MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

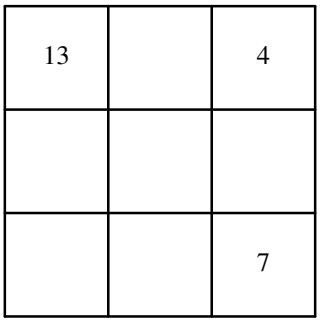
INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A 3×3 grid is filled with numbers so that it is a magic square: the three rows, the three columns and the two main diagonals all have the same sum. The top-left entry is 13, the top-right entry is 4, and the bottom-right entry is 7. What is the bottom-left entry?

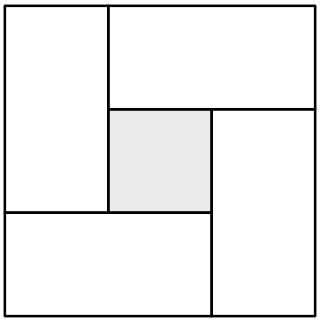


(A) 7 **(B)** 10 **(C)** 12 **(D)** 13 **(E)** 16

2. A glass jar is filled one-third of the way with water, and the jar together with its water weighs 440 grams. When the same jar is instead filled three-quarters of the way with water, the jar together with its water weighs 690 grams. How many grams does the empty jar weigh?

(A) 160 **(B)** 190 **(C)** 200 **(D)** 240 **(E)** 250

3. Four congruent rectangles, each with a longer side and a shorter side, are arranged in a pinwheel around a small square hole so that, with no overlaps and no other gaps, the four rectangles and the hole exactly fill a large square. Each side of the large square is made up of the longer side of one rectangle together with the shorter side of the next rectangle. The large square has area 81 and the hole has area 9. What is the perimeter of one rectangle?



(A) 12 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 36

4. How many three-digit positive integers have a middle (tens) digit that is strictly greater than both its hundreds digit and its units digit? For example, 172 and 391 qualify, but 535, 455 and 980 do not.

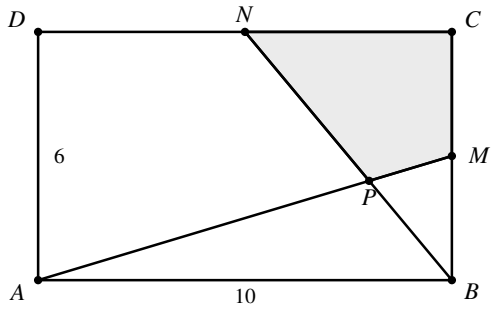
(A) 120 **(B)** 204 **(C)** 240 **(D)** 285 **(E)** 330

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5. For how many integers n with $1 \le n \le 2026$ is the remainder when n is divided by 3 equal to the remainder when n is divided by 5?

(A) 135 **(B)** 271 **(C)** 405 **(D)** 406 **(E)** 407

6. Rectangle $ABCD$ has $AB = 10$ and $BC = 6$, with its vertices labelled in order around the rectangle. Point M is the midpoint of side $\overline{BC}$ and point N is the midpoint of side $\overline{CD}$. Segments $\overline{AM}$ and $\overline{BN}$ cross at point P. What is the area of quadrilateral $PMCN$?



(A) 10 **(B)** 12 **(C)** $\frac{27}{2}$ **(D)** 15 **(E)** 18

7. A function $f(x) = ax + b$, where a and b are real constants, satisfies $f(f(f(x))) = 8x + 21$ for every real number x. What is $a + b$?

(A) -1 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 5

8. A fair six-sided die is rolled four times. What is the probability that each roll is greater than or equal to the roll before it (so the four results, in order, never decrease)?

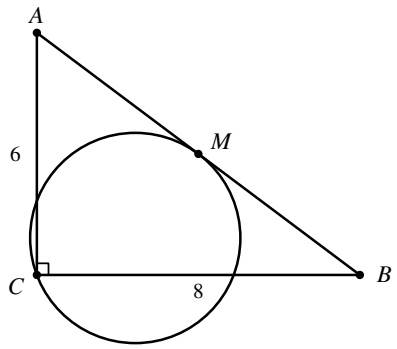
(A) $\frac{5}{432}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{35}{648}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{7}{72}$

9. Consider all fractions $\frac{a}{30}$, where a is a positive integer, that are in lowest terms (that is, a and 30 have no common factor greater than 1) and that lie strictly between 0 and 10. What is the sum of all of these fractions?

(A) 80 **(B)** 200 **(C)** 360 **(D)** 400 **(E)** 440

10. Triangle ABC has a right angle at C, with $AC = 6$ and $BC = 8$. Let M be the midpoint of the hypotenuse $\overline{AB}$. A circle passes through C and is tangent to line AB at the point M. What is the radius of this circle?

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(A) $\frac{125}{48}$ **(B)** $\frac{8}{3}$ **(C)** $\frac{25}{8}$ **(D)** $\frac{125}{36}$ **(E)** $\frac{25}{6}$

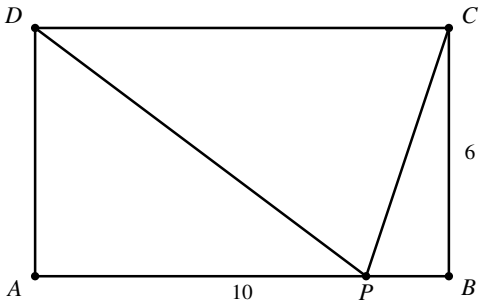
11. Leo climbs a staircase of 10 steps, starting at the bottom. With each stride he goes up either 1 step or 2 steps, but he never takes two 2-step strides in a row. He finishes exactly on the 10th step. In how many different ways (different sequences of strides) can he climb the staircase?

(A) 41 **(B)** 45 **(C)** 55 **(D)** 60 **(E)** 89

12. Call a four-digit number $\overline{abcd}$ *level* if the sum of its first two digits equals the sum of its last two digits, that is, $a+b=c+d$. For example, 2413 and 5005 are level. How many level four-digit numbers are *not* divisible by 11?

