MathCastle

AMC 10

Practice Contest

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

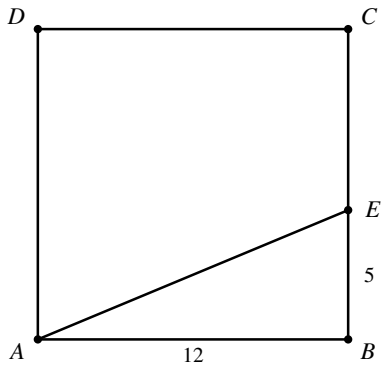
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The five members of the Ortiz family have ages, in whole years, that add up to exactly 100 today. Exactly five years ago, the ages of the family members who were alive at that time added up to 78. The youngest member of the family is Zoe, and each of the other four family members is at least 12 years old. How old is Zoe today?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 22

2. Square $ABCD$ has side length 12, with its vertices labelled in order around it. Point E lies on side $\overline{BC}$ with $BE = 5$. What is the distance from the corner D to the line AE?



(A) $\frac{60}{13}$ **(B)** $\frac{120}{13}$ **(C)** 11 **(D)** $\frac{144}{13}$ **(E)** 12

3. Sam has a jar of marbles. When he puts them into bags of 4, exactly 3 marbles are left over. When he puts them into bags of 5 instead, exactly 4 are left over, and when he uses bags of 6, exactly 5 are left over. But when he uses bags of 7, no marbles are left over. What is the smallest number of marbles Sam could have?

(A) 59 **(B)** 119 **(C)** 161 **(D)** 179 **(E)** 239

4. Let r and s be the two roots of the quadratic equation $x^2 - 7x + 3 = 0$. The roots are not whole numbers, but you do not need to find them. What is the value of the product $(r^2 - 4r + 5)(s^2 - 4s + 5)$?

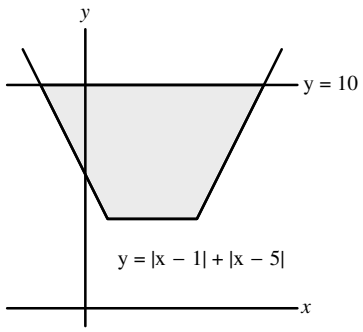
(A) 43 **(B)** 53 **(C)** 67 **(D)** 73 **(E)** 97

5. Ana and Ben leave their camp at the same moment to go to a village 12 km away along a straight road, and they share one bicycle. Ana walks at 4 km/h and Ben walks at 6 km/h; either of them rides the bicycle at 12 km/h. Only one person can ride the bicycle at a time, and it moves only when someone rides it. A rider may leave the bicycle at the roadside at any point and continue on foot, and the other person can pick it up on reaching that point; this may happen any number of times. Nobody ever moves back toward the camp. What is the least possible number of minutes until both of them have reached the village?

(A) 90 **(B)** 100 **(C)** 102 **(D)** 108 **(E)** 120

6. In the coordinate plane, what is the area of the region that lies above the graph of $y = |x - 1| + |x - 5|$ and below the horizontal line $y = 10$?

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(A) 42 **(B)** 48 **(C)** 50 **(D)** 60 **(E)** 84

7. The six letters of the word BANANA can be rearranged to form different strings; the three A's are identical to each other and the two N's are identical to each other, so, for example, BANANA and BNAAAN are two of the strings, while swapping two A's does not make a new string. How many of these strings have no two identical letters next to each other?

(A) 6 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 18

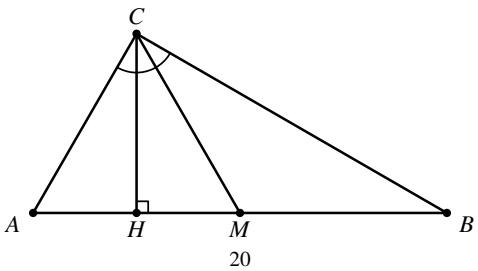
8. How many four-digit positive integers have four different digits whose sum is 10? For example, 2035 is one of them.

(A) 72 **(B)** 96 **(C)** 108 **(D)** 114 **(E)** 120

9. For how many integers n with $1 \le n \le 100$ is the greatest common divisor $\gcd(n, 100)$ a perfect square? (Note that 1 is a perfect square.)

(A) 23 **(B)** 61 **(C)** 63 **(D)** 64 **(E)** 68

10. Triangle ABC has a right angle at C, and its hypotenuse $\overline{AB}$ has length 20. From C two segments are drawn to the hypotenuse: the altitude $\overline{CH}$ (perpendicular to $\overline{AB}$) and the median $\overline{CM}$ (to the midpoint M of $\overline{AB}$). These two segments divide the right angle $\angle ACB$ into three equal angles. What is the area of triangle ABC?



(A) $50\sqrt{3}$ **(B)** 100 **(C)** $100\sqrt{3}$ **(D)** 200 **(E)** $200\sqrt{3}$

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11. An arithmetic sequence $a_1, a_2, a_3, \ldots$ and a geometric sequence $b_1, b_2, b_3, \ldots$ of real numbers are added term by term to form a new sequence $c_n = a_n + b_n$. The first four terms of the new sequence are $c_1 = 3$, $c_2 = 7$, $c_3 = 15$ and $c_4 = 35$. What is c_5?

(A) 75 **(B)** 83 **(C)** 85 **(D)** 91 **(E)** 95

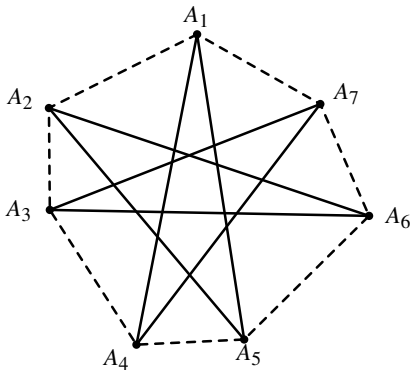
12. A quiz has 20 questions. Each correct answer earns 5 points, each question left blank earns 2 points, and each wrong answer earns 0 points; every question is either answered correctly, left blank, or answered wrongly. So the total score is always a whole number from 0 to 100. How many of the 101 whole numbers $0, 1, 2, \ldots, 100$ are possible total scores?

(A) 91 **(B)** 95 **(C)** 96 **(D)** 99 **(E)** 101

13. What is the least positive integer k such that every set of k different numbers chosen from $\{1, 2, 3, \ldots, 50\}$ must contain two numbers whose difference is exactly 6?

(A) 25 **(B)** 26 **(C)** 27 **(D)** 31 **(E)** 45

14. The points $A_1, A_2, \ldots, A_7$ are the vertices of a convex heptagon, labelled in order around it. The heptagon need not be regular. Draw the seven segments A_1A_4, A_4A_7, A_7A_3, A_3A_6, A_6A_2, A_2A_5, A_5A_1, so that each vertex is joined to the two vertices three steps away from it along the heptagon. This makes a seven-pointed star. The tip angle at A_k is the angle between the two star segments that meet at A_k. What is the sum of the seven tip angles, in degrees?



