9, 99 and 999 all divide 999999, it is enough to look at the divisors of $999999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, of $9999 = 3^2 \cdot 11 \cdot 101$ and of $99999 = 3^2 \cdot 41 \cdot 271$. Divisors of 999999 from 2 to 50: $3, 7, 9, 11, 13, 21, 27, 33, 37, 39$ (ten numbers). The divisors of 9999 in range, $3, 9, 11, 33$, are already listed, and 99999 adds only 41. That makes 11 values of n.

Check. $\frac{1}{41} = 0.\overline{02439}$ has a 5-digit block and $\frac{1}{37} = 0.\overline{027}$ a 3-digit block; $\frac{1}{17}$ has a 16-digit block and $\frac{1}{49}$ a 42-digit block, so they are rightly left out, and every n with a factor 2 or 5 has a decimal that terminates or starts repeating late.

12. Answer: (E) 6**. The idea.** Divisors come in pairs d and $\frac{N}{d}$, the smallest with the largest, and so on inwards, so the 4th and 5th divisors multiply to N. Hence $N = m(m + 1)$. Because m and $m + 1$ share no factor, the number of divisors of N is $d(m)\, d(m + 1)$, which must be 8.

Work it out. $m(m + 1) < 1000$ means $m \le 31$. Conversely, if $N = m(m + 1)$ has 8 divisors, then $m < \sqrt{N} < m + 1$, the four divisors below $\sqrt{N}$ are the four smallest and the largest of them is m, so m and $m + 1$ really are the 4th and 5th. Now $d(m)\, d(m + 1) = 8$ with $m \ge 2$ forces one of $m, m + 1$ to be a prime and the other to have exactly 4 divisors. The numbers up to 32 with 4 divisors are $6, 8, 10, 14, 15, 21, 22, 26, 27$; those with a prime neighbour give $5 \cdot 6$, $6 \cdot 7$, $7 \cdot 8$, $10 \cdot 11$, $13 \cdot 14$ and $22 \cdot 23$. So $N \in \{30, 42, 56, 110, 182, 506\}$: 6 numbers.

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Check. $506 = 2 \cdot 11 \cdot 23$ has divisors $1, 2, 11, 22, 23, 46, 253, 506$, and indeed $22, 23$ sit in the middle. By contrast $72 = 8 \cdot 9$ has 12 divisors, so it is not counted.

13. Answer: (B) 36**. The idea.** Do not expand nine squares blindly. Move everything to one side and collect: the m^2 terms, the linear terms and the constants each collapse to a single number.

Work it out. Each square $(m+k)^2 = m^2 + 2km + k^2$. The left side has five m^2 terms, linear part $2m(0+1+2+3+4) = 20m$ and constant $0+1+4+9+16 = 30$. The right side has four m^2 terms, linear part $2m(5+6+7+8) = 52m$ and constant $25+36+49+64 = 174$. Setting them equal: $m^2 + 20m + 30 = 52m + 174$, so $m^2 - 32m - 144 = 0$, i.e. $(m-36)(m+4) = 0$. The positive solution is $m = 36$.

Check. $36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 1296 + 1369 + 1444 + 1521 + 1600 = 7230$ and $41^2 + 42^2 + 43^2 + 44^2 = 1681 + 1764 + 1849 + 1936 = 7230$. (The same pattern gives $3^2 + 4^2 = 5^2$ and $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.)

14. Answer: (A) 100**. The idea.** Such a subset is decided entirely by its smallest and largest elements $a < c$: the middle element must be $\frac{a+c}{2}$, and that is an integer strictly between them exactly when a and c have the same parity. So we only need to count same-parity pairs.

Work it out. If a and c are both odd or both even with $a < c$, then $b = \frac{a+c}{2}$ is an integer with $a < b < c$, and $\{a, b, c\}$ is a valid subset. Conversely every valid subset arises once this way, from its smallest and largest elements. Among $1, \ldots, 21$ there are 11 odd and 10 even numbers, giving $\binom{11}{2} + \binom{10}{2} = 55 + 45 = 100$ subsets.