(A) 525 **(B)** 580 **(C)** 606 **(D)** 615 **(E)** 670

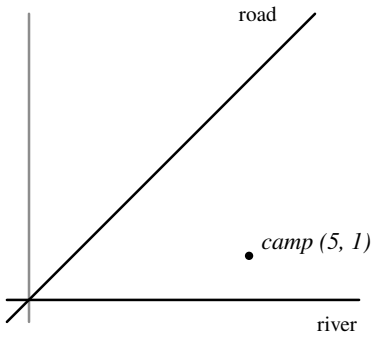
13. Rectangle $ABCD$ has $AB = 10$ and $BC = 6$. Point P lies on side $\overline{AB}$ so that segment $\overline{PC}$ bisects $\angle BPD$. What is the length PC?



(A) $2\sqrt{10}$ **(B)** $2\sqrt{13}$ **(C)** $\sqrt{61}$ **(D)** 8 **(E)** 10

14. On a map with coordinates measured in kilometres, a straight river runs along the x-axis and a straight road runs along the line $y = x$. A hiker's camp is at the point $(5, 1)$. The hiker wants to walk in straight segments from the camp to some point of the road, then from there to some point of the river, and then straight back to the camp. What is the shortest possible total length of such a walk, in kilometres?

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(A) $2\sqrt{11}$ **(B)** $2\sqrt{13}$ **(C)** $2\sqrt{14}$ **(D)** $6\sqrt{2}$ **(E)** $2\sqrt{26}$

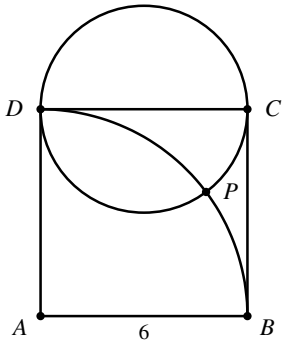
15. Three different vertices of a regular 12-gon are chosen, and they form a triangle. For how many of the $\binom{12}{3} = 220$ possible choices is the triangle isosceles? (Equilateral triangles count as isosceles.)

(A) 44 **(B)** 48 **(C)** 52 **(D)** 56 **(E)** 60

16. Positive real numbers a, b and c satisfy the three equations $ab + 2a + b = 16$, $bc + 3b + 2c = 30$ and $ca + c + 3a = 15$. What is $a + b + c$?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 15

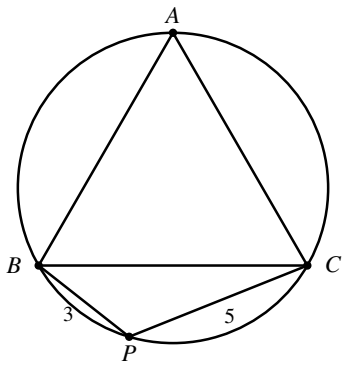
17. Square $ABCD$ has side length 6, with its vertices labelled in order around it. One circle has center A and radius 6. A second circle has side CD as a diameter. The two circles meet at D and at one other point P, which lies inside the square. What is the distance from P to side AB?



(A) $\frac{18}{5}$ **(B)** 4 **(C)** $\frac{9}{2}$ **(D)** $\frac{24}{5}$ **(E)** 6

18. Equilateral triangle ABC is inscribed in a circle. Point P lies on the arc BC that does not contain A, with $PB = 3$ and $PC = 5$. What is the area of quadrilateral $ABPC$?

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(A) $12\sqrt{3}$ **(B)** $\frac{49\sqrt{3}}{4}$ **(C)** $14\sqrt{3}$ **(D)** $16\sqrt{3}$ **(E)** $18\sqrt{3}$

19. For how many integers n with $1 \le n \le 2026$ is the number $n^2 - 1$ a multiple of 504? (The value $n = 1$ counts, because 0 is a multiple of 504.)

(A) 9 **(B)** 17 **(C)** 33 **(D)** 64 **(E)** 65

20. A ribbon 1 metre long is cut at two points, chosen independently and uniformly at random along its length, giving three pieces. What is the probability that every piece is at least 20 cm long and, in addition, the three pieces can be the side lengths of a triangle (with positive area)?

(A) $\frac{13}{200}$ **(B)** $\frac{7}{100}$ **(C)** $\frac{9}{100}$ **(D)** $\frac{1}{10}$ **(E)** $\frac{13}{100}$

21. For exactly one positive real number m, the equation $|x^2 - 8x + 12| = mx$ has exactly three distinct real solutions x. What is this value of m?

(A) $4 - 2\sqrt{3}$ **(B)** 1 **(C)** $8 - 4\sqrt{3}$ **(D)** $-2 + 2\sqrt{3}$ **(E)** 2

22. Three balls of radius 6 rest on a flat table, each touching the other two. In the space between them a column of smaller balls is built along the vertical line through the middle of the three: ball X rests on the table and touches all three big balls; ball Y rests on top of X and touches all three big balls; ball Z rests on top of Y and touches all three big balls. How high above the table is the top of ball Z?

(A) 7 **(B)** $\frac{15}{2}$ **(C)** 8 **(D)** 9 **(E)** 10

23. A fair six-sided die is rolled 5 times, and the results are written down in order, giving one of $6^5 = 7776$ equally likely sequences. In how many of these sequences are there two consecutive rolls that both show a 6?

(A) 671 **(B)** 751 **(C)** 776 **(D)** 864 **(E)** 1025

24. A country issues coins worth $1, 2, 4, 8, 16, \ldots$ cents (every power of 2). Mia wants to pay exactly 2026 cents, and she may use at most two coins of each value. Two payments are the same if they use the same number of coins of every value. In how many different ways can she pay?

(A) 20 **(B)** 33 **(C)** 45 **(D)** 53 **(E)** 64

25. A 6×6 board holds 36 lamps, all switched off. A move consists of choosing one whole row or one whole column and toggling every lamp in it (lamps that are off turn on, lamps that are on turn off). Moves may

be repeated and made in any order. How many different patterns of lit lamps with exactly 18 lamps on can be reached by some sequence of moves?

