## MathCastle

# AMC 10

Practice Contest

25 questions • 75 minutes

**Name** _____

**School** _____

**Grade** _____

**INSTRUCTIONS**

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

**DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.**

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

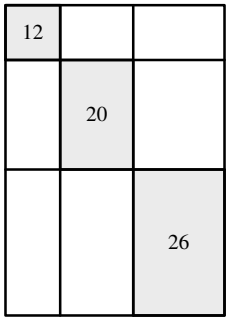
**1.** A rectangular photo measures 5 inches by 12 inches. It is set in a frame of the same width $x$ inches on all four sides, so that the photo and the frame together form a larger rectangle measuring $(5 + 2x)$ by $(12 + 2x)$ inches. The area of the frame alone (the larger rectangle with the photo removed) is exactly equal to the area of the photo. What is $x$?

**(A)** 1 **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{5}{2}$ **(E)** 3

**2.** Four consecutive positive integers $n$, $n + 1$, $n + 2$ and $n + 3$ are divisible by 7, 8, 9 and 10, respectively. How many such $n$ satisfy $1 \le n \le 10000$?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

**3.** A rectangle is cut into nine smaller rectangles by two straight cuts parallel to one pair of its sides and two straight cuts parallel to the other pair, like a tic-tac-toe board; the pieces need not be equal. The nine pieces form three rows and three columns. The piece in the top-left corner has perimeter 12, the piece in the centre has perimeter 20, and the piece in the bottom-right corner has perimeter 26. What is the perimeter of the original rectangle?




**(A)** 29 **(B)** 52 **(C)** 58 **(D)** 64 **(E)** 116

**4.** The digits of a five-digit positive integer add up to at most $5 \cdot 9 = 45$. How many five-digit positive integers have digits that add up to exactly 42? (For example, 99879 is one of them.)

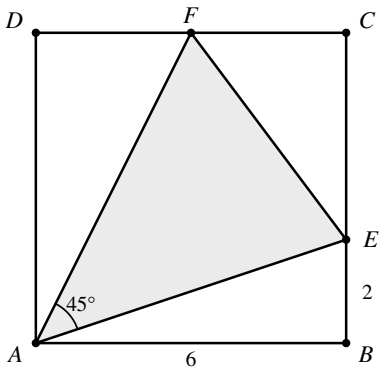
**(A)** 35 **(B)** 56 **(C)** 60 **(D)** 70 **(E)** 125

**5.** A cart has one front wheel and one back wheel, of different sizes. The circumference of the back wheel is 30 cm greater than the circumference of the front wheel. On a straight trip of 360 metres, with neither wheel slipping, the front wheel turns exactly 40 more times than the back wheel. What is the circumference of the front wheel, in centimetres?

**(A)** 90 **(B)** 120 **(C)** 150 **(D)** 180 **(E)** 240

**6.** Square $ABCD$ has side length 6, with its vertices labelled in order around the square. Point $E$ lies on side $\overline{BC}$ with $BE = 2$, and point $F$ lies on side $\overline{CD}$, placed so that $\angle EAF = 45^\circ$. What is the area of triangle $AEF$?

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**(A)** 15 **(B)** 16 **(C)** 18 **(D)** 21 **(E)** 24

**7.** Let $P = 1 \cdot 3 \cdot 5 \cdot 7 \cdots 97 \cdot 99$ be the product of all the odd positive integers less than 100. What two-digit number do the last two digits of $P$ form? (For example, the last two digits of $1 \cdot 3 \cdot 5 \cdot 7 = 105$ form the number 05.)

**(A)** 15 **(B)** 25 **(C)** 45 **(D)** 75 **(E)** 95

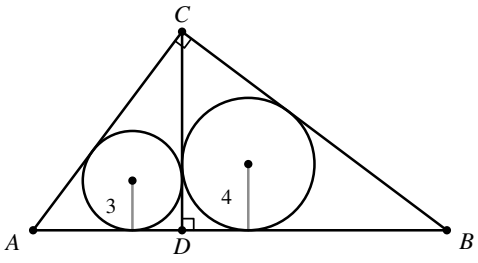
**8.** Nine cards are numbered 1 through 9. Three different cards are drawn at random, with every set of three cards equally likely. What is the probability that the average (mean) of the three numbers drawn is equal to one of the three numbers drawn?

**(A)** $\frac{1}{7}$ **(B)** $\frac{4}{21}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{5}{21}$ **(E)** $\frac{4}{9}$

**9.** As $x$ and $y$ range over all real numbers, what is the smallest value that the expression $x^2 + 2xy + 2y^2 - 4x + 2y + 30$ can take?

**(A)** 13 **(B)** 17 **(C)** 21 **(D)** 26 **(E)** 30

**10.** Triangle $ABC$ has a right angle at $C$. The altitude from $C$ meets the hypotenuse $\overline{AB}$ at $D$, splitting triangle $ABC$ into two smaller right triangles, $ACD$ and $BCD$. The circle inscribed in triangle $ACD$ has radius 3, and the circle inscribed in triangle $BCD$ has radius 4. What is the area of triangle $ABC$?