Check. Count by common difference d instead: the subset $\{b-d, b, b+d\}$ needs $1 \le b-d$ and $b+d \le 21$, so there are $21 - 2d$ of them for $d = 1, \ldots, 10$: $19 + 17 + \cdots + 1 = 100$. Counting ordered arrangements or all pairs would give 200 or 210; allowing $d = 0$ (a repeated element) gives 121.

15. Answer: (A) $\frac{9}{16}$**. The idea.** The points at distance at least 1 from all three sides form a smaller triangle whose sides are parallel to the original ones, each pushed 1 unit inward. That smaller triangle has the same incenter, and its inradius is 1 less. So it is a scaled copy about the incenter.

Work it out. The 13-14-15 triangle has semiperimeter $s = 21$ and, by Heron, area $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. Its inradius is $r = 84/21 = 4$. The incenter I is at distance 4 from each side; the inner triangle's sides are at distance $4 - 1 = 3$ from I, so the inner triangle is the image of the original under the dilation about I with ratio $\frac{3}{4}$. Areas scale by $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$, which is the probability.

Check. The inradius must be area over semiperimeter; dividing the area by the side 14 instead gives $r = 6$ and the wrong ratio $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$. Forgetting to square the ratio gives $\frac{3}{4}$.

16. Answer: (A) 3**. The idea.** If the line is $y = mx + k$, the x-coordinates of A, B, C are the three roots of one cubic equation, so Vieta fixes their sum. The midpoint condition then pins B down completely, whatever the line is: it must be the point about which the curve is symmetric. Around that point the algebra becomes very simple.

Work it out. Let the line be $y = mx + k$ and let a, b, c be the x-coordinates of A, B, C. They are the roots of $x^3 - 3x^2 + (5-m)x - k = 0$, so $a + b + c = 3$. Since B is the midpoint of $\overline{AC}$, $b = \frac{a+c}{2}$, so $3b = 3$ and $b = 1$: every such line passes through $B = (1, 3)$.

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Now shift to $u = x - 1$. Since $(x-1)^3 = x^3 - 3x^2 + 3x - 1$, the curve is $y - 3 = u^3 + 2u$, and the line through B is $y - 3 = mu$. They meet where $u^3 + 2u = mu$, i.e. $u = 0$ or $u^2 = m - 2$. Three distinct points need $m > 2$, and then A and C are at $u = \pm d$ with $d = \sqrt{m-2}$, i.e. at $B \pm (d, md)$. So

$AC = 2d\sqrt{1+m^2}$, and $AC^2 = 40$ gives $(m-2)(1+m^2) = 10$.

For $m > 2$ the left side increases with m, so there is only one solution, and $m = 3$ works: $1 \cdot 10 = 10$.

Check. The line of slope 3 through $(1, 3)$ is $y = 3x$. Then $x^3 - 3x^2 + 5x = 3x$ gives $x(x^2 - 3x + 2) = 0$, so $x = 0, 1, 2$: the points $(0, 0)$, $(1, 3)$, $(2, 6)$. Indeed $(1, 3)$ is the midpoint and $AC = \sqrt{4 + 36} = 2\sqrt{10}$. Using only the horizontal half-width, $2d = 2\sqrt{10}$, would give $m = 12$.

17. Answer: (C) $3\sqrt{5}$**. The idea.** The two curves are mirror images in the line $y = x$, so subtracting the equations should factor. It does, and it sorts the four points onto two perpendicular lines, which become the diagonals of the quadrilateral.

Work it out. Subtract $x = y^2 - 2$ from $y = x^2 - 2$: $y - x = x^2 - y^2 = (x - y)(x + y)$, so $(x - y)(x + y + 1) = 0$.