(A) 400 **(B)** 540 **(C)** 1080 **(D)** 1280 **(E)** 2160

Answer Key — AMC 10 Practice Contest 4

1. (E) 16

2. (D) 240

3. (B) 18

4. (C) 240

5. (D) 406

6. (B) 12

7. (E) 5

8. (E) $\frac{7}{72}$

9. (D) 400

10. (A) $\frac{125}{48}$

11. (A) 41

12. (A) 525

13. (A) $2\sqrt{10}$

14. (B) $2\sqrt{13}$

15. (C) 52

16. (B) 9

17. (A) $\frac{18}{5}$

18. (D) $16\sqrt{3}$

19. (E) 65

20. (E) $\frac{13}{100}$

21. (C) $8 - 4\sqrt{3}$

22. (C) 8

23. (B) 751

24. (D) 53

25. (C) 1080

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Solutions — AMC 10 Practice Contest 4

1. **Answer: (E)** 16**. The idea.** Every line through the centre (the middle row, middle column and both diagonals) contains the centre c. Adding those four lines counts every entry once except c, which is counted four times; that sum equals the whole grid ($3S$) plus $3c$, and it also equals $4S$. So $S = 3c$: the magic sum is three times the centre. Consequently two entries symmetric about the centre always add to $S - c = 2c$.

   Work it out. Top-left and bottom-right are symmetric about the centre, so $13 + 7 = 2c$, giving $c = 10$ and $S = 30$. Top-right and bottom-left are also symmetric about the centre, so the bottom-left entry is $2c - 4 = 20 - 4 = 16$.

   Check. The whole square can be completed: top row 13, 13, 4; middle row 1, 10, 19; bottom row 16, 7, 7. Rows: 30, 30, 30. Columns: $13+1+16 = 30$, $13+10+7 = 30$, $4+19+7 = 30$. Diagonals: $13 + 10 + 7 = 30$ and $4 + 10 + 16 = 30$. (The entries need not be distinct, and the answer 16 is forced by the three given numbers alone.)

2. **Answer: (D)** 240**. The idea.** The jar's own weight is the same in both weighings, so it cancels when you subtract. The difference in weight is due only to the extra water.

   Work it out. Going from one-third full to three-quarters full adds $\frac{3}{4} - \frac{1}{3} = \frac{5}{12}$ of a jarful of water, and that adds $690 - 440 = 250$ grams. So a full jar's worth of water weighs $250 \cdot \frac{12}{5} = 600$ grams, and one-third of a jarful weighs 200 grams. The empty jar weighs $440 - 200 = 240$ grams.

   Check. Three-quarters of a jarful is 450 grams of water, and $240 + 450 = 690$. The tempting $190 = 440 - 250$ treats the 250 grams as if it were the water in the first weighing.

3. **Answer: (B)** 18**. The idea.** Don't find the rectangle's sides separately — read the two squares' side lengths in terms of them. If a rectangle is a by b with $a > b$, then along each edge of the big square you see one long side plus one short side, so the big square has side $a + b$; the hole is what is left of a long side after a short side is taken away, so it has side $a - b$.

   Work it out. The big square has side $\sqrt{81} = 9$, so $a + b = 9$. The perimeter of a rectangle is $2(a + b) = 18$. (For completeness, the hole has side 3, so $a - b = 3$, giving $a = 6$ and $b = 3$.)

   Check. Area: four 6×3 rectangles cover 72, the hole 9, total 81, which matches the big square. The tempting wrong answers are 36 (the big square's perimeter) and 24 (four times the long side).

4. **Answer: (C)** 240**. The idea.** Fix the middle digit first; then the other two digits are chosen independently, each just has to be smaller than it. The only subtlety is that the hundreds digit cannot be 0 while the units digit can.

   Work it out. Let the middle digit be m. The hundreds digit is one of $1, \ldots, m - 1$ ($m - 1$ choices) and the units digit is one of $0, \ldots, m - 1$ (m choices). We need $m \geq 2$. The total is $\sum_{m=2}^{9} m(m-1) = 2 + 6 + 12 + 20 + 30 + 42 + 56 + 72 = 240$.

   Check. Using $m(m-1) = m^2 - m$: $\sum_{m=1}^{9} m^2 - \sum_{m=1}^{9} m = 285 - 45 = 240$. Forgetting that 0 is allowed in the units place gives 204; allowing 0 in the hundreds place gives 285.

5. **Answer: (D)** 406**. The idea.** If both remainders equal r, then r is a possible remainder for division by 3, so $r \in \{0, 1, 2\}$, and $n - r$ is divisible by both 3 and 5, i.e. by 15. Conversely $n \equiv r$

(mod 15) with $r \le 2$ gives remainder r on division by both 3 and 5. So the condition is exactly: n leaves remainder 0, 1 or 2 when divided by 15.

Work it out. Since $2025 = 15 \cdot 135$, the numbers $1, \dots, 2025$ form 135 complete runs of 15 consecutive integers, each containing exactly three good numbers: 405 so far. The extra number $2026 = 2025 + 1$ leaves remainder 1 on division by 15, so it is also good. Total 406.

Check. Count each class directly: multiples of 15 up to 2026: 135; numbers $15k + 1$ with $0 \le k \le 135$: 136; numbers $15k + 2 \le 2026$ with $0 \le k \le 134$: 135. Sum 406. Forgetting the remainder-0 class gives 271.

6. Answer: (B) 12**. The idea.** Triangle ABM and triangle BCN each cover exactly one quarter of the rectangle. Both of them contain triangle BPM; removing it from each leaves triangle ABP from the first and quadrilateral $PMCN$ from the second, so these two have equal area. Triangle ABP is easy once we know how high P sits.

