MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

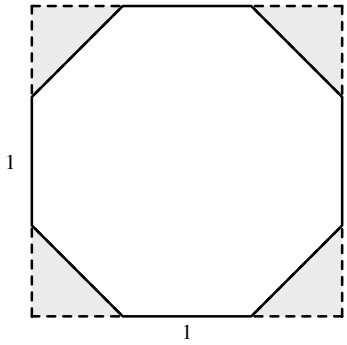
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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Identical plastic cups nest inside one another, so a stack of cups is only a little taller than a single cup. A stack of some number of these cups is 30 cm tall. A stack with twice as many cups is 46 cm tall, and a stack with 5 more cups than the first stack is 40 cm tall. How tall, in centimeters, is one cup?

(A) 2 **(B)** $\frac{15}{4}$ **(C)** 10 **(D)** 14 **(E)** 16

2. A square sheet of paper has side length 1. Four straight cuts are made, one across each corner of the square, and each cut removes a triangle from that corner. The piece that remains is a regular octagon (all eight sides equal and all eight angles equal). What is the area of this octagon?



(A) $3 - 2\sqrt{2}$ **(B)** $\sqrt{2} - 1$ **(C)** $2 - \sqrt{2}$ **(D)** $\frac{3}{4}$ **(E)** $2\sqrt{2} - 2$

3. Three fair six-sided dice, with faces numbered 1 through 6, are rolled. What is the probability that the three numbers rolled are all different and can be put in an order that forms an arithmetic sequence? (For example, the rolls 5, 1, 3 qualify, because 1, 3, 5 is an arithmetic sequence.)

(A) $\frac{1}{6}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{3}{10}$ **(E)** $\frac{1}{3}$

4. A grain of rice is placed on the first square of a very long board, two grains on the second square, four on the third, and so on, each square holding twice as many grains as the square before it. The board has 2027 squares. All the rice is then shared among 7 people, each getting the same whole number of grains, as many as possible. How many grains are left over? (In other words, what is the remainder when $1 + 2 + 2^2 + \cdots + 2^{2026}$ is divided by 7?)

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 6

5. Each of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 is used exactly once to write three three-digit numbers. What is the smallest possible sum of the three numbers?

(A) 747 **(B)** 774 **(C)** 855 **(D)** 1368 **(E)** 2556

6. A train travels along a straight track at a constant speed. A jogger runs on a path beside the track at 5 metres per second, heading toward the oncoming train. From the moment the front of the train is level with the jogger until the moment the back of the train is level with the jogger takes 8 seconds. Later, the same train crosses a bridge 400 metres long: from the moment its front reaches the start of the bridge until the moment its back leaves the end of the bridge takes 30 seconds. How long is the train, in metres?

(A) 160 **(B)** 180 **(C)** 200 **(D)** 220 **(E)** 240

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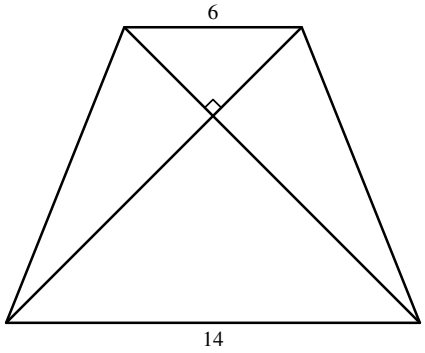
7. A six-digit number is formed by writing the digits $1, 2, 3, 4, 5, 6$ in some order, each used exactly once. (For example, 254316 is one such number.) How many of the 720 numbers formed this way are divisible by 24?

(A) 84 **(B)** 90 **(C)** 96 **(D)** 120 **(E)** 192

8. Kai cycles up a hill and then back down the same road to his starting point. He rides at one constant speed going up and at another constant speed going down, and his downhill speed is 20 km/h faster than his uphill speed. His average speed for the whole round trip (total distance divided by total time) is 15 km/h. What was his uphill speed, in km/h?

(A) 5 **(B)** $\frac{15}{2}$ **(C)** 10 **(D)** 12 **(E)** 15

9. An isosceles trapezoid has parallel sides of lengths 6 and 14, and its two diagonals are perpendicular to each other. What is the area of the trapezoid?



(A) 60 **(B)** 84 **(C)** 100 **(D)** 120 **(E)** 140

10. The triangular numbers are $T_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, so $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, and so on. How many of the first 100 triangular numbers, $T_1, T_2, \ldots, T_{100}$, are multiples of 6?

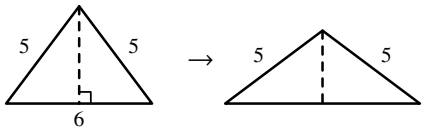
(A) 16 **(B)** 25 **(C)** 32 **(D)** 33 **(E)** 66

11. For some real numbers k, the two equations $x^2 + kx + 1 = 0$ and $x^2 + x + k = 0$ have at least one real solution x in common. What is the sum of all such real numbers k?

(A) -3 **(B)** -2 **(C)** -1 **(D)** 0 **(E)** 1

12. An isosceles triangle has sides 5, 5 and 6. It is cut along its altitude to the base into two right triangles, and the two pieces are glued back together along their other legs (without overlapping) to form a new isosceles triangle whose two equal sides still have length 5. By how much is the radius of the inscribed circle of the new triangle smaller than the radius of the inscribed circle of the original triangle?

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(A) $\frac{1}{12}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

13. How many different right triangles with all three side lengths whole numbers have a leg of length 24? (Congruent triangles count as the same triangle.)

(A) 6 **(B)** 7 **(C)** 8 **(D)** 14 **(E)** 15

14. A seven-digit number is formed by using each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, in some order. For example, 1234567 is one such number. How many of these 5040 numbers are divisible by 11?

(A) 144 **(B)** 288 **(C)** 480 **(D)** 576 **(E)** 720

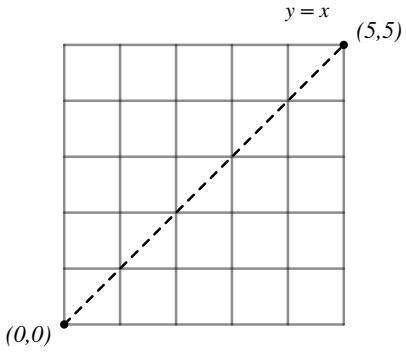
15. Real numbers x and y satisfy $2^x + 3^y = 11$ and $4^x + 9^y = 65$. What is the value of 3^{xy}?