On $y = x$: $x = x^2 - 2$ gives $x = 2$ or $x = -1$, the points $(2, 2)$ and $(-1, -1)$, at distance $3\sqrt{2}$ apart.

On $x + y = -1$: $-1 - x = x^2 - 2$ gives $x^2 + x - 1 = 0$, $x = \frac{-1 \pm \sqrt{5}}{2}$, the points $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$ and its mirror image $\left(\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$. Their distance is $\sqrt{2} \cdot \sqrt{5} = \sqrt{10}$.

The lines $y = x$ and $x + y = -1$ are perpendicular and meet at $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, which lies between $(-1, -1)$ and $(2, 2)$ and is the midpoint of the second pair. So the two segments cross, the four points form a convex kite with these segments as diagonals, and the area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot \sqrt{10} = \frac{1}{2} \cdot 3\sqrt{20} = 3\sqrt{5}$.

Check. $3\sqrt{5} \approx 6.71$. By the shoelace formula with vertices in order $(2, 2)$, $(0.618, -1.618)$, $(-1, -1)$, $(-1.618, 0.618)$ the area is again about 6.71. Forgetting the $\frac{1}{2}$ gives $3\sqrt{10}$.

18. Answer: (E) 35**. The idea.** If $5a + 7b = N$ is one way, the others come from trading seven 5-cent stamps for five 7-cent stamps, since $7 \cdot 5 = 5 \cdot 7 = 35$. So the ways to make N form a chain, and N has at least two ways exactly when $N - 35$ can be made. Everything reduces to knowing which amounts can be made at all.

Work it out. If $5a + 7b = 5a' + 7b'$, then $5(a - a') = 7(b' - b)$, so $a - a'$ is a multiple of 7. Hence the ways to make N are (a_0, b_0), $(a_0 + 7, b_0 - 5)$, $(a_0 + 14, b_0 - 10)$, ... where a_0 is the least number of 5-cent stamps; removing one trade from a way to make N gives a way to make $N - 35$ and back. So N has exactly two ways iff $M = N - 35$ can be made and $M - 35$ cannot.

Which amounts cannot be made? Listing: $1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23$, twelve of them; from 24 to 28 every amount can be made ($25, 24{=}10{+}14, 26{=}5{+}21, 27{=}20{+}7, 28$), and adding 5 repeatedly covers everything above. (This matches $\frac{(5-1)(7-1)}{2} = 12$.)

Now $M = N - 35$ runs over $0, 1, \ldots, 65$. The makeable values there: $66 - 12 = 54$. Remove those whose $M - 35$ is also makeable: $M - 35$ runs over $0, \ldots, 30$, and $31 - 12 = 19$ of those are makeable. So the answer is $54 - 19 = 35$.

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Check. The smallest amounts with two ways are $35, 40, 42, 45$; the first with three ways is 70. Among $1, \ldots, 100$: 34 amounts have exactly one way, 35 have two, 19 have three, and 12 have none; $34 + 35 + 19 + 12 = 100$.

19. Answer: (C) 35**. The idea.** Colour the vertices like a checkerboard: each edge joins a "black" vertex to a "white" one, and each colour class is a regular tetrahedron of 4 vertices. Count by size, and use the colouring to control the larger sets.

Work it out. Size 0: 1. Size 1: 8. Size 2: all $\binom{8}{2} = 28$ pairs minus the 12 edges gives 16. Size 3: suppose the set uses both colours, say black u and white w. The three neighbours of u are white and are not in the set; the white vertex not adjacent to u is its antipode u'. So the only white vertex allowed with u is u', and similarly the only black vertex allowed with u' is u; a third vertex is impossible. Hence every scattered triple is a single colour: $2\binom{4}{3} = 8$ triples. Size 4: the same argument shows a mixed set has at most 2 vertices, so only the two tetrahedra work: 2. Size 5 or more: impossible. Total $1 + 8 + 16 + 8 + 2 = 35$.