(A) 180 **(B)** 360 **(C)** 540 **(D)** 720 **(E)** 900

15. How many ordered triples (a, b, c) of positive integers satisfy all three conditions: a divides b, b divides c, and c divides 360? The three numbers need not be different; for example, $(1, 1, 1)$, $(2, 2, 360)$ and $(3, 15, 45)$ are three such triples.

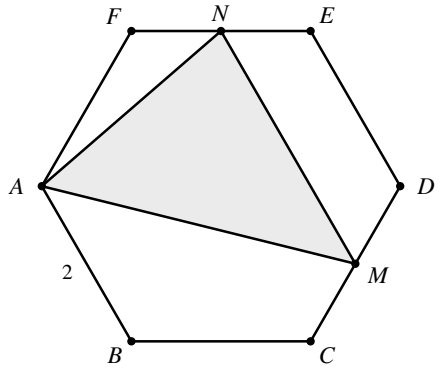
(A) 800 **(B)** 1200 **(C)** 4800 **(D)** 7000 **(E)** 13824

16. Consider the inequality $|x| + |y| + |x - y| \leq 8$, where $|u|$ is the absolute value of u. How many ordered pairs of integers (x, y) satisfy it? (Pairs such as $(1, 3)$ and $(3, 1)$ count as different; each coordinate may be positive, negative or zero.)

(A) 37 **(B)** 41 **(C)** 49 **(D)** 52 **(E)** 61

17. Regular hexagon $ABCDEF$ has side length 2, with its vertices labelled in order around it. Let M be the

midpoint of side $\overline{CD}$ and let N be the midpoint of side $\overline{EF}$. What fraction of the area of the hexagon lies inside triangle AMN?



(A) $\frac{1}{3}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

18. For a positive integer n, let $d(n)$ be the number of positive divisors of n. For instance, 24 has the eight divisors $1, 2, 3, 4, 6, 8, 12, 24$, so $d(24) = 8$; since $24 \neq 8 \cdot 8$, the number 24 is not one we want. Find the sum of all positive integers n with $n = 8\,d(n)$.

(A) 80 **(B)** 96 **(C)** 160 **(D)** 176 **(E)** 256

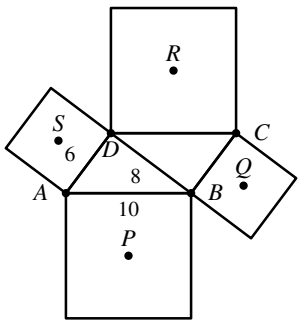
19. Ada, Ben and Cy take turns rolling a fair six-sided die in the order Ada, Ben, Cy, Ada, Ben, Cy, and so on. Ada wins if she rolls a 1. Ben wins if he rolls a 1 or a 2. Cy wins if he rolls a 1, 2 or 3. The game stops the moment someone wins. What is the probability that Cy wins?

(A) $\frac{3}{13}$ **(B)** $\frac{5}{18}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{5}{13}$ **(E)** $\frac{1}{2}$

20. How many ordered triples (a, b, c) of positive integers satisfy $abc = 3(a + b + c)$? (Triples that list the same numbers in a different order, such as $(2, 3, 5)$ and $(5, 3, 2)$, count as different.)

(A) 6 **(B)** 22 **(C)** 24 **(D)** 27 **(E)** 28

21. Parallelogram $ABCD$ has $AB = 10$, $AD = 6$ and diagonal $BD = 8$. On each of its four sides a square is built outward, so that the square has that side as one of its sides and lies outside the parallelogram. Let P, Q, R, S be the centres of the squares on AB, BC, CD, DA. What is the area of quadrilateral $PQRS$?



(A) 20 **(B)** 68 **(C)** 116 **(D)** 128 **(E)** 164

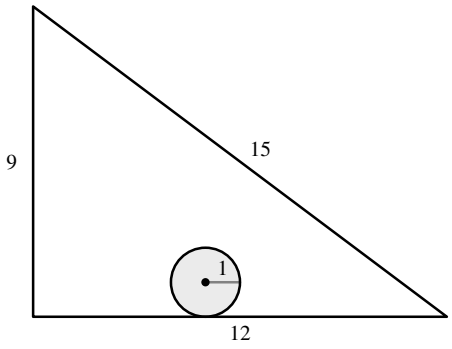
22. For each integer n from 1 to 2026, compute $\left\lfloor \frac{2026}{n} \right\rfloor$, the greatest integer less than or equal to $\frac{2026}{n}$. This gives a list of 2026 numbers, beginning $2026, 1013, 675, \ldots$ and ending $\ldots, 1, 1$. How many different numbers appear in the list?

(A) 45 **(B)** 88 **(C)** 89 **(D)** 90 **(E)** 1013

23. A sequence starts with $a_0 = \frac{5}{2}$, and each later term is $a_{n+1} = a_n^2 - 2$. So $a_1 = \frac{17}{4}$, $a_2 = \frac{257}{16}$, and so on. What is the exact value of $\frac{1}{a_0} + \frac{1}{a_0 a_1} + \frac{1}{a_0 a_1 a_2} + \frac{1}{a_0 a_1 a_2 a_3}$?

(A) $\frac{1}{2} - \frac{3}{2^5-2}$ **(B)** $\frac{1}{2} - \frac{3}{2^9-2}$ **(C)** $\frac{1}{2} - \frac{3}{2^{17}-2}$ **(D)** $\frac{1}{2} - \frac{3}{2^{32}-1}$ **(E)** $\frac{1}{2} - \frac{3}{2^{33}-2}$

24. A flat tray is shaped like a triangle with sides of lengths 9, 12 and 15. A coin of radius 1 lies flat on the tray, touching one of the tray's sides. The coin is rolled around the inside of the tray, always staying inside it and always touching at least one side, until it has gone once all the way around and is back where it started. What is the area of the part of the tray that the coin never covers?



(A) $6 - \pi$ **(B)** $12 - 2\pi$ **(C)** $9 - \pi$ **(D)** 6 **(E)** $12 - \pi$

25. Five coins lie on a table, all showing heads. On each turn, one of the five coins is chosen uniformly at random, independently of all earlier turns, and turned over (heads becomes tails, tails becomes heads). What is the expected number of turns until all five coins show tails for the first time?