(A) 7 **(B)** 14 **(C)** 28 **(D)** 49 **(E)** 196

16. Twelve points are marked on the graph of $y = x^3 - 15x^2 + 60x$: the points whose x-coordinates are $1, 2, 3, \ldots, 12$. How many sets of three of these twelve points lie on a single straight line?

(A) 12 **(B)** 14 **(C)** 19 **(D)** 24 **(E)** 72

17. A path from $(0, 0)$ to $(5, 5)$ in the coordinate plane is made of 10 unit steps, each one unit to the right or one unit up. How many such paths touch the line $y = x$ only at their two endpoints $(0, 0)$ and $(5, 5)$, and at no lattice point in between?



(A) 28 **(B)** 42 **(C)** 70 **(D)** 84 **(E)** 252

18. For a positive integer n, let $f(n) = \frac{n}{2}$ if n is even and $f(n) = n + 5$ if n is odd. Starting from a positive integer n, form the sequence $n, f(n), f(f(n)), \ldots$. For how many starting values n with $1 \le n \le 100$ does the number 1 appear in this sequence?

(A) 80 **(B)** 81 **(C)** 90 **(D)** 98 **(E)** 100

19. For how many integers n with $1 \le n \le 2026$ do n and $n + 2$ have the same number of 1s in their base-2 representations?

(A) 253 **(B)** 254 **(C)** 505 **(D)** 506 **(E)** 507

20. Alice flips 6 fair coins and Bob flips 4 fair coins, all independently. What is the probability that Alice gets strictly more heads than Bob?

(A) $\frac{11}{64}$ **(B)** $\frac{105}{512}$ **(C)** $\frac{193}{512}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{319}{512}$

21. Six players play a round-robin tournament: every two players play each other exactly once, and every game has a winner (there are no ties). A set of three players is called a *cycle* if each of the three beats exactly one of the other two. What is the largest possible number of cycles in such a tournament?

(A) 5 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 20

22. Eight children, all of different heights, line up in a row in a uniformly random order. A child can be *seen from the left end* if they are taller than every child to their left, and *seen from the right end* if they are taller than every child to their right. (The leftmost child can always be seen from the left end, and the rightmost child from the right end.) What is the expected number of children who can be seen from at least one end?

(A) $\frac{761}{280}$ **(B)** $\frac{481}{140}$ **(C)** 4 **(D)** $\frac{621}{140}$ **(E)** $\frac{761}{140}$

23. A square room $ABCD$ with side length 1 has mirrors along all four walls, with A and C at opposite corners. A laser beam is fired from corner A into the interior of the room, not along a wall, in a straight line. Whenever the beam hits a wall at a point that is not a corner, it reflects (angle of incidence equals angle of reflection) and continues; this counts as one bounce. As soon as the beam reaches any corner, it stops. For how many different firing directions does the beam stop at corner C after at most 20 bounces? (A direction that runs straight to C with 0 bounces counts.)

(A) 10 **(B)** 45 **(C)** 54 **(D)** 55 **(E)** 149

24. Seven friends hold a gift swap. Each writes their name on a slip, the slips are shuffled, and each friend draws one slip; the friend then buys a gift for the person named. If anyone draws their own name, all slips are returned and the drawing is done again, until nobody has their own name. Thus every assignment in which no one draws their own name is equally likely. What is the probability that the final assignment has no two friends who drew each other's names?

(A) $\frac{19}{84}$ **(B)** $\frac{119}{309}$ **(C)** $\frac{40}{103}$ **(D)** $\frac{155}{309}$ **(E)** $\frac{190}{309}$

25. For a positive integer n, let $S(n)$ denote the sum of the decimal digits of n. For example, $S(2026) = 10$. How many positive integers n less than 10000 satisfy $S(11n) = 2\,S(n)$?

(A) 624 **(B)** 2145 **(C)** 2529 **(D)** 2530 **(E)** 3024

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Answer Key — AMC 10 Practice Contest 3

1. (E) 16

2. (E) $2\sqrt{2} - 2$

3. (A) $\frac{1}{6}$

4. (C) 3

5. (B) 774

6. (C) 200

7. (A) 84

8. (C) 10

9. (C) 100

10. (D) 33

11. (B) -2

12. (B) $\frac{1}{6}$

13. (B) 7

14. (D) 576

15. (D) 49

16. (A) 12

17. (A) 28

18. (A) 80

19. (E) 507

20. (E) $\frac{319}{512}$

21. (B) 8

22. (D) $\frac{621}{140}$

23. (D) 55

24. (E) $\frac{190}{309}$

25. (C) 2529

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Solutions — AMC 10 Practice Contest 3

1. Answer: (E) 16**. The idea.** Height is not proportional to the number of cups. Every cup adds the same small step (the part of its rim that sticks out above the cup below), and the bottom cup also contributes its full body. So a stack of k cups has height $a + bk$, where b is the step and $a + b$ is the height of one cup.

Work it out. Let the first stack have n cups. Then $a+bn = 30$, $a+2bn = 46$ and $a+b(n+5) = 40$. Subtracting the first equation from the third gives $5b = 10$, so $b = 2$ cm. Subtracting the first from the second gives $bn = 16$, so $n = 8$ and $a = 30 - 16 = 14$. One cup is $a + b = 16$ cm tall.

Check. The heights $14 + 2k$ give 30 for 8 cups, 46 for 16 cups and 40 for 13 cups, as required, and 16 for one cup. Treating height as proportional gives $\frac{30}{8} = \frac{15}{4}$; the step alone is 2, the base part alone is 14, and the difference $40 - 30 = 10$ is five steps, not one cup.

2. Answer: (E) $2\sqrt{2} - 2$**. The idea.** Every angle of a regular octagon is 135°, so each cut meets the square's sides at 45° and every cut-off corner is an isosceles right triangle. The four triangles are congruent, and a triangle with legs t has a hypotenuse $t\sqrt{2}$, which is one side of the octagon.