Work it out. Put $A = (0, 0)$, $B = (10, 0)$, $C = (10, 6)$, $D = (0, 6)$, so $M = (10, 3)$ and $N = (5, 6)$. Line AM is $y = \frac{3}{10}x$. Line BN goes from $(10, 0)$ to $(5, 6)$: $y = -\frac{6}{5}(x - 10)$. Setting them equal, $\frac{3}{10}x = 12 - \frac{6}{5}x$, so $\frac{3}{2}x = 12$, $x = 8$ and $y = \frac{12}{5}$. Triangle ABP has base 10 and height $\frac{12}{5}$, so area 12. Hence $[PMCN] = [BCN] - [BPM] = [ABM] - [BPM] = [ABP] = 12$.

Check. Shoelace on $P(8, \frac{12}{5})$, $M(10, 3)$, $C(10, 6)$, $N(5, 6)$: the terms $x_i y_{i+1} - x_{i+1} y_i$ are 0, 30, 30 and $12 - 48 = -36$, summing to 24, so the area is 12. Also $[BPM] = 15 - 12 = 3$, which is $\frac{1}{2} \cdot 3 \cdot (10 - 8)$, as it should be. The option 15 is the area of triangle ABM (or BCN), forgetting to remove the overlap.

7. Answer: (E) 5**. The idea.** A composition of linear functions is linear, and we can track its slope and intercept separately: the slopes multiply, while the intercept collects b scaled by the later slopes.

Work it out. $f(f(x)) = a(ax+b)+b = a^2x+(a+1)b$, and $f(f(f(x))) = a(a^2x+(a+1)b)+b = a^3x + (a^2 + a + 1)b$. Matching with $8x + 21$: $a^3 = 8$, so $a = 2$ (the only real cube root), and $(4 + 2 + 1)b = 21$, so $b = 3$. Hence $a + b = 5$.

Check. $f(x) = 2x + 3$: $f(1) = 5$, $f(5) = 13$, $f(13) = 29$, and $8 \cdot 1 + 21 = 29$. Stopping at $b = 3$ or at $a = 2$ answers only half the question, and $a - b$ or $b - a$ is a sign slip; the intercept really is $(a^2 + a + 1)b$, since the three copies of b are scaled by a^2, a and 1.

8. Answer: (E) $\frac{7}{72}$**. The idea.** A never-decreasing sequence is completely determined by which values appear and how many times — that is, by a multiset of size 4 drawn from $\{1, \dots, 6\}$. Once you know the multiset there is exactly one way to write it in non-decreasing order. So count multisets with stars and bars.

Work it out. A multiset of size 4 from 6 values corresponds to an arrangement of 4 stars and 5 bars (the bars separate the six value-bins): $\binom{9}{4} = 126$ ways. All $6^4 = 1296$ ordered outcomes are equally likely, so the probability is $\frac{126}{1296} = \frac{7}{72}$.

Check. Count by the number of distinct values: all four distinct $\binom{6}{4} = 15$; three distinct values with one repeated $\binom{6}{3} \cdot 3 = 60$; two distinct values $\binom{6}{2} \cdot 3 = 45$ (patterns 3+1, 2+2, 1+3); one value 6. Total 126. The answer $\frac{1}{24} = \frac{1}{4!}$ wrongly assumes the four rolls are always different; $\frac{5}{432}$ counts only strictly increasing rolls.

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9. Answer: (D) 400**. The idea.** Whether a shares a factor with 30 depends only on the remainder of a on division by 30, so the good numerators repeat in blocks of 30. Within the first block, pair a with $30 - a$: any common factor of 30 and one of them also divides the other, so the good numerators pair off with sum 30. Every later block is the first block shifted up by a multiple of 30.

Work it out. The numerators satisfy $1 \le a \le 299$. In the first block, the numerators coprime to $30 = 2 \cdot 3 \cdot 5$ are $1, 7, 11, 13, 17, 19, 23, 29$: eight of them, in four pairs summing to 30, so those fractions add to 4. In the block $30k < a < 30k + 30$ the numerators are those same eight numbers increased by $30k$, so each fraction is increased by k and the block sums to $4 + 8k$. Adding the blocks $k = 0, 1, \ldots, 9$: $10 \cdot 4 + 8(0 + 1 + \cdots + 9) = 40 + 360 = 400$.

Check. Symmetry gives the same total quickly: the fractions come in pairs $\frac{a}{30}$ and $10 - \frac{a}{30} = \frac{300-a}{30}$ (again coprime to 30 together), there are 80 fractions, so 40 pairs each summing to 10: total 400. The distractor 80 is the number of fractions, 360 forgets the 4 inside each block, and 200 halves the total.

10. Answer: (A) $\frac{125}{48}$**. The idea.** The centre O lies on the line through M perpendicular to $\overline{AB}$ (a radius is perpendicular to the tangent at the point of contact), and it is equally far from M and C. Coordinates turn these two facts into one linear equation.

Work it out. Put $C = (0, 0)$, $A = (6, 0)$, $B = (0, 8)$, so $AB = 10$ and $M = (3, 4)$. The direction of $\overline{AB}$ is $(-6, 8)$, so a perpendicular direction is $(4, 3)$, and $O = (3 + 4t, 4 + 3t)$ for some t, with radius $OM = 5|t|$. Setting $OC^2 = OM^2$: $(3 + 4t)^2 + (4 + 3t)^2 = 25t^2$, i.e. $25 + 48t + 25t^2 = 25t^2$, so $t = -\frac{25}{48}$. The radius is $5 \cdot \frac{25}{48} = \frac{125}{48}$.

Check. A second route: $CM = 5$ (the midpoint of the hypotenuse is equidistant from all three vertices), and $\triangle CMB$ is isosceles with $MC = MB = 5$. The chord $\overline{CM}$ meets the tangent line AB at angle $\angle CMB = 180^\circ - 2\angle B$, and a chord of length c making angle θ with the tangent at its endpoint has $c = 2r \sin\theta$. Here $\sin\angle CMB = \sin 2B = 2 \cdot \frac{6}{10} \cdot \frac{8}{10} = \frac{24}{25}$, so $r = \frac{5}{2 \cdot 24/25} = \frac{125}{48}$. Using $\sin B = \frac{4}{5}$ in place of $\sin 2B$ gives $\frac{25}{8}$, and using $\sin A = \frac{3}{5}$ gives $\frac{25}{6}$.