(A) $\frac{137}{12}$ **(B)** $\frac{35}{3}$ **(C)** 31 **(D)** 32 **(E)** $\frac{128}{3}$

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Answer Key — AMC 10 Practice Contest 2

1. (A) 2

2. (D) $\frac{144}{13}$

3. (B) 119

4. (D) 73

5. (B) 100

6. (A) 42

7. (C) 10

8. (B) 96

9. (C) 63

10. (A) $50\sqrt{3}$

11. (D) 91

12. (B) 95

13. (C) 27

14. (A) 180

15. (A) 800

16. (E) 61

17. (B) $\frac{3}{8}$

18. (D) 176

19. (D) $\frac{5}{13}$

20. (E) 28

21. (C) 116

22. (C) 89

23. (E) $\frac{1}{2} - \frac{3}{2^{33}-2}$

24. (E) $12 - \pi$

25. (E) $\frac{128}{3}$

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Solutions — AMC 10 Practice Contest 2

1. Answer: (A) 2**. The idea.** If all five people had been alive five years ago, the total would have dropped by exactly $5 \times 5 = 25$, to 75. It only dropped to 78, so somebody was not yet born, and that person's missing years explain the difference.

Work it out. The four older members are all at least 12, so each of them was alive five years ago and is exactly 5 years younger in that count: together they lose 20 years. If Zoe had been alive five years ago too, the total would have lost another 5 and been 75, not 78. So Zoe was not yet born five years ago, and she contributes nothing to the old total, while she contributes her whole age z to today's total. Therefore $78 = (100 - z) - 20$, which gives $z = 2$.

Check. Take ages $40, 38, 12, 8$ and 2: they add up to 100. Five years ago the four older members were $35, 33, 7, 3$, adding up to 78, and Zoe was not born. The tempting answer 3 comes from $78 - 75$, which forgets that Zoe's own age was removed, not just five years of it.

2. Answer: (D) $\frac{144}{13}$**. The idea.** Look at triangle AED. Its base $\overline{AD}$ is a side of the square, and E lies on the opposite side, so its height to that base is 12 no matter where E is. That fixes its area; measuring the same area with $\overline{AE}$ as the base gives the distance we want.

Work it out. Triangle AED has base $AD = 12$ and height 12 (the distance from $\overline{BC}$ to $\overline{AD}$), so its area is $\frac{1}{2} \cdot 12 \cdot 12 = 72$, half of the square. In right triangle ABE the legs are 12 and 5, so $AE = \sqrt{144 + 25} = 13$. If h is the distance from D to line AE, then $\frac{1}{2} \cdot 13 \cdot h = 72$, so $h = \frac{144}{13}$.

Check. Let F be the foot of the perpendicular from D to $\overline{AE}$. Since $\overline{AD} \parallel \overline{BC}$, $\angle DAF = \angle AEB$, so right triangles DFA and ABE are similar and $\frac{DF}{AD} = \frac{AB}{AE}$, giving $DF = \frac{12 \cdot 12}{13} = \frac{144}{13} \approx 11.08$, just under the side 12, as it must be. The distractor $\frac{60}{13}$ is the distance from B to the line AE, not from D.

3. Answer: (B) 119**. The idea.** In each of the first three cases the number of marbles is one short of a full extra bag. So one more marble would divide evenly by 4, by 5 and by 6.

Work it out. If Sam has n marbles, then $n + 1$ is a multiple of 4, 5 and 6, hence a multiple of $\operatorname{lcm}(4, 5, 6) = 60$. So n is one of $59, 119, 179, 239, \ldots$ Now test divisibility by 7: $59 = 7 \cdot 8 + 3$ fails, and $119 = 7 \cdot 17$ works. So the smallest possibility is 119.

Check. $119 = 4 \cdot 29 + 3 = 5 \cdot 23 + 4 = 6 \cdot 19 + 5 = 7 \cdot 17$. The distractor 59 satisfies the first three conditions but not the last; $n + 1$ must be a multiple of 60, not of $4 \cdot 5 \cdot 6 = 120$, which would wrongly skip to 239.

4. Answer: (D) 73**. The idea.** Each root satisfies its own equation, $r^2 = 7r - 3$. Use that to replace r^2, which turns each quadratic factor into a linear one, and then Vieta's formulas finish the job.

Work it out. Since $r^2 - 7r + 3 = 0$, we have $r^2 = 7r - 3$, so $r^2 - 4r + 5 = 7r - 3 - 4r + 5 = 3r + 2$. In the same way $s^2 - 4s + 5 = 3s + 2$. The product is $(3r + 2)(3s + 2) = 9rs + 6(r + s) + 4$. By Vieta's formulas $r + s = 7$ and $rs = 3$, so the product is $9 \cdot 3 + 6 \cdot 7 + 4 = 27 + 42 + 4 = 73$.

Check. The roots are $\frac{7 \pm \sqrt{37}}{2}$, so $r \approx 6.541$ and $s \approx 0.459$. Then $r^2 - 4r + 5 \approx 42.78 - 26.16 + 5 = 21.62$ and $s^2 - 4s + 5 \approx 0.21 - 1.84 + 5 = 3.38$, and $21.62 \times 3.38 \approx 73$.

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5. Answer: (B) 100**. The idea.** The bicycle only moves forward, so every stretch of the road is ridden by exactly one of them and walked by the other. So all that matters is how many kilometres Ana rides in total. Riding more helps Ana and hurts Ben, so the best plan makes them arrive together.

Work it out. Say Ana rides x km in total. Then she walks $12 - x$ km, and Ben walks exactly the x km that Ana rides and rides the other $12 - x$ km. Waiting can only make things later, so Ana needs at least $\frac{12-x}{4} + \frac{x}{12} = 3 - \frac{x}{6}$ hours and Ben at least $\frac{x}{6} + \frac{12-x}{12} = 1 + \frac{x}{12}$ hours. The first decreases and the second increases as x grows, so the later of the two is smallest when they are equal: $3 - \frac{x}{6} = 1 + \frac{x}{12}$ gives $\frac{x}{4} = 2$, so $x = 8$, and both times are $1 + \frac{8}{12} = \frac{5}{3}$ hours, which is 100 minutes. For any other x one of them takes longer than that.

Check. The plan really works: Ana rides the first 8 km (40 minutes), leaves the bicycle and walks the last 4 km (60 minutes), arriving after 100 minutes. Ben walks 8 km (80 minutes), finds the bicycle waiting, and rides 4 km (20 minutes), also arriving after 100 minutes. Splitting the riding 6 km each would leave Ana arriving after 120 minutes, and making them walk for equal times ($x = 7.2$) gives 108 minutes.

6. Answer: (A) 42**. The idea.** $|x - 1| + |x - 5|$ is the sum of the distances from x to 1 and to 5 on the number line. For every x between 1 and 5 that sum is just the distance from 1 to 5, so the graph has a flat bottom rather than a sharp point, and the region is a trapezoid, not a triangle.