Work it out. Each side of the square keeps a middle piece $1-2t$, which is also a side of the octagon. Equal sides give $1 - 2t = t\sqrt{2}$, so $t = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}$. The four corner triangles have total area $4 \cdot \frac{1}{2}t^2 = 2t^2 = 2 \cdot \frac{(2-\sqrt{2})^2}{4} = \frac{6-4\sqrt{2}}{2} = 3-2\sqrt{2}$. The octagon's area is $1-(3-2\sqrt{2}) = 2\sqrt{2}-2 \approx 0.83$.

Check. The octagon's side is $s = t\sqrt{2} = \sqrt{2} - 1$, and a regular octagon of side s has area $2(1 + \sqrt{2})s^2 = 2(1 + \sqrt{2})(3 - 2\sqrt{2}) = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$. The option $\sqrt{2} - 1$ is the side length, not the area, and $3 - 2\sqrt{2}$ is the area cut away, not the area kept.

3. Answer: (A) $\frac{1}{6}$**. The idea.** Only the set of three numbers matters. List the sets of three different numbers from 1 to 6 that are evenly spaced, and then count the orders in which each set can be rolled.

Work it out. A set $\{a, a + d, a + 2d\}$ with $d \geq 1$ must have $a + 2d \leq 6$. With common difference $d = 1$ there are 4 sets, from $\{1, 2, 3\}$ to $\{4, 5, 6\}$. With $d = 2$ there are 2 sets, $\{1, 3, 5\}$ and $\{2, 4, 6\}$. A difference of 3 or more would need a largest number at least $1 + 6 = 7$. That makes 6 sets. The three dice are different, so each set of three different numbers can be rolled in $3! = 6$ orders, giving 36 favourable rolls out of $6^3 = 216$. The probability is $\frac{36}{216} = \frac{1}{6}$.

Check. Letting three equal numbers count as a (constant) arithmetic sequence would add 6 rolls and give $\frac{42}{216} = \frac{7}{36}$, but the problem asks for three different numbers. Dividing by the $6 \cdot 5 \cdot 4 = 120$ rolls with all numbers different, instead of all 216 rolls, gives $\frac{36}{120} = \frac{3}{10}$.

4. Answer: (C) 3**. The idea.** A sum of consecutive powers of 2 starting at 1 is one less than the next power of 2. Then only the remainder of a single power of 2 is needed, and the powers of 2 repeat with period 3 modulo 7 because $2^3 = 8 = 7 + 1$.

Work it out. Doubling the sum $S = 1+2+\cdots+2^{2026}$ gives $2S = 2+4+\cdots+2^{2027}$, and subtracting, $S = 2^{2027} - 1$. The remainders of $2^1, 2^2, 2^3, 2^4, \ldots$ on division by 7 are $2, 4, 1, 2, 4, 1, \ldots$. Since $2027 = 3 \cdot 675 + 2$, we get $2^{2027} = (2^3)^{675} \cdot 2^2$, which leaves remainder $1 \cdot 4 = 4$. So S leaves remainder $4 - 1 = 3$.

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Check. Group the 2027 terms in threes from the start: each group $2^{3k}(1+2+4) = 7 \cdot 2^{3k}$ is a multiple of 7. That uses 2025 terms, up to 2^{2024}, leaving $2^{2025} + 2^{2026}$, with remainders $1 + 2 = 3$. Forgetting the -1 gives 4; treating the last exponent as 2026 instead of 2027 gives $2 - 1 = 1$.

5. Answer: (B) 774**. The idea.** The sum does not depend on how the digits are grouped into numbers, only on which three digits sit in the hundreds places, which three in the tens places and which three in the units places. A digit in the hundreds place counts 100 times, so the smallest digits belong there.

Work it out. The sum equals 100·(sum of hundreds digits)+10·(sum of tens digits)+(sum of units digits). To make it as small as possible, put the three smallest digits 1, 2, 3 in the hundreds places, the next three 4, 5, 6 in the tens places, and 7, 8, 9 in the units places. The sum is $100 \cdot 6 + 10 \cdot 15 + 24 = 600 + 150 + 24 = 774$, for example $147 + 258 + 369 = 774$.

Check. Swapping any digit from a lower-weight place with a smaller one from a higher-weight place only increases the sum, so this is the minimum. Swapping the tens and units groups gives $600+240+15 = 855$, and simply writing $123+456+789 = 1368$ ignores place value. The largest possible sum is 2556.

6. Answer: (C) 200**. The idea.** Measure each event by how far the train must move relative to the thing it passes. Passing the jogger, the train covers its own length, but the jogger comes toward it, so the closing speed is the train's speed plus 5. Crossing the bridge, the front travels the bridge's length plus the train's length.

Work it out. Let the train have length L and speed v. The jogger gives $L = 8(v+5)$, and the bridge gives $L + 400 = 30v$. Substitute: $8v + 40 + 400 = 30v$, so $22v = 440$ and $v = 20$ metres per second. Then $L = 8 \cdot 25 = 200$ metres.

Check. At 20 m/s the train covers $600 = 200+400$ metres in 30 seconds, as needed, and it closes on the jogger at 25 m/s, covering its 200 metres in 8 seconds. Using $v-5$ instead (as if the jogger ran the same way) gives $L = 8(v-5)$, $22v = 360$, and no whole-number length at all; ignoring the jogger's speed gives $L = 8v$ and $L = \frac{1600}{11}$, not among the options.

7. Answer: (A) 84**. The idea.** $24 = 3 \cdot 8$. Every such number has digit sum $1 + 2 + \cdots + 6 = 21$, so all of them are divisible by 3 for free. The question is really about divisibility by 8, which depends only on the last three digits.