11. Answer: (A) 41**. The idea.** Classify a climb of n steps by how it ends. If it ends with a 1-stride, what came before is any valid climb of $n - 1$ steps. If it ends with a 2-stride, the stride before it (if any) must be a 1-stride, so it ends in "1, 2" and what came before is any valid climb of $n - 3$ steps — or the whole climb is the single stride "2" when $n = 2$. So $a_n = a_{n-1} + a_{n-3}$, with $a_0 = 1$ (the empty climb), $a_1 = 1$, $a_2 = 2$.

Work it out. $a_3 = a_2 + a_0 = 3$, $a_4 = 3 + 1 = 4$, $a_5 = 4 + 2 = 6$, $a_6 = 6 + 3 = 9$, $a_7 = 9 + 4 = 13$, $a_8 = 13 + 6 = 19$, $a_9 = 19 + 9 = 28$, $a_{10} = 28 + 13 = 41$.

Check. Count by the number m of 2-strides. There are $10 - 2m$ one-strides, making $11 - 2m$ gaps (including the ends), and the 2-strides go in distinct gaps: $\sum_m \binom{11-2m}{m} = \binom{11}{0} + \binom{9}{1} + \binom{7}{2} + \binom{5}{3} + \binom{3}{4} = 1 + 9 + 21 + 10 + 0 = 41$. Without the restriction the count would be the Fibonacci number 89.

12. Answer: (A) 525**. The idea.** Two separate steps. First count all level numbers by grouping on the common sum $s = a + b = c + d$. Then find which level numbers are multiples of 11: the test for 11 says $11 \mid (a + c) - (b + d)$, and together with $a + b = c + d$ a parity argument pins them down completely.

Work it out. Let $N(s)$ be the number of digit pairs (x, y), both from 0 to 9, with $x + y = s$: $N(s) = s + 1$ for $0 \le s \le 9$ and $N(s) = 19 - s$ for $9 \le s \le 18$. The pair (c, d) can be any of

these, while (a, b) must avoid $a = 0$, which removes one pair exactly when $s \le 9$. So the level count is $\sum_s N(s)^2 - \sum_{s=0}^{9} N(s) = (2(1^2 + \cdots + 9^2) + 10^2) - 55 = 670 - 55 = 615$. Now let $D = (a + c) - (b + d)$, so $|D| \le 17$ and $11 \mid D$ means $D \in \{-11, 0, 11\}$. Since $a + b - c - d = 0$, adding gives $D = 2(a - d)$, which is even; so $D = \pm 11$ is impossible and $D = 0$, i.e. $a = d$, and then $b = c$. The level multiples of 11 are exactly the palindromes $\overline{abba}$, all of which are level: $9 \cdot 10 = 90$ of them. The answer is $615 - 90 = 525$.

Check. 2413: $2-4+1-3 = -4$, not a multiple of 11, so it counts; $5005 = 11 \cdot 455$ is a palindrome and does not. Answer 615 forgets the 11 condition, 670 also lets $a = 0$, 580 $= 670 - 90$ makes both the leading-digit slip and the subtraction, and 606 removes only the repdigits $\overline{aaaa}$ instead of all palindromes.

13. Answer: (A) $2\sqrt{10}$**. The idea.** Parallel lines turn the angle bisector into an isosceles triangle. Since $\overline{AB} \parallel \overline{DC}$, the alternate angles $\angle BPC$ and $\angle PCD$ are equal. The bisector condition says $\angle BPC = \angle CPD$. So $\angle CPD = \angle PCD$, and triangle PDC has two equal base angles at P and C, which forces $DP = DC$.

Work it out. $DP = DC = AB = 10$. In right triangle DAP, $AD = 6$, so $AP = \sqrt{10^2 - 6^2} = 8$. Then $PB = 10 - 8 = 2$, and in right triangle PBC, $PC = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$.

Check. With $A = (0, 0)$, $P = (8, 0)$, $C = (10, 6)$, $D = (0, 6)$: $\tan \angle BPC = 6/2 = 3$, and $\angle BPD = 180^\circ - \angle APD$ with $\tan \angle APD = 6/8 = 3/4$, so $\tan \angle BPD = -3/4$. Doubling: $\tan(2\angle BPC) = \frac{2 \cdot 3}{1-9} = -\frac{3}{4}$, matching. (Taking P as the midpoint gives $\sqrt{61}$, and mixing up AP with PB gives 10 — both traps.)

14. Answer: (B) $2\sqrt{13}$**. The idea.** Reflections straighten a broken path. Reflect the camp $K = (5, 1)$ in the road to get $K_1 = (1, 5)$ and in the river to get $K_2 = (5, -1)$. For any road point R and river point S, $KR = K_1R$ and $SK = SK_2$, so the walk has the same length as the path $K_1 \to R \to S \to K_2$, which is at least the straight distance K_1K_2.

Work it out. $K_1K_2 = \sqrt{(5-1)^2 + (-1-5)^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. The bound is reached: the segment from $(1, 5)$ to $(5, -1)$ is $(1 + 4t,\ 5 - 6t)$; it meets $y = x$ at $t = \frac{2}{5}$, the road point $(2.6, 2.6)$, and later meets $y = 0$ at $t = \frac{5}{6}$, the river point $(\frac{13}{3}, 0)$, in the right order. Walking camp $\to (2.6, 2.6) \to (\frac{13}{3}, 0) \to$ camp has length exactly $2\sqrt{13}$.

Check. The two reflections together rotate the camp by twice the 45° angle between road and river, i.e. by 90° about the origin, so $K_1K_2 = \sqrt{2} \cdot OK = \sqrt{2} \cdot \sqrt{26} = 2\sqrt{13}$. The tempting $2\sqrt{26}$ (twice the distance to the crossing point) is the length of the walk through the origin, which is not optimal. A direct walk via the nearest points, $(3, 3)$ and $(5, 0)$, is $2\sqrt{2} + \sqrt{13} + 1 \approx 7.43 > 2\sqrt{13} \approx 7.21$.