Work it out. For $1 \le x \le 5$ the value is 4. For $x \ge 5$ it is $(x - 1) + (x - 5) = 2x - 6$, which reaches 10 at $x = 8$; for $x \le 1$ it is $(1 - x) + (5 - x) = 6 - 2x$, which reaches 10 at $x = -2$. So the region has vertices $(-2, 10)$, $(8, 10)$, $(5, 4)$ and $(1, 4)$: a trapezoid with parallel horizontal sides of lengths 10 and 4 and height $10 - 4 = 6$. Its area is $\frac{10+4}{2} \cdot 6 = 42$.

Check. The rectangle from $x = -2$ to $x = 8$ and from $y = 4$ to $y = 10$ has area $10 \cdot 6 = 60$; the two corner triangles below the slanted sides each have legs 3 and 6, area 9, so the region has area $60 - 18 = 42$. Treating the graph as a V with its point at $(3, 0)$, like $y = 2|x - 3|$, would give the triangle of area 50 instead.

7. Answer: (C) 10**. The idea.** The three A's are the hardest letters to separate, so place them first; then place B and the two N's in the three spots left, keeping the two N's apart.

Work it out. Number the positions 1 to 6. Three A's in non-adjacent positions can go in only four ways: $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 6\}$ and $\{2, 4, 6\}$. In each case the three remaining positions get B, N, N, and there are 3 ways to decide where B goes. For $\{1, 3, 5\}$ the free positions are $2, 4, 6$, no two adjacent, so all 3 ways work; likewise for $\{2, 4, 6\}$ (free positions $1, 3, 5$). For $\{1, 3, 6\}$ the free positions are $2, 4, 5$, and the N's must not take both 4 and 5, so B cannot go in position 2: 2 ways. Likewise $\{1, 4, 6\}$ leaves $2, 3, 5$ and gives 2 ways. Total $3 + 3 + 2 + 2 = 10$.

Check. Counting only the A condition gives $4 \times 3 = 12$ (also found by arranging B, N, N and dropping the A's into 3 of the 4 gaps); the two strings removed are the ones whose A's are apart but whose N's are together, ABANNA and ANNABA. $12 - 2 = 10$. A computer listing all 60 strings agrees.

8. Answer: (B) 96**. The idea.** First find which sets of four different digits add up to 10, then count the ways to arrange each set, remembering that a four-digit number cannot start with 0. There is very little room: the four smallest digits $0, 1, 2, 3$ already add up to 6.

Work it out. Sets without 0: four different positive digits add up to at least $1+2+3+4=10$, so the only such set is $\{1,2,3,4\}$. Sets containing 0: the other three are different positive digits $a<b<c$ with $a+b+c=10$. If $a=1$ then $b+c=9$ with $b\geq 2$: $\{2,7\}$, $\{3,6\}$, $\{4,5\}$. If $a=2$ then $b+c=8$ with $b\geq 3$: only $\{3,5\}$. If $a\geq 3$ then $b+c\geq 4+5=9>7$, impossible. So the sets with 0 are $\{0,1,2,7\}$, $\{0,1,3,6\}$, $\{0,1,4,5\}$, $\{0,2,3,5\}$. A set with 0 can be arranged in $4!=24$ ways, of which $3!=6$ start with 0, leaving 18; the set $\{1,2,3,4\}$ gives all 24. Total $4\cdot 18+24=72+24=96$.

Check. The example 2035 uses the set $\{0,2,3,5\}$. Forgetting the leading-zero rule gives $5\cdot 24=120$, and forgetting the set $\{1,2,3,4\}$ gives 72. A computer check of all four-digit numbers also finds 96.

9. Answer: (C) 63**. The idea.** $\gcd(n,100)$ is always a divisor of $100=2^2\cdot 5^2$, and only four of those divisors are perfect squares: 1, 4, 25 and 100. Each prime's exponent in the gcd must be even, so decide separately what n does with the prime 2 and the prime 5.

Work it out. The power of 2 in the gcd is 2^0 if n is odd, 2^1 if n is even but not a multiple of 4, and 2^2 if $4\mid n$; it is a square exactly when n is odd or $4\mid n$. Likewise the power of 5 is a square exactly when $5\nmid n$ or $25\mid n$. Now count the four cases. Gcd 1 (n odd, $5\nmid n$): the numbers coprime to 100, of which there are $100\cdot\frac{1}{2}\cdot\frac{4}{5}=40$. Gcd 4 ($4\mid n$, $5\nmid n$): 25 multiples of 4, minus the 5 multiples of 20, leaves 20. Gcd 25 ($25\mid n$, n odd): 25 and 75, so 2. Gcd 100: only $n=100$, so 1. Total $40+20+2+1=63$.

Check. Count the complement: the gcd fails to be a square when $n\equiv 2 \pmod 4$ (25 numbers) or when $5\mid n$ but $25\nmid n$ ($20-4=16$ numbers), and both happen for $n=10,30,70,90$, so $25+16-4=37$ numbers fail and $100-37=63$ succeed. Forgetting that 1 is a square gives 23; counting all multiples of 4 without removing multiples of 20 gives 68.

10. Answer: (A) $50\sqrt{3}$**. The idea.** The midpoint of the hypotenuse is the centre of the circle through A, B and C, so $MA=MB=MC$, making the median create isosceles triangles. The altitude creates angles equal to the triangle's acute angles. Matching these to three 30° angles pins down the triangle's shape.

Work it out. Say $\angle B\leq\angle A$. Because $MC=MB$, triangle MBC is isosceles and $\angle MCB=\angle B$. Because $\overline{CH}\perp\overline{AB}$, in right triangle AHC we get $\angle ACH=90^\circ-\angle A=\angle B$. So the two outer pieces of the right angle both equal $\angle B$, and the middle piece is $90^\circ-2\angle B$. All three are equal, so $\angle B=90^\circ-2\angle B$, giving $\angle B=30^\circ$ and $\angle A=60^\circ$. The triangle is a 30-60-90 triangle with hypotenuse 20, so its legs are 10 and $10\sqrt{3}$, and its area is $\frac{1}{2}\cdot 10\cdot 10\sqrt{3}=50\sqrt{3}$.

Check. In this triangle $\angle MCB=30^\circ$, $\angle ACH=90^\circ-60^\circ=30^\circ$, and the middle angle is $90^\circ-30^\circ-30^\circ=30^\circ$, as required. Also the altitude is $\frac{10\cdot 10\sqrt{3}}{20}=5\sqrt{3}$, and $\frac{1}{2}\cdot 20\cdot 5\sqrt{3}=50\sqrt{3}$ agrees. The distractor 100 is the largest possible area (an isosceles right triangle), which this triangle is not.

11. Answer: (D) 91**. The idea.** Taking differences of consecutive terms turns an arithmetic sequence into a constant, and taking differences again removes it completely. A geometric sequence stays geometric, with the same ratio, however many times you take differences. So the second differences of c form a pure geometric sequence.