Work it out. Write the last three digits as $\overline{abc}$. Since $100a = 96a + 4a$, the number $\overline{abc}$ is a multiple of 8 exactly when $4a + \overline{bc}$ is. So $\overline{bc}$ must be a multiple of 4; the two-digit multiples of 4 with different digits from 1 to 6 are $12, 16, 24, 32, 36, 52, 56, 64$. If $\overline{bc} \equiv 0 \pmod 8$ (that is $16, 24, 32, 56, 64$), then a must be even and unused: 16 allows $a = 2, 4$; 24 allows $a = 6$; 32 allows $a = 4, 6$; 56 allows $a = 2, 4$; 64 allows $a = 2$. That is 8 endings. If $\overline{bc} \equiv 4 \pmod 8$ (that is $12, 36, 52$), then a must be odd and unused: 12 allows $a = 3, 5$; 36 allows $a = 1, 5$; 52 allows $a = 1, 3$. That is 6 more. With 14 valid endings and $3! = 6$ ways to order the other three digits in front, the count is $14 \cdot 6 = 84$.

Check. Stopping at divisibility by 4 (that is, counting multiples of 12) gives $8 \cdot 4! = 192$. A direct check of the endings, for example $416 = 8 \cdot 52$ and $136 = 8 \cdot 17$, but $236 = 8 \cdot 29 + 4$ is not, confirms the parity rule for the hundreds digit.

8. Answer: (C) 10**. The idea.** The average speed is not the average of the two speeds, because

Kai spends much longer going up than coming down. For equal distances the average speed is the total distance over the total time, which is the harmonic mean of the two speeds.

Work it out. Let the uphill speed be u and the road length d. The round trip takes $\frac{d}{u} + \frac{d}{u+20}$ hours to cover $2d$, so $\frac{2}{\frac{1}{u}+\frac{1}{u+20}} = \frac{2u(u+20)}{2u+20} = \frac{u(u+20)}{u+10} = 15$. Then $u^2+20u = 15u+150$, so $u^2+5u-150 = 0$, which factors as $(u+15)(u-10) = 0$. The speed is positive, so $u = 10$ km/h (and he comes down at 30 km/h).

Check. Take a 30 km road: up takes 3 hours, down takes 1 hour, and 60 km in 4 hours is 15 km/h. Averaging the speeds, $\frac{u+(u+20)}{2} = 15$, gives the trap $u = 5$; with $u = 5$ the real average speed is $\frac{5\cdot 25}{15} = \frac{25}{3}$ km/h.

9. Answer: (C) 100**. The idea.** In an isosceles trapezoid the diagonals are mirror images of each other in the axis of symmetry, so they cross on that axis and make equal angles with the bases. If they are perpendicular, each makes a 45° angle with the bases, and the crossing point splits the trapezoid into pieces whose heights are easy to read off.

Work it out. Let the diagonals cross at P. Triangle P forms with the long base has its apex angle at P equal to 90° and is isosceles (by symmetry), so it is an isosceles right triangle. The height of an isosceles right triangle to its hypotenuse is half the hypotenuse, so P lies $\frac{14}{2} = 7$ above the long base. In the same way the triangle on the short base is an isosceles right triangle, and P lies $\frac{6}{2} = 3$ below the short base. The trapezoid's height is $7 + 3 = 10$, and its area is $\frac{6+14}{2} \cdot 10 = 100$.

Check. Use coordinates: long base from $(-7, 0)$ to $(7, 0)$, short base from $(-3, h)$ to $(3, h)$. The diagonal from $(-7, 0)$ to $(3, h)$ has direction $(10, h)$ and the other has direction $(-10, h)$; perpendicular means $-100 + h^2 = 0$, so $h = 10$. The general rule is: in an isosceles trapezoid with perpendicular diagonals the height equals the average of the bases, so the area is the height squared.

10. Answer: (D) 33**. The idea.** T_n is a multiple of 6 exactly when $n(n+1)$ is a multiple of 12, that is, of both 4 and 3. Because n and $n+1$ share no common factor, each of these conditions is a condition on n modulo 4 or 3, and together they pick out certain residues modulo 12.

Work it out. The prime power 4 must divide one of $n, n+1$ entirely (one of them is odd), so $n \equiv 0$ or $3 \pmod 4$. Similarly $3 \mid n(n+1)$ means $n \equiv 0$ or $2 \pmod 3$. Combining the 2×2 choices gives four residues modulo 12: $n \equiv 0, 3, 8, 11 \pmod{12}$. (Check: $T_3 = 6$, $T_8 = 36$, $T_{11} = 66$, $T_{12} = 78$.) Among $1, \ldots, 96$ there are 8 complete blocks of 12, with 4 values each: 32. Among $97, 98, 99, 100$, which are $1, 2, 3, 4 \pmod{12}$, only 99 works. The total is 33.

Check. The pattern of T_n mod 6 for $n = 1, \ldots, 12$ is $1, 3, 0, 4, 3, 3, 4, 0, 3, 1, 0, 0$: exactly four zeros per block, at $n = 3, 8, 11, 12$. Counting only multiples of 3 gives 66, which forgets that T_n must also be even.

11. Answer: (B) -2**. The idea.** A common root satisfies both equations, so it also satisfies their difference, which is only linear in x. That pins the common root down almost completely; the one exceptional value of k has to be checked separately.

Work it out. Subtracting the second equation from the first gives $(k-1)x + (1-k) = 0$, that is $(k-1)(x-1) = 0$. If $k = 1$, the two equations are both $x^2 + x + 1 = 0$, which has discriminant $1 - 4 < 0$ and so no real solutions at all: $k = 1$ does not work. Otherwise the common root must be $x = 1$. Putting $x = 1$ into $x^2 + x + k = 0$ gives $k = -2$, and then the first equation is

$x^2 - 2x + 1 = (x-1)^2 = 0$, which does have root 1. So $k = -2$ is the only value, and the sum is -2.

Check. For $k = -2$: $x^2 + x - 2 = (x-1)(x+2)$ and $x^2 - 2x + 1 = (x-1)^2$ share the root 1. Forgetting to test $k = 1$ (which makes the two equations identical) gives the trap $-2 + 1 = -1$.

12. Answer: (B) $\frac{1}{6}$**. The idea.** Regluing the same two pieces does not change the area, only the perimeter. For any triangle, the area equals the inradius times the semiperimeter (split the triangle into three triangles with apex at the incenter, each of height r), so with the area fixed the inradius is just the area divided by the semiperimeter.