15. Answer: (C) 52**. The idea.** Count an isosceles triangle by its apex (the vertex where the two equal sides meet): the other two vertices must be placed symmetrically about the diameter through the apex. The only overcount comes from equilateral triangles, which have three apexes.

Work it out. Fix the apex V. The pair of base vertices is k steps clockwise and k steps counterclockwise from V, for $k = 1, 2, 3, 4, 5$ ($k = 6$ would give the same opposite vertex twice). That is $12 \cdot 5 = 60$ (apex, triangle) pairs. A non-equilateral isosceles triangle has exactly one apex, so it is counted once. An equilateral triangle ($k = 4$) is counted three times, once from each vertex. There are $12/3 = 4$ equilateral triangles, each counted 2 times too many, so the number of isosceles triangles is $60 - 2 \cdot 4 = 52$.

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Check. Classify each triangle by its three arc-steps (p, q, r) with $p + q + r = 12$: the triangle is isosceles when two steps are equal. Choices $\{p, p, 12 - 2p\}$ with $p = 1, \ldots, 5$, $p \neq 4$ give 4 shapes, each appearing in 12 positions (48 triangles), and $\{4, 4, 4\}$ gives 4 more: 52.

16. Answer: (B) 9**. The idea.** Each left side is a product with a missing constant: $ab + 2a + b$ is $(a + 1)(b + 2)$ minus 2. The shifts must match across equations: a always appears as $a + 1$, b as $b + 2$, c as $c + 3$. Then multiplying the three equations gives the product of all three shifted numbers.

Work it out. Adding the constants: $(a+1)(b+2) = 18$, $(b+2)(c+3) = bc + 3b + 2c + 6 = 36$ and $(c+3)(a+1) = ca + c + 3a + 3 = 18$. Let $X = a + 1, Y = b + 2, Z = c + 3$, all positive. Then $(XYZ)^2 = 18 \cdot 36 \cdot 18 = 108^2$, so $XYZ = 108$. Dividing, $X = \frac{108}{36} = 3$, $Y = \frac{108}{18} = 6$, $Z = \frac{108}{18} = 6$. So $a = 2$, $b = 4$, $c = 3$ and $a + b + c = 9$.

Check. $ab+2a+b = 8+4+4 = 16$, $bc+3b+2c = 12+12+6 = 30$, $ca+c+3a = 6+3+6 = 15$. The distractor 15 is $X + Y + Z$, forgetting to undo the shifts. (The negative root $XYZ = -108$ would make $a = -4$, which is not positive.)

17. Answer: (A) $\frac{18}{5}$**. The idea.** Put the square on coordinates and write both circles as equations. Subtracting one equation from the other cancels the squared terms and leaves a straight line, the line through both meeting points (the common chord). Intersect that line with either circle.

Work it out. Take $A = (0, 0)$, $B = (6, 0)$, $C = (6, 6)$, $D = (0, 6)$. The first circle is $x^2 + y^2 = 36$. The second has center $(3, 6)$, the midpoint of CD, and radius 3: $(x - 3)^2 + (y - 6)^2 = 9$, which expands to $x^2 + y^2 - 6x - 12y + 36 = 0$. Substituting $x^2 + y^2 = 36$ gives $72 - 6x - 12y = 0$, that is, $x + 2y = 12$. Then $x = 12 - 2y$ and $(12 - 2y)^2 + y^2 = 36$, so $5y^2 - 48y + 108 = 0$, which factors as $(y - 6)(5y - 18) = 0$. The root $y = 6$ gives $x = 0$, the point D; the other gives $y = \frac{18}{5}$, $x = \frac{24}{5}$. The distance from P to side AB (the x-axis) is $\frac{18}{5}$.

Check. $P = (\frac{24}{5}, \frac{18}{5})$: $(\frac{24}{5})^2 + (\frac{18}{5})^2 = \frac{576+324}{25} = 36$, and $(\frac{24}{5} - 3)^2 + (\frac{18}{5} - 6)^2 = \frac{81}{25} + \frac{144}{25} = 9$. Also $\angle DPC = 90°$ since CD is a diameter, and indeed $\vec{PD} \cdot \vec{PC} = (-\frac{24}{5})(\frac{6}{5}) + (\frac{12}{5})(\frac{12}{5}) = 0$. The choice $\frac{24}{5}$ is the distance to side AD, and 6 is the height of D, the other meeting point.

18. Answer: (D) $16\sqrt{3}$**. The idea.** $ABPC$ is a cyclic quadrilateral, so $\angle BPC = 180° - \angle BAC = 120°$. That angle lets us find the side of the triangle from PB and PC, and then the quadrilateral splits into triangle ABC plus triangle BPC.

Work it out. In triangle BPC, with the 120° angle at P (law of cosines, or drop a perpendicular from B to line PC extended, making a 30-60-90 triangle with legs $\frac{3}{2}$ and $\frac{3\sqrt{3}}{2}$): $BC^2 = 3^2 + 5^2 + 3 \cdot 5 = 49$, so $BC = 7$. Then $[ABC] = \frac{\sqrt{3}}{4} \cdot 49 = \frac{49\sqrt{3}}{4}$. Triangle BPC has base $PC = 5$ and height $\frac{3\sqrt{3}}{2}$ from B, so $[BPC] = \frac{15\sqrt{3}}{4}$. Adding, $[ABPC] = \frac{64\sqrt{3}}{4} = 16\sqrt{3}$.

Check. Rotate triangle ABP by 60° about A so that B goes to C; P goes to a point P' with $CP' = 3$. Since $\angle ABP + \angle ACP = 180°$, P' lands on line PC extended, so $PP' = 8$ and APP' is equilateral. The rotation preserves area, so $[ABPC] = [APP'] = \frac{\sqrt{3}}{4} \cdot 64 = 16\sqrt{3}$. (This also shows $PA = PB + PC = 8$.)