Work it out. Write $a_n=p+(n-1)d$ and $b_n=qr^{n-1}$. Then $c_{n+1}-c_n=d+q(r-1)r^{n-1}$, and the second differences are $q(r-1)^2r^{n-1}$. From $3,7,15,35$ the first differences are $4,8,20$ and

the second differences are 4, 12. Their ratio is the common ratio: $r = 3$. Then $q(3-1)^2 = 4$ gives $q = 1$; the first difference $4 = d + q(r-1) = d + 2$ gives $d = 2$; and $c_1 = p + q = 3$ gives $p = 2$. So $a_n = 2n$ and $b_n = 3^{n-1}$, and $c_5 = 10 + 81 = 91$.

Check. $2 + 1 = 3$, $4 + 3 = 7$, $6 + 9 = 15$, $8 + 27 = 35$. Also, the next second difference is $12 \cdot 3 = 36$, so the next first difference is $20 + 36 = 56$ and $c_5 = 35 + 56 = 91$. Guessing that the second differences grow by a constant 8 (to 20) would give the distractor $35 + 40 = 75$.

12. Answer: (B) 95**. The idea.** With c correct answers the score is $5c$ plus an even number from 0 up to $2(20-c)$. The parity of the score is the parity of c, so handle even and odd scores separately. The ranges for consecutive values of c of the same parity overlap in the middle, but at the bottom and at the top, where few blanks are left, gaps appear.

Work it out. With c correct and b blank ($b \le 20 - c$), the score is $5c + 2b$, which ranges over the numbers $5c, 5c + 2, \ldots, 5c + 2(20 - c) = 40 + 3c$ of the same parity as c. Even scores come from even c: $c = 0$ gives 0 to 40; $c = 2$ gives 10 to 46; $c = 4$ gives 20 to 52, and so on, each range reaching past where the next begins, up to $c = 16$ (80 to 88), $c = 18$ (90 to 94) and $c = 20$ (just 100). So the even scores are $0, 2, \ldots, 94$ (48 of them) and 100: 49 in all; 96 and 98 are impossible. Odd scores come from odd c: $c = 1$ gives 5 to 43, and the ranges again chain together up to $c = 17$ (85 to 91), then $c = 19$ gives only 95 and 97. So the odd scores are $5, 7, \ldots, 91$ (44 of them) plus 95 and 97: 46 in all; 1, 3, 93 and 99 are impossible. Total $49 + 46 = 95$.

Check. The impossible scores are exactly $1, 3, 93, 96, 98, 99$: six of the 101 numbers, and $101 - 6 = 95$. For instance 99 would need c odd and $5c \le 99$ with $5c + 2(20 - c) \ge 99$, i.e. $c \ge 19\frac{2}{3}$, which is impossible. Missing only the small gaps (1 and 3) would give the tempting 99.

13. Answer: (C) 27**. The idea.** Two numbers can differ by 6 only if they leave the same remainder when divided by 6. So split $\{1, \ldots, 50\}$ into six chains by remainder, each chain going up in steps of 6. A set avoids a difference of 6 exactly when it never takes two neighbours in the same chain. Find the biggest such set; k is one more than its size.

Work it out. The chains are $1, 7, 13, \ldots, 49$ and $2, 8, \ldots, 50$ (nine numbers each), and $3, 9, \ldots, 45$, then $4, \ldots, 46$, then $5, \ldots, 47$, then $6, \ldots, 48$ (eight numbers each). In a chain of 9 with no two neighbours taken, you can take at most 5: pair the chain up as (1st, 2nd), (3rd, 4th), …, (7th, 8th) plus the lone 9th, and each pair gives at most one. The same pairing shows a chain of 8 gives at most 4. So a set with no difference 6 has at most $5 + 5 + 4 + 4 + 4 + 4 = 26$ numbers.

And 26 is possible: take every other member of each chain, starting from the first. That is exactly the numbers 1 to 6, 13 to 18, 25 to 30, 37 to 42 and 49, 50, which is $6 + 6 + 6 + 6 + 2 = 26$ numbers, and no two of them differ by 6. So some 26-number set avoids the difference, but every 27-number set contains it. The answer is $k = 27$.

Check. The same method on $\{1, \ldots, 12\}$ with difference 6 gives six chains of length 2, so at most 6 numbers, and $\{1, \ldots, 6\}$ achieves it: $k = 7$, which is right, since 7 numbers from 1 to 12 must include a pair $n, n + 6$. Answering with the size of the biggest safe set gives 26; treating all six chains as length 9 gives 31; and 45 counts the 44 pairs $(n, n + 6)$, a much weaker bound. A computer scan over all subsets (one number at a time, remembering the last six choices) also finds that the largest safe set has 26 numbers.

14. Answer: (A) 180**. The idea.** Walk along the star and keep track of how far you turn. At a tip with angle t you turn through $180° - t$. The total turning over one full circuit is $360°$ times the

number of times the path winds around the centre.

Work it out. Each segment moves you 3 vertices forward around the heptagon. After the seven segments you have moved 21 steps, which is 3 full laps. Because the vertices are in convex position, every turn is in the same direction. So the path winds around 3 times and the total turning is $3 \cdot 360^\circ = 1080^\circ$. Therefore $\sum(180^\circ - t_k) = 1080^\circ$, and $\sum t_k = 7 \cdot 180^\circ - 1080^\circ = 180^\circ$.

Check. For a regular heptagon, each tip angle is an inscribed angle standing on one seventh of the circle, so it equals $\frac{1}{2} \cdot \frac{360^\circ}{7} = \frac{180^\circ}{7}$. Seven of them add to 180°. The ordinary five-pointed star works the same way: it winds twice, so its tips add to $5 \cdot 180^\circ - 720^\circ = 180^\circ$.

15. Answer: (A) 800**. The idea.** Divisibility can be checked one prime at a time. Write $360 = 2^3 \cdot 3^2 \cdot 5$. A chain $a \mid b \mid c \mid 360$ is the same as choosing, separately for each prime, a non-decreasing list of three exponents that stays at or below that prime's exponent in 360.

Work it out. Say a, b, c contain the prime 2 to the powers x, y, z. The conditions on this prime say exactly $0 \le x \le y \le z \le 3$, and likewise for 3 (exponents up to 2) and 5 (up to 1). The choices for different primes are independent, so we multiply.

A non-decreasing list $x \le y \le z$ of values from $\{0, 1, \ldots, e\}$ is just a choice of 3 values from $e + 1$ with repetition allowed, and there are $\binom{e+3}{3}$ of those (stars and bars: 3 stars and e bars). So the count is $\binom{6}{3} \cdot \binom{5}{3} \cdot \binom{4}{3} = 20 \cdot 10 \cdot 4 = 800$.