**(A)** 75 **(B)** 120 **(C)** 150 **(D)** 225 **(E)** 300

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**11.** Call a positive divisor $d$ of 2520 *clean* if $d$ and $\frac{2520}{d}$ have no common factor greater than 1. For example, 9 is clean because $\frac{2520}{9} = 280$ and $\gcd(9, 280) = 1$, but 6 is not clean because $\frac{2520}{6} = 420$ is also divisible by 6. What is the sum of all the clean divisors of 2520?

**(A)** 576 **(B)** 1799 **(C)** 2520 **(D)** 4320 **(E)** 9360

**12.** A cube has 12 edges. Three different edges are chosen at random, with every set of three edges equally likely. What is the probability that the three edges are pairwise skew, meaning that no two of the chosen edges are parallel and no two of them share a vertex?

**(A)** $\frac{1}{55}$ **(B)** $\frac{2}{55}$ **(C)** $\frac{4}{55}$ **(D)** $\frac{6}{55}$ **(E)** $\frac{16}{55}$

**13.** A farmer has 100 metres of fencing and wants to build a rectangular pen against the straight outside wall of a barn. The wall is 30 metres long. One side of the pen lies along the wall. Wherever the pen's boundary runs along the wall, no fence is needed; every other part of the boundary (including any part of that side that sticks out past the end of the wall) must be fenced. The farmer does not have to use all of the fencing. What is the largest possible area of the pen, in square metres?

**(A)** $1056\frac{1}{4}$ **(B)** 1100 **(C)** 1250 **(D)** 1500 **(E)** $2112\frac{1}{2}$

**14.** In a regular polygon with $n$ sides, where $n \geq 4$, all of the diagonals from one vertex are drawn. Together with the two sides at that vertex, they split the interior angle at that vertex into $n - 2$ smaller angles. For how many values of $n$ is every one of these smaller angles a whole number of degrees?

**(A)** 15 **(B)** 16 **(C)** 18 **(D)** 21 **(E)** 22

**15.** Some of the integers $1, 2, 3, \ldots, 60$ are chosen so that the product of any two different chosen numbers is never a perfect square. (For example, 2 and 18 cannot both be chosen, because $2 \cdot 18 = 36$.) What is the largest possible number of chosen integers?

**(A)** 37 **(B)** 38 **(C)** 40 **(D)** 45 **(E)** 53

**16.** The nine cards numbered 1 through 9 are shuffled and laid out in a row, with all orders equally likely. A card is called a peak if its number is larger than the number on each card next to it. (A card at either end of the row has only one neighbour, so it only has to beat that one.) What is the expected number of peaks?

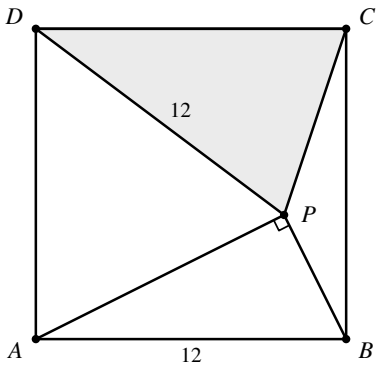
**(A)** $\frac{7}{3}$ **(B)** 3 **(C)** $\frac{10}{3}$ **(D)** $\frac{7}{2}$ **(E)** $\frac{11}{3}$

**17.** Real numbers $a$ and $b$ satisfy $a + b = 6$. What is the smallest possible value of $(a^2 + 7)(b^2 + 7)$?

**(A)** 84 **(B)** 196 **(C)** 203 **(D)** 245 **(E)** 252

**18.** Square $ABCD$ has side length 12. A point $P$ inside the square satisfies $\angle APB = 90^\circ$ and $PD = 12$. What is the area of triangle $PCD$?

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**(A)** $\frac{144}{5}$ **(B)** 36 **(C)** $\frac{216}{5}$ **(D)** 48 **(E)** 54

**19.** For $n = 11$, each of the three numbers $n = 11$, $n + 2 = 13$ and $n + 6 = 17$ has no prime factor in common with 210. For $n = 13$ this fails, because $n + 2 = 15$ shares the factor 3 (and 5) with 210. For how many integers $n$ with $1 \leq n \leq 2100$ are all three of $n$, $n + 2$ and $n + 6$ relatively prime to 210?

**(A)** 40 **(B)** 80 **(C)** 120 **(D)** 160 **(E)** 480

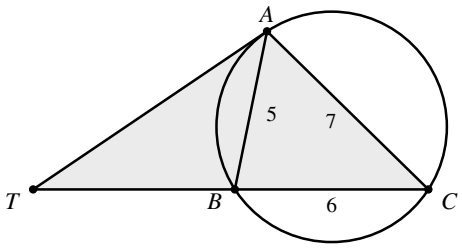
**20.** A positive divisor $d$ of 360 is chosen at random, with all 24 positive divisors of 360 equally likely. Then a positive divisor $e$ of $d$ is chosen at random, with all positive divisors of $d$ equally likely. What is the probability that $e = 1$?