Check. By the colouring, the scattered sets are: subsets of the black class (16), subsets of the white class (16), the empty set counted twice (subtract 1), and the 4 antipodal black-white pairs: $16 + 16 - 1 + 4 = 35$.

20. Answer: (D) 1444**. The idea.** Translate trailing zeros into prime exponents. Write $n = 2^a 3^b m$ with m coprime to 6. The number of base-4 zeros is the largest k with $4^k \mid n$, which is $\lfloor a/2 \rfloor$. The number of base-6 zeros is the largest k with 2^k and 3^k both dividing n, which is $\min(a, b)$. So we need $\min(a, b) = \lfloor a/2 \rfloor$.

Work it out. If $b \geq a$, then $\min(a, b) = a = \lfloor a/2 \rfloor$ forces $a = 0$: every odd n works. If $b < a$, we need $b = \lfloor a/2 \rfloor$: $(a, b) = (1, 0), (2, 1), (3, 1), (4, 2), (5, 2), (6, 3), \ldots$ For each pair count $m \leq 2026/(2^a 3^b)$ coprime to 6; among $1, \ldots, X$ there are $X - \lfloor X/2 \rfloor - \lfloor X/3 \rfloor + \lfloor X/6 \rfloor$ of them.

Odd n: 1013. $(1, 0)$: $n = 2m, m \leq 1013$: $1013-506-337+168 = 338$. $(2, 1)$: $n = 12m, m \leq 168$: $168 - 84 - 56 + 28 = 56$. $(3, 1)$: $n = 24m$, $m \leq 84$: $84 - 42 - 28 + 14 = 28$. $(4, 2)$: $n = 144m$, $m \leq 14$: $14 - 7 - 4 + 2 = 5$. $(5, 2)$: $n = 288m$, $m \leq 7$: $m = 1, 5, 7$, so 3. $(6, 3)$: $n = 1728$: 1. $(7, 3)$: $3456 > 2026$, and larger pairs are bigger still. Total $1013+338+56+28+5+3+1 = 1444$.

Check. Small cases: $n = 2$ (2_4, 2_6): both 0 zeros, counted in $(1, 0)$. $n = 4 = 10_4 = 4_6$: 1 vs 0, fails, and indeed $(a, b) = (2, 0)$ is not on the list. $n = 1728 = 2^6 3^3$: $\lfloor 6/2 \rfloor = 3 = \min(6, 3)$. Stopping after the first two cases gives 1351; forgetting $a \geq 4$ gives 1435.

21. Answer: (D) 31**. The idea.** Subtract 4 from every element. A set has average 4 exactly when its shifted elements sum to 0. After the shift, the only negative numbers available are $-1, -2, -3$, whose total is at most 6 in size, so the positive part can never exceed 6. That kills every element above 10 and leaves a small count.

Work it out. Shifted, a good set consists of: an optional 0 (the original 4), a subset N of $\{1, 2, 3\}$ used as negatives, and a subset P of $\{1, 2, \ldots, 96\}$ with the same sum s as N. Since $s \leq 1+2+3 = 6$, the set P lies inside $\{1, \ldots, 6\}$. For $s = 1, \ldots, 6$:

Subsets of $\{1, 2, 3\}$ with sum s: 1, 1, 2, 1, 1, 1 (for $s = 3$: $\{3\}$, $\{1, 2\}$). Subsets of $\{1, \ldots, 6\}$ with sum s: 1, 1, 2, 2, 3, 4 (e.g. $s = 6$: $\{6\}$, $\{1, 5\}$, $\{2, 4\}$, $\{1, 2, 3\}$). Products: $1+1+4+2+3+4 = 15$. Also $s = 0$ gives $N = P = \varnothing$ once. Each choice may include or omit 0, giving $2(15 + 1) = 32$ zero-sum sets, and removing the empty set leaves 31.