Check. For the prime 5 the four exponent patterns are 000, 001, 011, 111, and indeed $\binom{4}{3} = 4$. For $c \mid p$ with p prime the rule predicts 4 triples, and they are $(1, 1, 1)$, $(1, 1, p)$, $(1, p, p)$, (p, p, p). Using $\binom{e+4}{3}$ instead of $\binom{e+3}{3}$ (an off-by-one in stars and bars) gives $35 \cdot 20 \cdot 10 = 7000$, multiplying by $3! = 6$ as if the triples could be rearranged gives 4800, and $24^3 = 13824$ counts all triples of divisors with no chain condition. A computer check over all divisor triples of 360 finds 800.

16. Answer: (E) 61**. The idea.** Look at the three numbers $0, x, y$. The three absolute values $|x - 0|$, $|y - 0|$, $|x - y|$ are the three distances between them. For any three numbers on a line, the two short distances add up to the long one, so the sum of all three distances is twice the long one: $|x| + |y| + |x - y| = 2(\text{largest} - \text{smallest})$ of $0, x, y$. The condition just says that 0, x and y all fit in a window of length 4.

Work it out. We need $\max(0, x, y) - \min(0, x, y) \le 4$. Sort the pairs by the smallest of the three numbers, $m = \min(0, x, y)$, which is one of $0, -1, -2, -3, -4$.

If $m = 0$: x and y are both in $\{0, 1, 2, 3, 4\}$, giving 25 pairs.

If $m = -k$ with $1 \le k \le 4$: all three numbers lie in the window $\{-k, \ldots, 4 - k\}$ (which contains 0), giving $5 \cdot 5 = 25$ pairs, but we must remove the $4 \cdot 4 = 16$ pairs that avoid $-k$ entirely, since their smallest number is larger. That leaves 9 pairs for each of the four values of k.

The total is $25 + 4 \cdot 9 = 61$.

Check. The region is the hexagon with corners $(4, 0)$, $(4, 4)$, $(0, 4)$, $(-4, 0)$, $(-4, -4)$, $(0, -4)$. It is made of the squares $[0, 4]^2$ and $[-4, 0]^2$ (area 16 each) and two right triangles with legs 4 (area 8 each), so its area is 48. It has 24 lattice points on its boundary, so Pick's theorem gives $48 - 12 + 1 = 37$ inside and $37 + 24 = 61$ in all. Counting only the pairs with $|x| + |y| \le 4$ gives 41; forgetting the case $m = -4$ gives $25 + 3 \cdot 9 = 52$.

17. Answer: (B) $\frac{3}{8}$**. The idea.** Put the hexagon on coordinates with its centre at the origin. Then

every vertex and midpoint has simple coordinates, and the shoelace (cross product) formula gives the triangle's area directly.

Work it out. The vertices are $A = (2, 0)$, $B = (1, \sqrt{3})$, $C = (-1, \sqrt{3})$, $D = (-2, 0)$, $E = (-1, -\sqrt{3})$, $F = (1, -\sqrt{3})$. So $M = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $N = (0, -\sqrt{3})$. From A, the vectors to M and N are $\left(-\frac{7}{2}, \frac{\sqrt{3}}{2}\right)$ and $(-2, -\sqrt{3})$. Their cross product is $\left(-\frac{7}{2}\right)(-\sqrt{3}) - \frac{\sqrt{3}}{2}(-2) = \frac{7\sqrt{3}}{2} + \sqrt{3} = \frac{9\sqrt{3}}{2}$, so $[AMN] = \frac{9\sqrt{3}}{4}$. The hexagon is six equilateral triangles of side 2, with area $6 \cdot \sqrt{3} = 6\sqrt{3}$. The fraction is $\dfrac{9\sqrt{3}/4}{6\sqrt{3}} = \dfrac{9}{24} = \dfrac{3}{8}$.

Check. Add up the three pieces of the hexagon outside the triangle. Quadrilateral $ABCM$ has area $2\sqrt{3}$, quadrilateral $MDEN$ has area $\frac{5\sqrt{3}}{4}$, and triangle NFA (base $NF = 1$, height $\sqrt{3}$) has area $\frac{\sqrt{3}}{2}$. Together they make $\frac{15\sqrt{3}}{4}$, and $6\sqrt{3} - \frac{15\sqrt{3}}{4} = \frac{9\sqrt{3}}{4}$, as found. Forgetting the $\frac{1}{2}$ in the triangle's area gives $\frac{3}{4}$, and reporting the outside part instead gives $\frac{5}{8}$.

18. Answer: (D) 176**. The idea.** Divisors come in pairs $(e, n/e)$ with one member at most $\sqrt{n}$, so $d(n) \le 2\sqrt{n}$. A number cannot be a large multiple of its divisor count unless it is small, and then the powers of 2 in n pin everything down.

Work it out. From $n = 8d(n) \le 16\sqrt{n}$ we get $n \le 256$, and n is a multiple of 8. Write $n = 2^{b+3}t$ with t odd and $b \ge 0$. Then $d(n) = (b + 4)\, d(t)$ and $n/8 = 2^b t$, so we need $(b + 4)\, d(t) = 2^b t$.

$b = 0$: $4d(t) = t$ is impossible, as t is odd.

$b = 1$: $5d(t) = 2t$, so $t = \frac{5}{2}d(t)$ and $5 \mid t$. With $d(t) \le 2\sqrt{t}$ this gives $t \le 25$, so $t \in \{5, 15, 25\}$; only $t = 5$ works ($d(5) = 2$). This gives $n = 2^4 \cdot 5 = 80$.

$b = 2$: $6d(t) = 4t$, so $t = \frac{3}{2}d(t)$, $3 \mid t$ and $t \le 9$. $t = 3$ works ($d(3) = 2$) but $t = 9$ does not ($d(9) = 3$). This gives $n = 2^5 \cdot 3 = 96$.

$b = 3$: $7d(t) = 8t$ forces $d(t) > t$, impossible since $d(t) \le t$.

$b \ge 4$: now $2^b > b + 4$, so $d(t) = \frac{2^b}{b+4}t > t$, impossible again.

The solutions are 80 and 96, and their sum is 176.

Check. $80 = 2^4 \cdot 5$ has $5 \cdot 2 = 10$ divisors and $80 = 8 \cdot 10$. $96 = 2^5 \cdot 3$ has $6 \cdot 2 = 12$ divisors and $96 = 8 \cdot 12$. A computer search up to 20000 finds no others.

19. Answer: (D) $\frac{5}{13}$**. The idea.** If a full round of three rolls produces no winner, the game starts over exactly as before. So only the first round matters: the chance that each player wins is proportional to the chance that they win in round one.

Work it out. In one round, Ada wins with probability $\frac{1}{6} = \frac{36}{216}$. Ben wins with probability $\frac{5}{6} \cdot \frac{2}{6} = \frac{60}{216}$. Cy wins with probability $\frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{60}{216}$. The round has no winner with probability $\frac{60}{216}$, and then everything repeats. Given that the game ends in some round, the three win chances are in the ratio $36 : 60 : 60 = 3 : 5 : 5$. So Cy wins with probability $\frac{5}{13}$. Surprisingly, Ben and Cy have exactly the same chance.