**(A)** $\frac{1}{24}$ **(B)** $\frac{65}{864}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{2}{15}$ **(E)** $\frac{275}{1152}$

**21.** For a real number $x$, let $\lfloor x \rfloor$ be the greatest integer less than or equal to $x$. What is the sum of the squares of all real numbers $x$ that satisfy $x^2 - 10\lfloor x \rfloor + 9 = 0$ ?

**(A)** 82 **(B)** 133 **(C)** 152 **(D)** 213 **(E)** 214

**22.** Triangle $ABC$ has $AB = 5$, $BC = 6$ and $CA = 7$. The line tangent to the circle through $A$, $B$ and $C$ at the point $A$ meets line $BC$ at a point $T$. What is the area of triangle $ATC$?




**(A)** $6\sqrt{6}$ **(B)** $\frac{25\sqrt{6}}{4}$ **(C)** $7\sqrt{6}$ **(D)** $\frac{35\sqrt{6}}{4}$ **(E)** $\frac{49\sqrt{6}}{4}$

**23.** Some positive integers can be written in base 3 using only the digits 0 and 1; for example, $12 = 110_3$

and $28 = 1001_3$. How many positive integers less than $3^{10} = 59049$ can be written in base 3 using only the digits 0 and 1, and are also divisible by 4?

**(A)** 251 **(B)** 252 **(C)** 256 **(D)** 271 **(E)** 272

**24.** A convex pentagon is built from five straight rods whose lengths are positive whole numbers of centimetres adding up to 20, joined end to end in a closed loop. The pentagon's pattern is the list of the five rod lengths read around the loop, starting at any rod and going in either direction. So two lists give the same pattern if one can be turned into the other by starting at a different rod, by reading in the opposite direction, or both. For example, $(2, 5, 3, 4, 6)$, $(3, 4, 6, 2, 5)$ and $(6, 4, 3, 5, 2)$ are all the same pattern, but $(2, 5, 4, 3, 6)$ is a different one. How many different patterns are possible?

**(A)** 270 **(B)** 271 **(C)** 283 **(D)** 290 **(E)** 296

**25.** Each of the 36 cells of a $6 \times 6$ grid is coloured black or white. Two cells are neighbours if they share a side. A colouring is called balanced if every cell, black or white, has an odd number of black neighbours. How many balanced colourings are there?

**(A)** 0 **(B)** 1 **(C)** 32 **(D)** 64 **(E)** 4096

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# Answer Key — AMC 10 Practice Contest 1

**1.** (B) $\frac{3}{2}$

**2.** (D) 4

**3.** (C) 58

**4.** (A) 35

**5.** (C) 150

**6.** (A) 15

**7.** (D) 75

**8.** (B) $\frac{4}{21}$

**9.** (B) 17

**10.** (C) 150

**11.** (D) 4320

**12.** (B) $\frac{2}{55}$

**13.** (A) $1056\frac{1}{4}$

**14.** (A) 15

**15.** (A) 37

**16.** (C) $\frac{10}{3}$

**17.** (E) 252

**18.** (C) $\frac{216}{5}$

**19.** (B) 80

**20.** (E) $\frac{275}{1152}$

**21.** (E) 214

**22.** (E) $\frac{49\sqrt{6}}{4}$

**23.** (D) 271

**24.** (E) 296

**25.** (D) 64

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# Solutions — AMC 10 Practice Contest 1

**1. Answer: (B)** $\frac{3}{2}$**. The idea.** If the frame has the same area as the photo, the whole framed picture has twice the photo's area. So set the outer rectangle's area equal to $2 \cdot 60 = 120$ and solve the resulting quadratic.

**Work it out.** The photo has area $5 \cdot 12 = 60$, so we need $(5 + 2x)(12 + 2x) = 120$. Expanding, $60+34x+4x^2 = 120$, so $4x^2+34x-60 = 0$, or $2x^2+17x-30 = 0$. This factors as $(2x-3)(x+10) = 0$. The width is positive, so $x = \frac{3}{2}$.

**Check.** With $x = \frac{3}{2}$ the outer rectangle is 8 by 15, of area 120; removing the photo leaves $120 - 60 = 60$, equal to the photo's area. (A common slip is to add $x$ only once to each dimension: $(5 + x)(12 + x) = 120$ gives $x = 3$, but the frame sits on both sides.)

**2. Answer: (D)** 4**. The idea.** Each divisor is exactly 7 more than its offset: $n + 1$ is divisible by $8 = 1 + 7$, $n + 2$ by $9 = 2 + 7$, and so on. So subtract 7 from $n$ throughout, and all four conditions become the same kind of condition on $n - 7$.