19. Answer: (E) 65**. The idea.** Split the modulus into coprime pieces: $504 = 8 \cdot 9 \cdot 7$, so 504 divides $n^2 - 1$ exactly when 8, 9 and 7 each do. Count the good remainders for each piece separately. By

the Chinese Remainder Theorem every choice of one remainder mod 8, one mod 9 and one mod 7 is the remainder of exactly one number mod 504, so the counts multiply. The trap is the power of 2: there $n^2 \equiv 1$ has more than the two obvious solutions ± 1.

Work it out. Write $n^2 - 1 = (n-1)(n+1)$. Mod 8: if n is odd, $n-1$ and $n+1$ are consecutive even numbers, so one of them is a multiple of 4 and the product is a multiple of 8; if n is even the product is odd. So all four odd remainders $1, 3, 5, 7$ work. Mod 9: the factors differ by 2, so 3 cannot divide both, and 9 must divide one of them: $n \equiv 1$ or $8 \pmod 9$, two remainders. Mod 7 (7 is prime): $n \equiv 1$ or $6 \pmod 7$, two remainders. So $4 \cdot 2 \cdot 2 = 16$ remainders mod 504 work. Since $2016 = 4 \cdot 504$, the numbers 1 to 2016 contain $4 \cdot 16 = 64$ good values. The last ten numbers $2017, \ldots, 2026$ have remainders $1, \ldots, 10$ mod 504, and among $1, \ldots, 10$ only 1 is odd and $\equiv \pm 1$ modulo both 9 and 7. So $n = 2017$ adds one more: the total is 65.

Check. The 16 good remainders are $1, 55, 71, 125, 127, 181, 197, 251, 253, 307, 323, 377, 379, 433, 449, 503$, and they pair up as r and $504 - r$; for instance $55^2 - 1 = 3024 = 6 \cdot 504$. Using only $n \equiv \pm 1 \pmod{504}$ gives 9; allowing only ± 1 mod 8 gives 8 per block and 33; forgetting 2017 gives 64.

20. Answer: (E) $\frac{13}{100}$**. The idea.** Measure lengths in metres. For two uniform cuts, the three piece lengths (x, y, z) with $x + y + z = 1$ are uniformly spread over the triangle of all such triples, so probabilities are area ratios, and a condition like "every piece at least c" cuts out a smaller similar triangle. The triangle inequality says each piece is less than $\frac{1}{2}$.

Work it out. Write $x = \frac{1}{5} + x'$ and so on. "Every piece $\geq \frac{1}{5}$" means $x', y', z' \geq 0$ with $x' + y' + z' = \frac{2}{5}$: a triangle similar to the full one, scaled by $\frac{2}{5}$, of relative area $\frac{4}{25}$. Inside it we must remove the points where some piece is $\geq \frac{1}{2}$, i.e. $x' \geq \frac{3}{10}$: a corner triangle scaled by $\frac{2}{5} - \frac{3}{10} = \frac{1}{10}$, of relative area $\frac{1}{100}$. Two pieces cannot both be $\geq \frac{1}{2}$, so the three corners are disjoint. The probability is $\frac{4}{25} - \frac{3}{100} = \frac{13}{100}$.

Check. In the square of cut positions $u < v$ (then double), pieces are u, $v - u$, $1 - v$. The good region is the hexagon $\frac{1}{5} \leq u \leq \frac{1}{2}$, $\frac{1}{2} \leq v \leq \frac{4}{5}$, $\frac{1}{5} \leq v - u \leq \frac{1}{2}$: the $\frac{3}{10}$-by-$\frac{3}{10}$ square minus the corner triangles cut by $v - u \geq \frac{1}{5}$ and $v - u \leq \frac{1}{2}$ (legs $\frac{1}{10}$ and $\frac{1}{5}$), area $\frac{9}{100} - \frac{1}{200} - \frac{4}{200} = \frac{13}{200}$; doubled, $\frac{13}{100}$. Forgetting to double gives $\frac{13}{200}$, and stopping at the $\frac{3}{10}$-square gives $\frac{9}{100}$.

21. Answer: (C) $8 - 4\sqrt{3}$**. The idea.** Picture it: the graph of $y = |x^2 - 8x + 12| = |(x-2)(x-6)|$ is the parabola with its part between $x = 2$ and $x = 6$ flipped up into an arch, and $y = mx$ is a line through the origin. The outer branches always give two intersections; the arch gives two, one or none. Three solutions means the line is tangent to the arch.

Work it out. For $m > 0$ there are no solutions with $x \leq 0$ ($mx \leq 0$ but the left side is positive at $x = 0$ and nonnegative elsewhere). Outer part ($x \leq 2$ or $x \geq 6$): $x^2 - (8+m)x + 12 = 0$. At $x = 2$ and $x = 6$ this quadratic equals $-2m < 0$ and $-6m < 0$, so it has one root below 2 and one above 6: always two solutions. Arch ($2 \leq x \leq 6$): $-(x^2 - 8x + 12) = mx$, i.e. $x^2 + (m-8)x + 12 = 0$. Its values at $x = 2$ and $x = 6$ are $2m > 0$ and $6m > 0$, and its vertex $\frac{8-m}{2}$ lies in $(2, 6)$ for $0 < m < 4$, so its real roots (if any) lie in $[2, 6]$. The discriminant $(m-8)^2 - 48$ is zero at $m = 8 - 4\sqrt{3} \approx 1.07$: tangency, one arch solution, three in all. For smaller m there are 4 solutions and for larger m (up to $8 + 4\sqrt{3}$, whose tangency point $x = -2\sqrt{3}$ is not on the arch) only 2. So $m = 8 - 4\sqrt{3}$.

Check. The tangency point is $x = \frac{8-m}{2} = 2\sqrt{3} \approx 3.46$, inside $[2, 6]$, and there $-(x^2 - 8x + 12) = -12 + 16\sqrt{3} - 12 = 16\sqrt{3} - 24$ while $mx = (8 - 4\sqrt{3}) \cdot 2\sqrt{3} = 16\sqrt{3} - 24$. They agree.