Check. Summing the series directly gives $P(\text{Cy}) = \frac{60}{216}\left(1 + \frac{60}{216} + \left(\frac{60}{216}\right)^2 + \cdots\right) = \frac{60/216}{156/216} = \frac{60}{156} = \frac{5}{13}$. The three answers $\frac{3}{13} + \frac{5}{13} + \frac{5}{13}$ add to 1.

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20. Answer: (E) 28**. The idea.** A product grows much faster than a sum, so the numbers cannot all be large. First sort them, $a \le b \le c$. Then $3(a+b+c) \le 9c$, so $abc \le 9c$ and $ab \le 9$. That leaves only a few choices for the two smallest numbers, and for each one the equation factors (Simon's favourite factoring trick). At the end, count the orderings.

Work it out. Since $ab \le 9$ and $a \le b$, we have $a \le 3$.

$a = 1$: $bc = 3 + 3b + 3c$, so $bc - 3b - 3c + 9 = 12$, that is, $(b-3)(c-3) = 12$. Both factors are at least -2, so they cannot both be negative (their product would be at most 4). The positive factorizations with $b - 3 \le c - 3$ are $1 \cdot 12$, $2 \cdot 6$, $3 \cdot 4$, giving $(1, 4, 15)$, $(1, 5, 9)$, $(1, 6, 7)$.

$a = 2$: $2bc = 6 + 3b + 3c$. Multiply by 2: $4bc - 6b - 6c = 12$, so $(2b-3)(2c-3) = 21$. With $2 \le b \le c$ the factors are positive, and $1 \cdot 21$, $3 \cdot 7$ give $(2, 2, 12)$, $(2, 3, 5)$.

$a = 3$: $3bc = 9 + 3b + 3c$, so $bc - b - c = 3$ and $(b-1)(c-1) = 4$. With $b \ge 3$ only $2 \cdot 2$ works, giving $(3, 3, 3)$.

That is 6 sorted triples. Now count orderings: a triple of three different numbers can be ordered in 6 ways, one with exactly two equal numbers in 3 ways, and $(3, 3, 3)$ in 1 way. The total is $4 \cdot 6 + 3 + 1 = 28$.

Check. $1 \cdot 4 \cdot 15 = 60 = 3 \cdot 20$, $1 \cdot 5 \cdot 9 = 45 = 3 \cdot 15$, $1 \cdot 6 \cdot 7 = 42 = 3 \cdot 14$, $2 \cdot 2 \cdot 12 = 48 = 3 \cdot 16$, $2 \cdot 3 \cdot 5 = 30 = 3 \cdot 10$ and $27 = 3 \cdot 9$. Stopping at the sorted count gives 6; counting only the four all-different triples gives 24; missing $(1, 4, 15)$ gives 22; and forgetting $(3, 3, 3)$ gives 27. A computer that solves $c(ab-3) = 3(a+b)$ for every pair $a, b < 400$ finds 28 ordered triples.

21. Answer: (C) 116**. The idea.** First read the shape: $6^2 + 8^2 = 10^2$, so $\angle ADB = 90^\circ$ and the parallelogram has area $2 \cdot \frac{1}{2} \cdot 6 \cdot 8 = 48$. Next, the four centres always form a square. Coordinates prove this and give its side.

Work it out. Put $A = (0, 0)$ and $B = (10, 0)$. D is 6 from A and 8 from B, so $D = \left(\frac{18}{5}, \frac{24}{5}\right)$ and $C = B + D = \left(\frac{68}{5}, \frac{24}{5}\right)$. The centre of the outward square on a side is its midpoint moved outward, perpendicular to the side, by half the side length.

On AB: $P = (5, -5)$. On BC: the midpoint is $(11.8, 2.4)$ and the outward shift is $(2.4, -1.8)$, so $Q = (14.2, 0.6)$. On CD: $R = (8.6, 4.8 + 5) = (8.6, 9.8)$. On DA: the midpoint is $(1.8, 2.4)$ and the shift is $(-2.4, 1.8)$, so $S = (-0.6, 4.2)$.

Then $\overrightarrow{PQ} = (9.2, 5.6)$, $\overrightarrow{QR} = (-5.6, 9.2)$, $\overrightarrow{RS} = (-9.2, -5.6)$ and $\overrightarrow{SP} = (5.6, -9.2)$. Each is the previous one turned by 90°, so $PQRS$ is a square with side squared $9.2^2 + 5.6^2 = 84.64 + 31.36 = 116$. Its area is 116.

Check. Doing the same computation with general sides a, b and area K gives area $\frac{a^2+b^2}{2} + K$, here $\frac{100+36}{2} + 48 = 116$. For a rectangle ($K = ab$) this becomes $\frac{(a+b)^2}{2}$, which matches the easy picture: the centres sit $\frac{a+b}{2}$ from the middle along both axes. Treating this parallelogram as a 10×6 rectangle would wrongly give 128.

22. Answer: (C) 89**. The idea.** Split the list at $\sqrt{2026} \approx 45.01$. For small n, neighbouring quotients $\frac{2026}{n}$ are more than 1 apart, so their floors are all different. For large n, the quotients are less than 1 apart and all small, so the floors hit every value from 1 up to their maximum.

Work it out. $\frac{2026}{n} - \frac{2026}{n+1} = \frac{2026}{n(n+1)}$, which is more than 1 exactly when $n(n+1) < 2026$, that is, for $n \le 44$ (since $44 \cdot 45 = 1980$ and $45 \cdot 46 = 2070$). So the floors for $n = 1, \ldots, 45$ are strictly

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decreasing. That gives 45 different values, the smallest being $\lfloor 2026/45 \rfloor = 45$. The value for $n = 44$ is 46.

For $n \geq 45$, consecutive quotients differ by less than 1, so the floors drop by at most 1 at each step. They fall from 45 at $n = 45$ to 1 at $n = 2026$ and hit every value $45, 44, \ldots, 1$. Only 45 is shared with the first group. The total is $44 + 45 = 89$.

Check. Try the same method on 12 in place of 2026. Here $n(n+1) < 12$ only for $n \leq 2$, and $\lfloor 12/3 \rfloor = 4$, so the count should be $2 + 4 = 6$. The list $12, 6, 4, 3, 2, 2, 1, 1, 1, 1, 1, 1$ does have exactly 6 values. A computer run for 2026 counts 89.