**Work it out.** Since $n + 1 = (n - 7) + 8$, the number 8 divides $n + 1$ exactly when 8 divides $n - 7$. In the same way, $9 \mid n + 2 \iff 9 \mid n - 7$, and $10 \mid n + 3 \iff 10 \mid n - 7$, and of course $7 \mid n \iff 7 \mid n - 7$. So $n - 7$ must be a common multiple of $7, 8, 9, 10$, that is, a multiple of $\mathrm{lcm}(7, 8, 9, 10) = 2520$. Then $n = 7 + 2520k$ with $k \geq 0$, and $n \leq 10000$ allows $k = 0, 1, 2, 3$: the values 7, 2527, 5047, 7567. That is 4 values.

**Check.** For $n = 2527$: $2527 = 7 \cdot 361$, $2528 = 8 \cdot 316$, $2529 = 9 \cdot 281$, $2530 = 10 \cdot 253$. The next value, 10087, is too large. Forgetting the smallest solution $n = 7$ (where the four numbers are simply $7, 8, 9, 10$) gives the trap answer 3.

**3. Answer: (C)** 58**. The idea.** Every piece in a given column has the same width, and every piece in a given row has the same height. The three diagonal pieces use each column exactly once and each row exactly once, so between them they contain every width and every height exactly once, just like the big rectangle.

**Work it out.** Call the column widths $w_1, w_2, w_3$ and the row heights $h_1, h_2, h_3$ (top to bottom, left to right). The piece in row $i$ and column $j$ has perimeter $2(w_j + h_i)$. So the three given pieces satisfy $2(w_1 + h_1) = 12$, $2(w_2 + h_2) = 20$ and $2(w_3 + h_3) = 26$. The original rectangle has width $w_1 + w_2 + w_3$ and height $h_1 + h_2 + h_3$, so its perimeter is $2(w_1 + w_2 + w_3 + h_1 + h_2 + h_3) = 12 + 20 + 26 = 58$.

**Check.** The individual widths are not determined, but the answer does not depend on them: with widths $2, 4, 6$ and heights $4, 6, 7$ the diagonal pieces are $2 \times 4$, $4 \times 6$ and $6 \times 7$, with perimeters $12, 20, 26$, and the big rectangle is $12 \times 17$, with perimeter 58. Halving to 29 forgets the factor 2 in the perimeter; 116 counts each side twice.

**4. Answer: (A)** 35**. The idea.** A digit sum of 42 is only 3 short of the maximum 45. Instead of choosing the digits, choose how far each digit falls short of 9: the five shortfalls are nonnegative and add to 3.

**Work it out.** Write the digits as $9 - s_1, 9 - s_2, \ldots, 9 - s_5$. The digit sum is $45 - (s_1 + \cdots + s_5)$, so we need $s_1 + s_2 + s_3 + s_4 + s_5 = 3$ with each $s_i \geq 0$. Every shortfall is at most 3, so every digit

is at least 6: the digits are automatically valid and the leading digit is never 0. By stars and bars, the number of solutions is $\binom{3+4}{4} = \binom{7}{4} = 35$.

**Check.** Count by the pattern of shortfalls: one digit short by 3 (5 ways), one short by 2 and another by 1 ($5 \cdot 4 = 20$ ways), or three digits short by 1 ($\binom{5}{3} = 10$ ways). Total $5 + 20 + 10 = 35$.

**5. Answer: (C)** 150**. The idea.** A wheel that rolls without slipping turns (distance)/(circumference) times. Write both turn counts in terms of the front circumference $c$ and set their difference to 40; clearing denominators gives a quadratic.

**Work it out.** The trip is 36000 cm. The front wheel turns $\frac{36000}{c}$ times and the back wheel $\frac{36000}{c+30}$ times, so $\frac{36000}{c} - \frac{36000}{c+30} = 40$. The left side is $\frac{36000 \cdot 30}{c(c+30)}$, so $c(c+30) = \frac{1080000}{40} = 27000$. Then $c^2 + 30c - 27000 = 0$, which factors as $(c - 150)(c + 180) = 0$. So $c = 150$.

**Check.** The front wheel turns $36000/150 = 240$ times, the 180 cm back wheel turns $36000/180 = 200$ times, and $240 - 200 = 40$. Note 180 is the back wheel, a tempting wrong pick.

**6. Answer: (A)** 15**. The idea.** Since $\angle EAF = 45°$, the two leftover angles $\angle BAE$ and $\angle DAF$ also add to 45°. Rotate triangle $ABE$ about $A$ by 90° so that $AB$ lands on $AD$. The rotated copy of $E$ lands on line $CD$ beyond $D$, and the new triangle it forms with $A$ and $F$ turns out to be congruent to triangle $AEF$. That gives $EF = BE + DF$.