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22. Answer: (C) 8**. The idea.** Everything is on the central vertical axis. The big centres are at height 6 and at horizontal distance d from the axis, where d is the circumradius of the equilateral triangle of side 12 they form: $d = \frac{12}{\sqrt{3}} = 4\sqrt{3}$, so $d^2 = 48$. A ball of radius r centred on the axis at height z touches a big ball when $48 + (z-6)^2 = (6+r)^2$.

Work it out. Ball X: $z = r$, so $48 + (r-6)^2 = (r+6)^2$, i.e. $48 = 24r$, $r = 2$; its top is at height 4. Ball Y: $z = 4 + r$, so $48 + (r-2)^2 = (r+6)^2$, i.e. $52 - 4r = 12r + 36$, $r = 1$; centre at 5, top at 6. Ball Z: $z = 6 + r$, so $48 + r^2 = (r+6)^2$, i.e. $48 = 12r + 36$, $r = 1$; centre at 7, top at 8.

Check. The pattern is symmetric about the plane of the big centres (height 6): Y and Z are mirror images, and one more ball would be the mirror image of X (radius 2, from 8 to 12). For X, the tangent-length fact for two touching balls on a table, $2\sqrt{6r} = 4\sqrt{3}$, again gives $r = 2$.

23. Answer: (B) 751**. The idea.** Counting sequences that contain a "66" directly runs into overlaps (a sequence can contain several such pairs, as in 6 6 6 1 2). It is easier to count the complement, sequences with no two consecutive 6s, by building them one roll at a time and keeping track only of whether the last roll was a 6.

Work it out. Among sequences of length n with no two consecutive 6s, let a_n be the number ending in a roll other than 6 and b_n the number ending in a 6. Any good sequence can be followed by one of the 5 non-6 faces, but a 6 may only follow a sequence ending in a non-6. So $a_{n+1} = 5(a_n + b_n)$ and $b_{n+1} = a_n$. Starting from $a_1 = 5$, $b_1 = 1$: $(a_2, b_2) = (30, 5)$, $(a_3, b_3) = (175, 30)$, $(a_4, b_4) = (1025, 175)$, $(a_5, b_5) = (6000, 1025)$. So $6000 + 1025 = 7025$ sequences avoid a "66", and $7776 - 7025 = 751$ contain one.

Check. Inclusion-exclusion on the four places $i = 1, 2, 3, 4$ where rolls $i, i+1$ could both be 6. Single places: $4 \cdot 6^3 = 864$. Pairs of places: three overlapping pairs force three 6s (6^2 each) and three disjoint pairs force four 6s (6 each), total $3 \cdot 36 + 3 \cdot 6 = 126$. Triples: $\{1, 2, 3\}$ and $\{2, 3, 4\}$ force four 6s (6 each), the other two force all five (1 each), total 14. All four: 1. So $864 - 126 + 14 - 1 = 751$. The choice 864 forgets the overlaps entirely.

24. Answer: (D) 53**. The idea.** Let $h(n)$ be the number of ways to pay n cents. Look at how many 1-cent coins are used: the rest of the payment uses only even coins, and halving every coin turns it into a payment of the same kind. This gives $h(2m+1) = h(m)$ (an odd amount needs exactly one 1-cent coin) and $h(2m) = h(m) + h(m-1)$ (no 1-cent coins, or two of them), with $h(0) = 1$.

Work it out. Track the pair $(h(n), h(n-1))$ while reading the binary digits of n from the left. If $n = 2m+1$, the new pair is $(h(m),\ h(m)+h(m-1))$; if $n = 2m$, it is $(h(m)+h(m-1),\ h(m-1))$. In binary $2026 = 11111101010_2$, and the prefixes are $1, 3, 7, 15, 31, 63, 126, 253, 506, 1013, 2026$. Starting from $(h(1), h(0)) = (1, 1)$: the five odd steps give $(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)$ at $n = 63$; then 126 (even): $(7, 6)$; 253 (odd): $(7, 13)$; 506 (even): $(20, 13)$; 1013 (odd): $(20, 33)$; 2026 (even): $(53, 33)$. So $h(2026) = 53$.

Check. Small cases by hand: $h(2) = 2$ (2 or $1+1$), $h(3) = 1$ ($2+1$), $h(4) = 3$ (4, $2+2$, $2+1+1$), matching the recursion. The distractor 33 is $h(2025)$, and 20 is $h(1013)$, the values met along the way.

25. Answer: (C) 1080**. The idea.** Only the parity of the number of times each line is toggled matters. If R is the set of rows and C the set of columns toggled an odd number of times, lamp (i, j) is on exactly when one, but not both, of "row $i \in R$" and "column $j \in C$" holds. Every reachable pattern comes from exactly two choices of (R, C): a pair and its complement.

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Work it out. Why exactly two: if (R, C) and (R', C') give the same pattern, then for every i, j we have $[i \in R] + [j \in C] \equiv [i \in R'] + [j \in C'] \pmod 2$, so $[i \in R] - [i \in R']$ is the same for every row and equals $[j \in C'] - [j \in C]$ for every column: either $R' = R, C' = C$ or both are complemented. With $|R| = a$ and $|C| = b$, the number of lit lamps is $a(6-b) + (6-a)b = 6a + 6b - 2ab$, and $18 - (6a + 6b - 2ab) = 2(a-3)(b-3)$. So exactly 18 lamps are on iff $a = 3$ or $b = 3$. Count pairs: $a = 3$ gives $\binom{6}{3} \cdot 2^6 = 1280$; $b = 3$ likewise 1280; both gives $20 \cdot 20 = 400$. So $1280 + 1280 - 400 = 2160$ pairs, and $\frac{2160}{2} = 1080$ patterns.

Check. Complementing both R and C sends $a \to 6 - a$ and $b \to 6 - b$, which keeps "$a = 3$ or $b = 3$", so the pairs really do split into 1080 complementary couples. Forgetting that couple gives 2160; counting only $a = b = 3$ gives 400. A computer search over all 2^{12} sets of moves finds 1080 reachable patterns with 18 lamps on.

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