Work it out. The altitude to the base 6 bisects it, so each piece is a right triangle with hypotenuse 5 and one leg 3, hence other leg 4: two 3-4-5 triangles, and the area is $2 \cdot \frac{1}{2} \cdot 3 \cdot 4 = 12$. The original triangle was glued along the legs of length 4 (the altitude). Gluing instead along the legs of length 3 makes a triangle with sides 5, 5 and $4 + 4 = 8$, still of area 12. The semiperimeters are $\frac{16}{2} = 8$ and $\frac{18}{2} = 9$, so the inradii are $\frac{12}{8} = \frac{3}{2}$ and $\frac{12}{9} = \frac{4}{3}$. The decrease is $\frac{3}{2} - \frac{4}{3} = \frac{1}{6}$.

Check. In the new triangle put the base from $(-4, 0)$ to $(4, 0)$ and the apex at $(0, 3)$. The incenter is $(0, \frac{4}{3})$: its distance to the base is $\frac{4}{3}$, and its distance to the line $3x - 4y + 12 = 0$ through $(-4, 0)$ and $(0, 3)$ is $\frac{|0 - \frac{16}{3} + 12|}{5} = \frac{20/3}{5} = \frac{4}{3}$. Using the full perimeter instead of the semiperimeter gives $\frac{12}{16} - \frac{12}{18} = \frac{1}{12}$; the difference of the two heights, $4 - 3 = 1$, halved gives the trap $\frac{1}{2}$.

13. Answer: (B) 7**. The idea.** Call the other leg b and the hypotenuse c. Then $c^2 - b^2 = 24^2$, and a difference of squares factors: $(c - b)(c + b) = 576$. So each triangle is a way of splitting 576 into two factors, and the only question is which splittings are allowed.

Work it out. The factors $c - b$ and $c + b$ differ by $2b$, an even number, so they have the same parity. Their product 576 is even, so both are even. Write $c - b = 2u$ and $c + b = 2v$. Then $4uv = 576$, so $uv = 144$, and $b = v - u$, $c = u + v$. We need $b > 0$, so $u < v$.

Every factor pair $uv = 144$ with $u < v$ gives a triangle, and different pairs give different b. The number $144 = 2^4 \cdot 3^2$ has $5 \cdot 3 = 15$ divisors. One of them pairs with itself ($u = v = 12$, which would make $b = 0$), and the other 14 form 7 pairs with $u < v$: $(1, 144)$, $(2, 72)$, $(3, 48)$, $(4, 36)$, $(6, 24)$, $(8, 18)$, $(9, 16)$. So there are 7 triangles.

Check. The pairs give $(b, c) = (143, 145), (70, 74), (45, 51), (32, 40), (18, 30), (10, 26), (7, 25)$. For example $24^2 + 7^2 = 576 + 49 = 625 = 25^2$ and $24^2 + 45^2 = 576 + 2025 = 2601 = 51^2$. The trap 8 keeps the degenerate pair $u = v$, 15 counts divisors instead of pairs, and 14 counts each pair in both orders.

14. Answer: (D) 576**. The idea.** A number is divisible by 11 exactly when its alternating digit sum is. Here the digits are fixed, so parity forces that alternating sum to be exactly 0, and the count reduces to choosing which three digits go in the even positions.

Work it out. Number the positions 1 to 7 from the left. Let O be the sum of the four digits in the odd positions $(1, 3, 5, 7)$ and E the sum of the three digits in the even positions $(2, 4, 6)$. Since $10 \equiv -1 \pmod{11}$, the number is divisible by 11 exactly when $O - E$ is. We know $O + E = 1 + 2 + \cdots + 7 = 28$, so $O - E = 28 - 2E$ is even. Three different digits from 1 to 7 sum to between 6 and 18, so $O - E$ lies between $28 - 36 = -8$ and $28 - 12 = 16$. The only even multiple of 11 in that range is 0 (the values ± 11 are odd and 22 is too big). So we need $E = 14$.

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The three-digit sets from $\{1,\dots,7\}$ with sum 14: with 7, the other two sum to 7: $\{1,6\}$, $\{2,5\}$, $\{3,4\}$. Without 7 but with 6, the other two sum to 8 from $\{1,\dots,5\}$: $\{3,5\}$. Without 6 or 7, the largest sum is $3+4+5=12$. So there are 4 sets: $\{1,6,7\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$.

For each set, arrange its digits in the even positions ($3!=6$ ways) and the other four digits in the odd positions ($4!=24$ ways). The total is $4\cdot 6\cdot 24=576$.

Check. One example: even-position digits 1, 6, 7 and odd-position digits 2, 3, 4, 5 give 2136475, and $2136475 = 11\cdot 194225$. As a sanity check, $\frac{576}{5040}\approx\frac{1}{8.75}$, close to the $\frac{1}{11}$ we would expect for a random number. Choosing $720=5\cdot 144$ would mean inventing a fifth set.

15. Answer: (D) 49**. The idea.** Name the two powers: $a=2^x$ and $b=3^y$. Since $4^x=a^2$ and $9^y=b^2$, the system is really about $a+b$ and a^2+b^2, which pin down a and b up to order. Then $x=\log_2 a$ and $y=\log_3 b$, and the product xy turns out not to care which order we picked.

Work it out. We have $a+b=11$ and $a^2+b^2=65$. Then $2ab=(a+b)^2-(a^2+b^2)=121-65=56$, so $ab=28$. So a and b are the roots of $t^2-11t+28=0$, that is, 4 and 7.

Case 1: $a=4$, $b=7$. Then $x=2$ and $y=\log_3 7$, so $xy=2\log_3 7$.

Case 2: $a=7$, $b=4$. Then $x=\log_2 7$ and $y=\log_3 4=2\log_3 2$. So $xy=2\log_2 7\cdot\log_3 2=2\cdot\frac{\ln 7}{\ln 2}\cdot\frac{\ln 2}{\ln 3}=2\log_3 7$.

Either way $xy=2\log_3 7=\log_3 49$, so $3^{xy}=49$.