**Work it out.** Let $DF = d$. Rotating by 90° about $A$ carries $B$ to $D$ and $E$ to a point $E'$ on line $CD$, on the far side of $D$ from $F$, with $DE' = BE = 2$ and $AE' = AE$. Then $\angle E'AF = \angle E'AD + \angle DAF = \angle BAE + \angle DAF = 90° - 45° = 45° = \angle EAF$. Triangles $E'AF$ and $EAF$ share side $AF$, have $AE' = AE$, and have equal angles at $A$, so they are congruent (SAS). Hence $EF = E'F = 2 + d$. Now use right triangle $ECF$, with $CE = 6 - 2 = 4$ and $CF = 6 - d$: $4^2 + (6 - d)^2 = (2 + d)^2$, so $16 + 36 - 12d = 4 + 4d$, and $d = 3$. The area of $AEF$ equals the area of $AE'F$, whose base $E'F = 5$ lies on line $CD$ with height $AD = 6$: the area is $\frac{1}{2} \cdot 5 \cdot 6 = 15$.

**Check.** Subtract the corner triangles from the square: $36 - \frac{1}{2}\cdot 6\cdot 2 - \frac{1}{2}\cdot 6\cdot 3 - \frac{1}{2}\cdot 4\cdot 3 = 36 - 6 - 9 - 6 = 15$. And with $A = (0, 0)$, $E = (6, 2)$, $F = (3, 6)$: $\overrightarrow{AE} \cdot \overrightarrow{AF} = 30$ and $|AE|\,|AF| = \sqrt{40}\sqrt{45} = 30\sqrt{2}$, so $\cos \angle EAF = \frac{1}{\sqrt{2}}$, confirming the 45° angle.

**7. Answer: (D)** 75**. The idea.** The last two digits are the remainder when $P$ is divided by $100 = 4 \cdot 25$. Find the remainder modulo 25 and modulo 4 separately; each is easy, and together they pin down the remainder modulo 100.

**Work it out.** Modulo 25: the product contains the factors 5 and 15, so $25 \mid P$. Modulo 4: every odd number is $\equiv 1$ or $\equiv 3 \equiv -1 \pmod 4$. The odd numbers below 100 that are $\equiv 3 \pmod 4$ are $3, 7, 11, \ldots, 99$, which is 25 numbers, an odd count, so $P \equiv (-1)^{25} = -1 \equiv 3 \pmod 4$. The multiples of 25 below 100 are $0, 25, 50, 75$, with remainders $0, 1, 2, 3$ modulo 4. Only 75 works, so $P$ ends in 75.

**Check.** Knowing only that $P$ is an odd multiple of 5 leaves all of the choices open, and knowing that $25 \mid P$ still leaves 25 and 75; it is the count of numbers $\equiv 3 \pmod 4$ that decides. A small case: $1 \cdot 3 \cdot 5 \cdots 15 = 2027025$ contains 5 and 15, and has four factors $\equiv 3 \pmod 4$ $(3, 7, 11, 15)$, so it should end in 25, and it does.

**8. Answer: (B)** $\frac{4}{21}$**. The idea.** The mean of three different numbers lies strictly between the smallest and the largest, so it can only equal the middle number. Mean = middle means the three

numbers form an arithmetic progression, and such a progression is fixed by its two ends, which must have the same parity (so that their average is a whole number).

**Work it out.** If the numbers are $a < b < c$ and $\frac{a+b+c}{3} = b$, then $a+c = 2b$, so $b = \frac{a+c}{2}$. Conversely any pair $a < c$ with $a + c$ even gives exactly one valid set, since $\frac{a+c}{2}$ is a whole number strictly between them. Pairs of odd numbers from $\{1, 3, 5, 7, 9\}$: $\binom{5}{2} = 10$. Pairs of even numbers from $\{2, 4, 6, 8\}$: $\binom{4}{2} = 6$. That is 16 good sets out of $\binom{9}{3} = 84$, a probability of $\frac{16}{84} = \frac{4}{21}$.

**Check.** Count the progressions by common difference: difference 1 gives 7 (123 up to 789), difference 2 gives 5, difference 3 gives 3, difference 4 gives 1 (159): $7 + 5 + 3 + 1 = 16$.

**9. Answer: (B)** 17**. The idea.** The quadratic part $x^2 + 2xy + 2y^2$ is $(x + y)^2 + y^2$, a sum of two squares. Switch to the new variable $u = x + y$ (so $x = u - y$); then the expression separates into a part in $u$ alone and a part in $y$ alone, and each can be minimised by completing the square.

**Work it out.** With $x = u - y$: the quadratic part is $u^2 + y^2$, and $-4x + 2y = -4u + 4y + 2y = -4u + 6y$. So the expression equals $u^2 - 4u + y^2 + 6y + 30 = (u-2)^2 + (y+3)^2 + 30 - 4 - 9 = (u-2)^2 + (y+3)^2 + 17$. As $x$ and $y$ range over all reals, so do $u$ and $y$ independently, so the smallest value is 17, reached when $u = 2$ and $y = -3$, that is, at $x = 5$, $y = -3$.