23. Answer: (E) $\frac{1}{2} - \frac{3}{2^{33}-2}$**. The idea.** Two telescopes, one inside the other. First, $a_0 = 2 + \frac{1}{2} = t + \frac{1}{t}$ with $t = 2$, and $(t^m + t^{-m})^2 - 2 = t^{2m} + t^{-2m}$, so $a_n = t^{2^n} + t^{-2^n}$ and each partial product $a_0 a_1 \cdots a_k$ collapses by repeated difference of squares. Second, the reciprocals of those products split into differences that telescope again.

Work it out. Write $u_m = 2^{2^m}$, so $u_0 = 2$, $u_1 = 4$, $u_2 = 16$, and $u_{m+1} = u_m^2$. Then $a_m = u_m + \frac{1}{u_m}$. Since $\left(t - \frac{1}{t}\right)\left(t + \frac{1}{t}\right) = t^2 - \frac{1}{t^2}$, multiplying $a_0 a_1 \cdots a_k$ by $u_0 - \frac{1}{u_0} = \frac{3}{2}$ collapses it step by step to $u_{k+1} - \frac{1}{u_{k+1}}$. So $a_0 a_1 \cdots a_k = \dfrac{2}{3} \cdot \dfrac{u_{k+1}^2 - 1}{u_{k+1}}$ and $\dfrac{1}{a_0 \cdots a_k} = \dfrac{3}{2} \cdot \dfrac{u}{u^2 - 1}$ with $u = u_{k+1}$.

Now $\dfrac{u}{u^2-1} = \dfrac{(u+1)-1}{u^2-1} = \dfrac{1}{u-1} - \dfrac{1}{u^2-1}$, and $u^2 = u_{k+2}$. So the k-th term is $\frac{3}{2}\left(\frac{1}{u_{k+1}-1} - \frac{1}{u_{k+2}-1}\right)$. Adding $k = 0, 1, 2, 3$, the middle terms cancel: $\dfrac{3}{2}\left(\dfrac{1}{u_1 - 1} - \dfrac{1}{u_5 - 1}\right) = \dfrac{3}{2}\left(\dfrac{1}{3} - \dfrac{1}{2^{32} - 1}\right) = \dfrac{1}{2} - \dfrac{3}{2^{33} - 2}$.

Check. One term: the formula gives $\frac{3}{2}\left(\frac{1}{3} - \frac{1}{15}\right) = \frac{2}{5} = \frac{1}{a_0}$. Two terms: $\frac{2}{5} + \frac{8}{85} = \frac{42}{85}$, and the formula gives $\frac{3}{2}\left(\frac{1}{3} - \frac{1}{255}\right) = \frac{126}{255} = \frac{42}{85}$. Stopping after one, two or three terms gives $\frac{1}{2} - \frac{3}{2^5-2}$, $\frac{1}{2} - \frac{3}{2^9-2}$ or $\frac{1}{2} - \frac{3}{2^{17}-2}$, and the infinite sum is exactly $\frac{1}{2}$. Multiplying the last piece by 3 instead of $\frac{3}{2}$ gives $\frac{1}{2} - \frac{3}{2^{32}-1}$, which differs from the answer by less than 10^{-9}, so no estimate can separate the two: only the exact telescoping decides.

24. Answer: (E) $12 - \pi$**. The idea.** Follow the centre of the coin, not the coin. It always stays exactly 1 from the side it touches, so it runs around a smaller triangle whose sides are 1 unit inside the tray's sides. The coin covers exactly the points within 1 of that path. Two kinds of region are missed: a triangle in the middle, and a small pocket in each corner of the tray.

Work it out. Since $9^2 + 12^2 = 15^2$, the tray is a right triangle with area 54 and inradius $\frac{9+12-15}{2} = 3$. Moving every side inward by 1 shrinks the triangle about its incentre by the factor $\frac{3-1}{3} = \frac{2}{3}$. So the centre's path is a 6-8-10 triangle, with area 24, perimeter 24 and inradius 2.

The middle: the points inside the path triangle that are more than 1 from its sides form a triangle shrunk again, by $\frac{2-1}{2} = \frac{1}{2}$. That is a 3-4-5 triangle of area 6, never covered.

The corners: the part of the tray outside the path triangle has area $54 - 24 = 30$. The coin covers the points of it within 1 of the path triangle. These form a rectangle of width 1 on each side, of total area $24 \cdot 1 = 24$, plus a circular sector of radius 1 at each corner of the path. The three sectors' angles are the exterior angles of the triangle, which add to $360°$, so they make one full unit disc, of area π. The uncovered corner pockets therefore have total area $30 - 24 - \pi = 6 - \pi$.

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The total uncovered area is $6 + (6 - \pi) = 12 - \pi \approx 8.86$.

Check. The pocket at the right-angle corner is a unit square with a quarter of a unit disc taken out, area $1 - \frac{\pi}{4} \approx 0.21$, which is positive and small as it should be. Forgetting the pockets gives 6, forgetting the middle triangle gives $6 - \pi$, and treating every corner pocket like the right-angle one (a unit square minus a sector) gives $6 + 3 - \pi = 9 - \pi$. Cutting the tray into thin horizontal slices and adding up the uncovered length of each (by computer) gives 8.8584.

25. Answer: (E) $\frac{128}{3}$. The idea. Only the number of heads matters. The number of heads can only drop by one at a time, so the time to reach 0 is the sum of $T_5, T_4, \ldots, T_1$, where T_k is the expected time to first go from k heads to $k-1$ heads. Each T_k comes from the one above it.

Work it out. With k heads, a turn lowers the count with probability $\frac{k}{5}$. Otherwise it raises the count to $k+1$, and then we need on average T_{k+1} turns to get back to k and T_k more after that. So $T_k = 1 + \frac{5-k}{5}(T_{k+1} + T_k)$, which rearranges to $T_k = \dfrac{5 + (5-k)T_{k+1}}{k}$. Starting from $T_5 = 1$ (every flip turns a head to tails): $T_4 = \frac{5+1}{4} = \frac{3}{2}$, $T_3 = \frac{5+2\cdot\frac{3}{2}}{3} = \frac{8}{3}$, $T_2 = \frac{5+3\cdot\frac{8}{3}}{2} = \frac{13}{2}$, $T_1 = 5 + 4 \cdot \frac{13}{2} = 31$. The total is $1 + \frac{3}{2} + \frac{8}{3} + \frac{13}{2} + 31 = \frac{6+9+16+39+186}{6} = \frac{256}{6} = \frac{128}{3}$.

Check. With one coin the answer is 1. With two coins the same method gives $T_2 = 1$ and $T_1 = 2 + 1 \cdot 1 = 3$, a total of 4. Listing the 32 head/tail patterns and solving their expected-time equations exactly (by computer) also gives $\frac{128}{3}$. The tempting $5\left(1 + \frac{1}{2} + \cdots + \frac{1}{5}\right) = \frac{137}{12}$ ignores coins turned back to heads. The last head is the slow part: $T_1 = 31$, and leaving it out gives only $\frac{35}{3}$.

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