Check. Case 1: $2^2+3^{\log_3 7}=4+7=11$ and $4^2+9^{\log_3 7}=16+7^2=65$. In general $xy=\frac{\ln a}{\ln 2}\cdot\frac{\ln b}{\ln 3}$ is symmetric in a and b, which is why both cases agree. The traps: 7 forgets that $x=2$ in case 1, 14 writes $3^{2\log_3 7}$ as $2\cdot 7$, and 28 stops at ab.

16. Answer: (A) 12**. The idea.** Do not compute slopes. A line $y=mx+k$ meets the graph where $x^3-15x^2+60x=mx+k$, that is, where $x^3-15x^2+(60-m)x-k=0$. This is a cubic whose x^2-coefficient is -15 no matter which line we took, so by Vieta the x-coordinates of its (at most three) crossing points always add up to 15.

Work it out. Suppose three of the points, with x-coordinates $a<b<c$, lie on a line $y=mx+k$. Then a, b, c are three different roots of the cubic above, so they are all of its roots, and $a+b+c=15$. Conversely, if $a+b+c=15$, draw the line through the points above a and b. Its cubic has roots a, b and a third root r with $a+b+r=15$, so $r=c$: the line also passes through the point above c. So the collinear sets are exactly the sets of three different numbers from 1 to 12 with sum 15.

Count them by the smallest number a. With $a=1$: $b+c=14$, b from 2 to 6, giving 5 sets. With $a=2$: $b+c=13$, b from 3 to 6, giving 4. With $a=3$: $b+c=12$, $b=4$ or 5, giving 2. With $a=4$: only $\{4,5,6\}$. With $a\ge 5$ the sum is at least $5+6+7=18$. The total is $5+4+2+1=12$. (The largest number used is 12, in $\{1,2,12\}$, so every set really uses marked points.)

Check. Take $\{4,5,6\}$: the points are $(4,64)$, $(5,50)$ and $(6,36)$, and both steps have slope -14, so they are collinear. Take $\{1,2,3\}$, with sum 6: the points $(1,46)$, $(2,68)$, $(3,72)$ have slopes 22 and 4, so they are not. The traps: 19 also counts sums like $5+5+5$ that repeat a number (one point cannot be used twice), and $72=12\cdot 3!$ counts each set once for every order.

17. Answer: (A) 28**. The idea.** A path that never returns to the diagonal stays strictly on one side of it. Its first and last steps are forced, and what is left in between is a path that may touch a shifted diagonal but not cross it. That is a Catalan count.

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Work it out. By symmetry (swap the roles of x and y), count the paths that stay strictly below the line, with $x > y$ at every interior lattice point, and then double.

A path below the line must start with a step right to $(1, 0)$ and end with a step up from $(5, 4)$. In between it runs from $(1, 0)$ to $(5, 4)$ and must stay on or below the line $y = x - 1$, i.e. never have $y > x - 1$. Shift everything left by 1. This becomes a path from $(0, 0)$ to $(4, 4)$ that never rises above $y = x$. The number of those is the Catalan number $C_4 = \frac{1}{5}\binom{8}{4} = 14$.

Doubling for paths above the line gives $2 \cdot 14 = 28$.

Check. A smaller case: to $(2, 2)$ the only such paths are RRUU and UURR, which is $2 = 2C_1$. To $(3, 3)$ we get $2C_2 = 4$: RRURUU, RRRUUU and their mirror images. The choice $42 = C_5$ counts paths that may touch the diagonal but stay on one side, and $252 = \binom{10}{5}$ counts all paths.

18. Answer: (A) 80**. The idea.** Two separate facts settle it. First, something never changes: f keeps a multiple of 5 a multiple of 5, and keeps a non-multiple a non-multiple, so multiples of 5 can never reach 1. Second, everything else keeps shrinking until it falls into a small loop that contains 1.

Work it out. *The invariant.* If n is odd, $n + 5 \equiv n \pmod 5$. If n is even, $n = 2 \cdot \frac{n}{2}$, and since $\gcd(2, 5) = 1$, $5 \mid n$ exactly when $5 \mid \frac{n}{2}$. So each term is divisible by 5 exactly when n is. Since 1 is not, the 20 multiples of 5 up to 100 never reach 1.

The descent. Take n not divisible by 5. An even term is halved, so it drops. An odd term m goes to $m + 5$, which is even, and then to $\frac{m+5}{2}$, which is less than m whenever $m > 5$. So every one or two steps the sequence drops, as long as it is not at an odd number at most 5. Positive integers cannot drop forever, so the sequence reaches an odd number at most 5 that is not a multiple of 5: that is 1 or 3. And $3 \to 8 \to 4 \to 2 \to 1$. So every one of the $100 - 20 = 80$ other starting values reaches 1.

Check. Examples: $7 \to 12 \to 6 \to 3 \to 8 \to 4 \to 2 \to 1$, and 1 itself lies on the loop $1 \to 6 \to 3 \to 8 \to 4 \to 2 \to 1$, so it counts. A multiple of 5 gets stuck in a different loop: $5 \to 10 \to 5$, and $15 \to 20 \to 10 \to 5$. The traps: 98 excludes only the loop $5 \to 10 \to 5$ itself, 90 excludes only multiples of 10, 81 is an off-by-one in counting the non-multiples of 5, and 100 ignores the invariant.

19. Answer: (E) 507**. The idea.** First understand adding 1 in binary: if m ends in a block of exactly t ones (preceded by a 0), adding 1 turns those t ones into zeros and that 0 into a 1. So the number of 1s changes by $1 - t$, and it stays the same exactly when $t = 1$, that is, when m ends in $\ldots 01$, i.e. $m \equiv 1 \pmod 4$. Adding 2 to n leaves the last binary digit alone and adds 1 to the number formed by the other digits, so the question reduces to the +1 case.

Work it out. Write $n = 2m + r$ with $r \in \{0, 1\}$ the last binary digit. Then $n + 2 = 2(m + 1) + r$. Deleting the last digit, n and $n + 2$ have equally many 1s exactly when m and $m + 1$ do, that is, when $m \equiv 1 \pmod 4$.

If n is even, $n = 2m$ with $m \equiv 1 \pmod 4$, so $n \equiv 2 \pmod 8$: $n = 2, 10, \ldots, 2026$, which is $\frac{2026-2}{8} + 1 = 254$ numbers.