**Check.** At $(x, y) = (5, -3)$: $25 - 30 + 18 - 20 - 6 + 30 = 17$. Completing the square in $x$ first gives the same result: the expression is $(x + y - 2)^2 + y^2 + 6y + 26 = (x + y - 2)^2 + (y + 3)^2 + 17$. Minimising in $x$ alone at $y = 0$ gives only 26, and dropping one of the subtracted constants gives 21 or 26.

**10. Answer: (C)** 150**. The idea.** The two small triangles are similar to the big one, and each is a scaled copy whose hypotenuse is one of the big triangle's legs. Every length, inradius included, scales by the same factor, so the three inradii behave exactly like the three sides: $r_1^2 + r_2^2 = r^2$.

**Work it out.** Let $a = BC$, $b = AC$, $c = AB$ and let $r$ be the inradius of $ABC$. Triangle $ACD$ is similar to $ABC$ (they share angle $A$ and each has a right angle) with ratio $\frac{b}{c}$, so its inradius is $\frac{b}{c}r = 3$. Likewise triangle $BCD$ has ratio $\frac{a}{c}$ and inradius $\frac{a}{c}r = 4$. Squaring and adding, $\frac{a^2+b^2}{c^2}r^2 = 9 + 16$, so $r^2 = 25$ and $r = 5$. Also $b : a = 3 : 4$, so $ABC$ is a 3-4-5 triangle scaled by some $k$: legs $3k, 4k$, hypotenuse $5k$. A right triangle's inradius is $\frac{\text{leg}+\text{leg}-\text{hypotenuse}}{2} = \frac{3k+4k-5k}{2} = k$, so $k = 5$. The legs are 15 and 20 and the area is $\frac{1}{2} \cdot 15 \cdot 20 = 150$.

**Check.** With legs $AC = 15$, $BC = 20$: $CD = \frac{15 \cdot 20}{25} = 12$, $AD = \frac{15^2}{25} = 9$, $BD = 16$. Triangle $ACD$ is 9-12-15 with inradius $\frac{9+12-15}{2} = 3$, and $BCD$ is 16-12-20 with inradius $\frac{16+12-20}{2} = 4$. Both match.

**11. Answer: (D)** 4320**. The idea.** Factor $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. If a prime $p$ divides both $d$ and $\frac{2520}{d}$, the divisor is not clean. So for each prime, the whole prime power must go to one side: $d$ takes all of $2^3$ or none of it, all of $3^2$ or none of it, and so on. The clean divisors are exactly the products of a subset of the four blocks $8, 9, 5, 7$.

**Work it out.** Choosing, for each block, either 1 or the block itself gives every clean divisor exactly once, so expanding the product $(1 + 8)(1 + 9)(1 + 5)(1 + 7)$ lists each clean divisor as one term. Their sum is therefore $9 \cdot 10 \cdot 6 \cdot 8 = 4320$. There are $2^4 = 16$ clean divisors, far fewer than the 48 divisors of 2520.

**Check.** The clean divisors pair up as $d$ and $\frac{2520}{d}$, for example $1 + 2520$, $8 + 315$, $9 + 280$, $5 + 504$, $7 + 360$, $72 + 35$, $40 + 63$, $56 + 45$, and these eight pair sums add to $2521 + 323 + 289 + 509 +$

$367 + 107 + 103 + 101 = 4320$. Summing all 48 divisors gives $\sigma(2520) = 15 \cdot 13 \cdot 6 \cdot 8 = 9360$; using the primes instead of the full prime powers, $(1+2)(1+3)(1+5)(1+7) = 576$, sums only the squarefree divisors; forgetting 1 and 2520 gives 1799.

**12. Answer: (B)** $\frac{2}{55}$**. The idea.** The 12 edges point in three directions, four edges in each. Since no two chosen edges may be parallel, the three edges must use all three directions, one each. Then count carefully, choosing one direction at a time and throwing out edges that touch what is already chosen.

**Work it out.** Put the cube at $[0,1]^3$. Choose the edge parallel to the $x$-axis: 4 ways; it sits at some fixed $y = y_0$, $z = z_0$. A $y$-edge sits at some $x = x_1$, $z = z_1$ and meets the $x$-edge exactly when $z_1 = z_0$ (they then share the corner $(x_1, y_0, z_0)$). So the $y$-edge must have $z_1 \neq z_0$: 2 ways. A $z$-edge sits at some $x = x_2$, $y = y_2$; it meets the $x$-edge when $y_2 = y_0$ and the $y$-edge when $x_2 = x_1$. So we need $y_2 \neq y_0$ and $x_2 \neq x_1$, which leaves exactly 1 edge. That is $4 \cdot 2 \cdot 1 = 8$ good sets out of $\binom{12}{3} = 220$, so the probability is $\frac{8}{220} = \frac{2}{55}$.