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Check. The largest element ever used is $6 + 4 = 10$, e.g. $\{1, 2, 3, 10\}$ with mean 4; an element like 11 would need the others to sit 7 below 4 in total, but at most $3 + 2 + 1 = 6$ is available. So the answer for $\{1, \ldots, 10\}$ and for $\{1, \ldots, 100\}$ is the same, 31; forgetting the singleton $\{4\}$ gives 30, forgetting the optional 4 entirely gives 15 or 16.

22. Answer: (E) 276**. The idea.** A common divisor of a, b and c also divides $a+b+c = 30$, so the only primes that can spoil a triple are 2, 3 and 5. Count all triples and use inclusion–exclusion over these primes. For any $d \mid 30$, the triples with d dividing all three numbers are (da', db', dc') with $a' + b' + c' = \frac{30}{d}$, and there are $\binom{30/d-1}{2}$ of them by stars and bars.

Work it out. All triples: $\binom{29}{2} = 406$. All even: $\binom{14}{2} = 91$; all multiples of 3: $\binom{9}{2} = 36$; of 5: $\binom{5}{2} = 10$. Multiples of 6: $\binom{4}{2} = 6$; of 10: $\binom{2}{2} = 1$; of 15: $\binom{1}{2} = 0$; of 30: 0. So the count is $406 - 91 - 36 - 10 + 6 + 1 - 0 + 0 = 276$.

Check. Sort all 406 triples by their gcd g instead. The same method gives 276 for $g = 1$, 84 for $g = 2$ (coprime triples adding to 15: $91 - 6 - 1$), 30 for $g = 3$ (adding to 10: $36 - 6$), 9 for $g = 5$ (adding to 6: $10 - 1$), 6 for $g = 6$, 1 for $g = 10$ and 0 for $g = 15, 30$. Indeed $276 + 84 + 30 + 9 + 6 + 1 = 406$.

23. Answer: (D) 105**. The idea.** $f(f(x)) = 0$ means $f(x) = r_1$ or $f(x) = r_2$, where r_1, r_2 are the roots of f. Each of these two equations has two solutions placed symmetrically about the vertex $x = h$ of the parabola. So the four solutions form two pairs with a common midpoint h, and the farther pair belongs to the larger root.

Work it out. Write $f(x) = (x - h)^2 - s^2$ with $s > 0$, so $r_{1,2} = h \pm s$. The equation $f(x) = h \pm s$ becomes $(x - h)^2 = s^2 + h \pm s$, so the two pairs have half-gaps $d_\pm$ with $d_+^2 = s^2 + s + h$ and $d_-^2 = s^2 - s + h$, and $d_+^2 - d_-^2 = 2s$.

Two of the three given numbers form one pair; its midpoint is h.

Pair $\{-10, -12\}$: $h = -11$, the other pair is $\{-8, -14\}$. Half-gaps 1 and 3: $9 - 1 = 2s$ gives $s = 4$, and $d_-^2 = 16 - 4 - 11 = 1$. Consistent!

Pair $\{-8, -10\}$: $h = -9$, other pair $\{-12, -6\}$, half-gaps 1 and 3, $s = 4$, but $d_-^2 = 16-4-9 = 3 \neq 1$. Impossible.

Pair $\{-8, -12\}$: $h = -10$, and then -10 is its own partner, a double root, contradicting four distinct solutions.

So $f(x) = (x + 11)^2 - 16 = x^2 + 22x + 105$, and $c = 105$.

Check. The roots of f are -7 and -15. $f(x) = -7$ gives $(x+11)^2 = 9$: $x = -8, -14$. $f(x) = -15$ gives $(x+11)^2 = 1$: $x = -10, -12$. Four distinct solutions including the three given. The rejected pairing would give $(x + 9)^2 - 16$, i.e. $c = 65$.

24. Answer: (D) 53**. The idea.** Build the covering from the right-hand end. A tromino at the end leaves a rectangle with one extra square sticking out, so keep track of two counts: t_n, the coverings of a $2 \times n$ rectangle, and p_n, the coverings of a $2 \times n$ rectangle with one extra square attached to the right end of its top row (the bottom-row version has the same count, by reflection).