If n is odd, $n = 2m + 1$ with $m \equiv 1 \pmod 4$, so $n \equiv 3 \pmod 8$: $n = 3, 11, \ldots, 2019$, which is $\frac{2019-3}{8} + 1 = 253$ numbers.

The total is $254 + 253 = 507$.

Check. $n = 2 = 10_2$ and $4 = 100_2$ each have one 1; $n = 3 = 11_2$ and $5 = 101_2$ each have two. But $n = 1$ and $3 = 11_2$ differ, and so do $n = 6 = 110_2$ and $8 = 1000_2$. The traps: 253 or 254 keeps only one parity, 505 forgets the +1 in both endpoint counts, and 506 is an off-by-one at the ends.

20. Answer: (E) $\frac{319}{512}$**. The idea.** Instead of comparing two counts, count Bob's tails. Alice's heads beat Bob's heads exactly when Alice's heads plus Bob's tails is more than 4. That sum is just the number of "good" faces among 10 fair coins.

Work it out. Let a be Alice's heads and b Bob's heads, so Bob has $t = 4 - b$ tails. Then $a > b$ is the same as $a > 4 - t$, i.e. $a + t \geq 5$. Now $a + t$ counts Alice's heads and Bob's tails. Each of the 10 coins independently shows its counted face with probability $\frac{1}{2}$, so $a + t$ is the number of successes in 10 fair flips.

So the probability is $P(a+t \geq 5) = \frac{1}{1024}\left(\binom{10}{5} + \binom{10}{6} + \cdots + \binom{10}{10}\right)$. The terms from 6 to 10 are half of $1024 - \binom{10}{5} = 772$, which is 386. Adding $\binom{10}{5} = 252$ gives 638. The probability is $\frac{638}{1024} = \frac{319}{512}$.

Check. The same trick gives $P(a = b) = P(a + t = 4) = \frac{\binom{10}{4}}{1024} = \frac{105}{512}$ and $P(a < b) = P(a + t \leq 3) = \frac{1+10+45+120}{1024} = \frac{11}{64}$. Then $\frac{319}{512} + \frac{105}{512} + \frac{88}{512} = 1$. Turning $a > b$ into $a + t \geq 6$ instead of $a + t \geq 5$ (an off-by-one) would give only $\frac{386}{1024} = \frac{193}{512}$. Direct check: $\sum_b \binom{4}{b} \sum_{a>b} \binom{6}{a} = 1 \cdot 63 + 4 \cdot 57 + 6 \cdot 42 + 4 \cdot 22 + 1 \cdot 7 = 638$.

21. Answer: (B) 8**. The idea.** Cycles are hard to count directly, so count the other triples. In a set of three players that is not a cycle, someone won both of their games inside the set, and exactly one player can do that. So every non-cycle has a unique "top" player, and a player with w wins is the top of exactly $\binom{w}{2}$ non-cycles: pick any two of the players they beat.

Work it out. Let the players have $w_1, \ldots, w_6$ wins. There are $\binom{6}{3} = 20$ sets of three, so the number of cycles is $20 - \sum \binom{w_i}{2}$. The 15 games give $w_1 + \cdots + w_6 = 15$, and we want $\sum \binom{w_i}{2}$ as small as possible.

If two players have win totals w and u with $w \geq u+2$, moving one win from the first to the second changes the sum by $-(w-1)+u < 0$. So the sum is smallest when all totals are within 1 of each other. Six totals adding to 15 must then be $2, 2, 2, 3, 3, 3$, giving $\sum \binom{w_i}{2} = 3 \cdot 1 + 3 \cdot 3 = 12$. So there are at most $20 - 12 = 8$ cycles.

To reach 8, seat the players $1, \ldots, 6$ around a circle and let each beat the next two clockwise (1 beats 2 and 3, ..., 5 beats 6 and 1, 6 beats 1 and 2). That settles 12 games. In the remaining three, 1 beats 4, 2 beats 5 and 3 beats 6. Now players 1, 2, 3 have 3 wins and 4, 5, 6 have 2, so this tournament has exactly 8 cycles.

Check. The 8 cycles of this tournament are $\{1,2,5\}, \{1,3,5\}, \{1,3,6\}, \{1,4,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,6\}, \{2,5,$ for instance 1 beats 2, 2 beats 5 and 5 beats 1. With five players the same count gives at most $10 - 5\binom{2}{2} = 5$, a trap for anyone who forgets that six players cannot all have 2.5 wins. The traps 12 and 10 are the number of non-cycles and a guess of "half the triples".

22. Answer: (D) $\frac{621}{140}$**. The idea.** Use linearity of expectation position by position. The child in position k is seen from the left exactly when they are the tallest of the first k children, which has probability $\frac{1}{k}$. The two ends overlap only for the tallest child, so inclusion–exclusion is easy.

Work it out. Among the first k children every one is equally likely to be the tallest, so position k is seen from the left with probability $\frac{1}{k}$. The expected number seen from the left is $H_8 =$

$1 + \frac{1}{2} + \cdots + \frac{1}{8} = \frac{761}{280}$. By symmetry the same number of children, on average, are seen from the right.

A child seen from both ends is taller than everyone on both sides, so it must be the tallest child, and the tallest child is always seen from both ends. So exactly one child is counted twice in every lineup. By linearity, the expected number seen from at least one end is $2H_8 - 1 = \frac{761}{140} - 1 = \frac{621}{140}$.

Check. With 3 children the formula gives $2(1 + \frac{1}{2} + \frac{1}{3}) - 1 = \frac{8}{3}$. Directly: if the tallest is in the middle, both others are seen (3 seen, 2 of the 6 orders). If the tallest is at an end, the far end child is seen and the middle child is seen from that end only when it beats the far-end child ($\frac{1}{2}$ of the time), giving 2 or 3. The average is $\frac{2\cdot 3+4\cdot 2.5}{6} = \frac{16}{6} = \frac{8}{3}$. The choice $\frac{761}{140}$ counts the tallest child twice.

23. Answer: (D) 55**. The idea.** Unfold the room. Instead of reflecting the beam, reflect the room, tiling the plane with mirror-image unit squares. The beam becomes a straight ray from the origin, every grid line it crosses is one bounce, and it stops at the first lattice point it meets. The parity of that lattice point's coordinates tells which corner it is.