**Check.** Count ordered choices instead: the first edge is any of 12; the second must be non-parallel and disjoint from it, 4 choices (of the 8 non-parallel edges, 4 touch it, 2 at each end); the third must avoid both, 1 choice. That gives $\frac{12 \cdot 4 \cdot 1}{12 \cdot 11 \cdot 10} = \frac{2}{55}$. Forgetting the "no shared vertex" condition would give $\frac{4^3}{220} = \frac{16}{55}$.

**13. Answer: (A)** $1056\frac{1}{4}$**. The idea.** If the wall were endless, the best pen would have its long side 50 along the wall and area 1250. But the wall is only 30 long, so that pen is impossible, and the question becomes what to do with the side along the wall: stop at 30, or run past the end of the wall and pay for the overhang. Treat the two cases separately.

**Work it out.** Let $x$ be the side along the wall and $y$ the two sides perpendicular to it. Using all the fence is never worse, since a longer $y$ gives more area.

Case $x \leq 30$: the fence covers the opposite side and the two ends, so $x + 2y = 100$ and the area is $\frac{x(100-x)}{2}$. This parabola increases all the way up to $x = 50$, so on $x \leq 30$ it is largest at $x = 30$: area $30 \cdot 35 = 1050$.

Case $x \geq 30$: now the overhang $x - 30$ also needs fence, so $x + (x - 30) + 2y = 100$, which simplifies to $x + y = 65$. With a fixed sum, the product $xy$ is largest when $x = y = 32.5$, and $32.5 \geq 30$ is allowed. The area is $32.5^2 = 1056.25 = 1056\frac{1}{4}$.

The larger of the two cases is $1056\frac{1}{4}$.

**Check.** At $x = y = 32.5$ the fence used is $32.5 + 2.5 + 2 \cdot 32.5 = 100$. Nearby choices do worse: $x = 32$, $y = 33$ gives 1056, and $x = 30$ gives 1050. The tempting 1250 needs a 50-metre wall that is not there, and forgetting to fence the side opposite the wall gives $2112\frac{1}{2}$.

**14. Answer: (A)** 15**. The idea.** All the vertices of a regular polygon lie on one circle, and each of the small angles at the chosen vertex is an inscribed angle. Each one cuts off exactly one side of the polygon, that is, an arc of $\frac{360^\circ}{n}$, so every small angle equals half of that: $\frac{180^\circ}{n}$. They are all equal.

**Work it out.** The small angles are whole numbers of degrees exactly when $n$ divides 180. Since $180 = 2^2 \cdot 3^2 \cdot 5$, it has $3 \cdot 3 \cdot 2 = 18$ divisors: 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180. Remove 1, 2 and 3, which are not allowed because $n \geq 4$. That leaves 15 values.

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**Check.** The $n-2$ equal angles add up to $(n-2)\cdot\frac{180^\circ}{n} = 180^\circ - \frac{360^\circ}{n}$, which is the interior angle, as it should be. For the square ($n=4$), one diagonal splits the $90^\circ$ corner into $45^\circ + 45^\circ$. For the regular 8-gon the pieces are $22.5^\circ$, so 8 fails even though its interior angle $135^\circ$ is whole. That is exactly the trap behind the answer 21, which only asks for the interior angle $\frac{(n-2)\cdot 180^\circ}{n}$ to be whole, that is, counts the 21 divisors of 360 that are at least 4. Counting all 18 divisors of 180, without removing 1, 2 and 3, gives 18.

**15. Answer: (A)** 37**. The idea.** Write every number as a perfect square times a squarefree number (one with no repeated prime factor): $18 = 9\cdot 2$, $50 = 25\cdot 2$, $12 = 4\cdot 3$. Call the squarefree factor the *core*. A product is a square exactly when every prime appears to an even power, so two numbers multiply to a square exactly when their cores are equal. The numbers therefore fall into families by core, and the question is how many families there are.

**Work it out.** Two numbers from the same family always multiply to a square, so at most one number can be taken from each family. Two numbers from different families never do, so one number from every family works. The answer is the number of families.

Every core is itself a squarefree number in $1, \ldots, 60$ (it divides its number), and every squarefree number is its own core. So we count squarefree numbers up to 60. Remove the multiples of 4 (15 of them), of 9 (6), of 25 (2) and of 49 (1); only 36 was removed twice (it is a multiple of both 4 and 9). That leaves $60 - 24 + 1 = 37$.

**Check.** Taking exactly the 37 squarefree numbers works: a product of two different squarefree numbers has a prime appearing exactly once, so it is not a square. Tempting slips: forgetting the multiples of 49 gives 38; removing only multiples of 4 and 9 gives 40; removing only multiples of 4 gives 45; and avoiding just the 7 perfect squares gives 53.

**16. Answer: (C)** $\frac{10}{3}$**. The idea.** Don't try to find the distribution of the number of peaks. By linearity of expectation, the expected number of peaks is the sum, over the nine positions, of the probability that the card in that position is a peak. Each of those probabilities depends only on a few cards.

**Work it out.** For one of the 7 interior positions, look at that card and its two neighbours. All $3! = 6$ relative orders of these three numbers are equally likely, and the middle card is the largest of the three with probability $\frac{1}{3}$.