Work it out. In a $2 \times n$ rectangle, the two squares of the last column are covered by one vertical domino (leaving t_{n-1}), by two horizontal dominoes (leaving t_{n-2}), or by one tromino that also takes one square of column $n - 1$, top or bottom (leaving a p_{n-2} shape, in 2 mirror ways). No

other option works: a tromino covering just one square of the last column would strand the other one. So $t_n = t_{n-1} + t_{n-2} + 2p_{n-2}$. In the p_n shape, the extra square is covered by a horizontal domino (leaving the mirror image of a p_{n-1} shape) or by a tromino that also takes both squares of column n (leaving t_{n-1}), so $p_n = p_{n-1} + t_{n-1}$. With $t_0 = t_1 = 1$, $p_0 = 0$, $p_1 = 1$: $p_2 = 2$, $p_3 = 4$, $p_4 = 9$, and $t_2 = 2$, $t_3 = 5$, $t_4 = 11$, $t_5 = 24$, $t_6 = 24 + 11 + 2 \cdot 9 = 53$.

Check. $t_3 = 5$ by hand: a 2×3 rectangle has 3 domino coverings, and two trominoes can fill it in 2 ways (the two L's interlock along either diagonal of the middle column). With dominoes alone the counts would be Fibonacci numbers, 13 for 2×6.

25. Answer: (E) 66**. The idea.** Since $m^2 + n^2 = (m+n)^2 - 2mn$, the condition is $m+n \mid 2mn$. Pull out the gcd: $m = ga$, $n = gb$ with $\gcd(a, b) = 1$. Then $a + b$ shares no prime with a or with b, hence none with ab, so all the divisibility has to be paid for by $2g$.

Work it out. $g(a+b) \mid 2g^2ab \iff a+b \mid 2g\,ab \iff a+b \mid 2g$ (since $\gcd(a+b, ab) = 1$). Now parity matters. If $a + b$ is odd, $a + b \mid 2g$ means $a + b \mid g$. If a and b are both odd (they cannot both be even), $a + b$ is even and the condition is $\frac{a+b}{2} \mid g$. Write $c = a + b$ in the first case and $c = \frac{a+b}{2}$ in the second; then the solutions are exactly $(m, n) = (tac, tbc)$ for positive integers t.

$a = b = 1$: $c = 1$, so every pair $m = n$ works: 30 pairs. For $a < b$ we need $tbc \le 30$ (then count both orders):

$(1, 2)$, $c = 3$: $(3, 6)t$, $t \le 5$: 5. $(1, 3)$, $c = 2$: $(2, 6)t$: 5. $(1, 4)$, $c = 5$: $(5, 20)$: 1. $(1, 5)$, $c = 3$: $(3, 15)t$: 2. $(1, 7)$, $c = 4$: $(4, 28)$: 1. $(2, 3)$, $c = 5$: $(10, 15)t$: 2. $(3, 4)$, $c = 7$: $(21, 28)$: 1. $(3, 5)$, $c = 4$: $(12, 20)$: 1.

Everything else is too big: $(1, 6)$ needs $n \ge 42$, $(2, 5)$ and $(3, 7)$ need 35, $(1, 9)$ needs 45, $(5, 7)$ needs 42, $(1, 8)$ needs 72. That gives $5 + 5 + 1 + 2 + 1 + 2 + 1 + 1 = 18$ pairs with $m < n$, so 36 with $m \ne n$, and $30 + 36 = 66$ in all.

Check. $(4, 28)$: $32 \mid 16 + 784 = 800$? $800 = 25 \cdot 32$, yes. $(12, 20)$: $32 \mid 144 + 400 = 544 = 17 \cdot 32$, yes. If you forget the halving when a, b are both odd you lose $(2, 6)$-type pairs and get 54; forgetting the diagonal gives 36; counting unordered pairs gives 48.

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