Work it out. Put $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, $D = (0, 1)$. Reflect the square across its walls again and again to tile the plane. The beam's path unfolds into a straight ray from $(0, 0)$ into the first quadrant. It reaches a corner exactly when the ray hits a lattice point. So the beam stops at all only when the ray points at a lattice point (a, b) with $a, b \geq 1$, and it stops at the first one, where $\gcd(a, b) = 1$. Before that it crosses the vertical lines $x = 1, \ldots, a - 1$ and the horizontal lines $y = 1, \ldots, b - 1$. It never crosses two at once, since that would be an earlier lattice point. So it makes $a + b - 2$ bounces.

Folding back, the lattice point (a, b) becomes the corner with x-coordinate a mod 2 and y-coordinate b mod 2. So the beam ends at C exactly when a and b are both odd.

We need coprime $a, b \geq 1$, both odd, with $a + b - 2 \leq 20$, so $s = a + b$ is even and $s \leq 22$. For even s, a pair with $a + b = s$ is coprime exactly when $\gcd(a, s) = 1$, and then a and b are automatically odd. So each even s contributes $\varphi(s)$ directions: $\varphi(2) + \varphi(4) + \cdots + \varphi(22) = 1 + 2 + 2 + 4 + 4 + 4 + 6 + 8 + 6 + 8 + 10 = 55$.

Check. Small cases: $s = 2$ is the diagonal straight to C with 0 bounces. For $s = 4$ the directions toward $(1, 3)$ and $(3, 1)$ each bounce twice before reaching C. For example, toward $(3, 1)$ the beam hits the right wall at height $\frac{1}{3}$ and the left wall at height $\frac{2}{3}$, then arrives at $(1, 1)$. Leaving out the diagonal gives 54. Counting every corner (all coprime pairs with $a + b \leq 22$) gives 149, and exactly 20 bounces gives only $\varphi(22) = 10$.

24. Answer: (E) $\frac{190}{309}$**. The idea.** An assignment is a permutation of the seven friends: follow "gives to" arrows and they split into cycles. Nobody drawing their own name means no cycles of length 1. Two friends drawing each other means a cycle of length 2. So count permutations by their cycle lengths.

Work it out. The ways to split 7 into parts all at least 2 are 7, 5 + 2, 4 + 3 and 3 + 2 + 2. Count each.

One 7-cycle: $6! = 720$. A 5-cycle and a 2-cycle: choose the pair ($\binom{7}{2} = 21$), then arrange the other five in a cycle ($4! = 24$): 504. A 4-cycle and a 3-cycle: choose the three ($\binom{7}{3} = 35$), with $2! = 2$ cycles on them and $3! = 6$ on the other four: 420. A 3-cycle and two 2-cycles: choose and cycle the three ($35 \cdot 2 = 70$), then split the other four into two pairs (3 ways): 210.

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The total is $720+504+420+210 = 1854$ assignments with nobody drawing their own name. The ones with no 2-cycle are the types 7 and $4+3$: $720+420 = 1140$. The probability is $\frac{1140}{1854} = \frac{190}{309}$.

Check. The number of derangements of 7 from inclusion–exclusion is $7!\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots-\frac{1}{7!}\right) = 1854$, which matches. The answer is about 0.615, close to $e^{-1/2} \approx 0.607$, the limiting chance of no 2-cycles given no 1-cycles. The choice $\frac{19}{84} = \frac{1140}{5040}$ forgets that the drawing is redone. The choice $\frac{40}{103}$ keeps only the 7-cycle and forgets the $4+3$ type, and $\frac{119}{309} = \frac{714}{1854}$ is the chance that some pair does swap.

25. Answer: (C) 2529**. The idea.** Compute $11n$ as $10n+n$ by column addition. Shifting does not change the digit sum, so $S(10n)+S(n) = 2S(n)$. Each carry in a column addition lowers the digit sum by exactly 9. So the equation holds exactly when adding n to $10n$ produces no carry at all. That turns into a condition on neighbouring digits of n, which we count one digit at a time.

Work it out. In general $S(x+y) = S(x)+S(y)-9c$, where c is the number of carries. A carry replaces 10 in one column by 1 in the next. So $S(11n) = 2S(n)-9c$, and we need $c = 0$.

Write n with leading zeros as $d_3d_2d_1d_0$. Adding $10n = d_3d_2d_1d_00$ to n, the units column holds d_0+0, and column i holds d_i+d_{i-1} for $i = 1,2,3$. The top column holds d_3. With no incoming carry, a carry first appears in a column exactly when its two digits add to 10 or more. So there are no carries exactly when every two neighbouring digits of $d_3d_2d_1d_0$ add to at most 9. (Leading zeros are harmless: $0+d \le 9$ always.)

Count strings $d_3d_2d_1d_0$ with each neighbouring sum at most 9, building from the left. Let $f_j(d)$ be the number of valid strings of length j ending in digit d. Then $f_1(d) = 1$ and $f_{j+1}(d) = \sum_{e \le 9-d} f_j(e)$. This gives $f_2(d) = 10-d$. Next, $f_3(d) = \sum_{e=0}^{9-d}(10-e)$, which for $d = 0,1,\ldots,9$ is $55, 54, 52, 49, 45, 40, 34, 27, 19, 10$. Finally, the total count of length-4 strings is $\sum_d f_4(d) = \sum_e f_3(e)\,(10-e)$, since digit e can be followed by the $10-e$ digits 0 through $9-e$. That is $550+486+416+343+270+200+136+81+38+10 = 2530$. Remove the all-zero string (which is not positive): 2529.

Check. Small cases: $n = 45$ gives $11 \cdot 45 = 495$ with $S = 18 = 2 \cdot 9$ (neighbours sum to 9, no carry). But $n = 55$ gives 605 with $S = 11 = 20-9$, one carry. For one-digit n all 9 work, matching $10-1$ strings of length 1. For $n < 100$ it gives $55-1 = 54$, and these are exactly the two-digit strings ab with $a+b \le 9$. The choice 2530 forgets to remove $n = 0$.

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