For an end position, only two cards matter, and the end card is the larger with probability $\frac{1}{2}$.

So the expected number of peaks is $7\cdot\frac{1}{3} + 2\cdot\frac{1}{2} = \frac{7}{3} + 1 = \frac{10}{3}$.

**Check.** Try the same method on 3 cards: $1\cdot\frac{1}{3} + 2\cdot\frac{1}{2} = \frac{4}{3}$. Listing all 6 orders, 123 and 321 have one peak each; 132, 231, 213 and 312 have $1, 1, 2, 2$ peaks. The total is 8 peaks over 6 orders, which is $\frac{4}{3}$. The formula matches.

**17. Answer: (E)** 252**. The idea.** The expression is symmetric in $a$ and $b$, so rewrite it using only $a+b$ and $p = ab$. Then it becomes a quadratic in the single variable $p$, and the only care needed is to check which values of $p$ are actually possible.

**Work it out.** Expanding, $(a^2+7)(b^2+7) = a^2b^2 + 7(a^2+b^2) + 49$. Since $a^2+b^2 = (a+b)^2 - 2ab = 36 - 2p$, the expression equals $p^2 + 7(36-2p) + 49 = p^2 - 14p + 301 = (p-7)^2 + 252$.

For real $a$, $b$ with sum 6, they are the roots of $t^2 - 6t + p = 0$, which needs $36 - 4p \geq 0$, that is $p \leq 9$. The value $p = 7$ is allowed, so the minimum is 252, reached at $a, b = 3 \pm \sqrt{2}$.

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**Check.** With $a = 3+\sqrt{2}$: $a^2 = 11+6\sqrt{2}$, and $b^2 = 11-6\sqrt{2}$, so the product is $(18+6\sqrt{2})(18-6\sqrt{2}) = 324-72 = 252$. The natural guess $a = b = 3$ gives $16\cdot 16 = 256$, which is larger: making $a$ and $b$ equal maximises $p$ to 9, overshooting the best $p = 7$. Dropping the constant 49 from the expansion gives 203, and writing $a^2+b^2$ without its factor 7 gives 84.

**18. Answer: (C)** $\frac{216}{5}$**. The idea.** Each condition puts $P$ on a circle. Since $\angle APB = 90°$, $P$ lies on the circle with diameter $\overline{AB}$, centred at the midpoint $M$ of $\overline{AB}$. Since $PD = 12 = DA$, $P$ lies on the circle centred at $D$ through $A$. Both circles pass through $A$, so their other common point $P$ is the reflection of $A$ across the line of centres $DM$.

**Work it out.** Put $A = (0,0)$, $B = (12,0)$, $C = (12,12)$, $D = (0,12)$, so $M = (6,0)$ and line $DM$ is $2x+y = 12$. The foot of the perpendicular from $A$ to this line is $\left(\frac{24}{5}, \frac{12}{5}\right)$ (it lies on the line and in the direction $(2,1)$ from $A$). Doubling gives $P = \left(\frac{48}{5}, \frac{24}{5}\right)$.

Side $\overline{CD}$ lies on the line $y = 12$, so the height of triangle $PCD$ from $P$ is $12-\frac{24}{5} = \frac{36}{5}$, and the area is $\frac{1}{2}\cdot 12\cdot\frac{36}{5} = \frac{216}{5}$.

**Check.** $PD^2 = \left(\frac{48}{5}\right)^2+\left(\frac{36}{5}\right)^2 = \frac{3600}{25} = 144$, and $\overrightarrow{PA}\cdot\overrightarrow{PB} = \frac{48}{5}\left(\frac{48}{5}-12\right)+\left(\frac{24}{5}\right)^2 = -\frac{576}{25}+\frac{576}{25} = 0$, so the angle at $P$ is right. The centre of the square also has $\angle APB = 90°$ and would give area 36, but its distance to $D$ is only $6\sqrt{2}$. The value $\frac{144}{5}$ is the area of triangle $PAB$, not $PCD$.

**19. Answer: (B)** 80**. The idea.** Since $210 = 2\cdot 3\cdot 5\cdot 7$, a number is relatively prime to 210 exactly when it is divisible by none of 2, 3, 5, 7. Each of those conditions depends only on $n$ modulo that prime, so handle the four primes one at a time and multiply, using the Chinese remainder theorem: residues mod 2, 3, 5, 7 combine to exactly one residue mod 210.

**Work it out.** For each prime, cross out the residues of $n$ that make one of $n$, $n+2$, $n+6$ divisible by it.

Mod 2: all three numbers have the same parity as $n$, so $n$ must be odd. That leaves 1 residue out of 2.

Mod 3: $n+6 \equiv n$, so only $n \equiv 0$ and $n \equiv 1$ (which makes $3 \mid n+2$) are forbidden. That leaves 1 residue, $n \equiv 2$.

Mod 5: forbidden are $n \equiv 0$, $n \equiv 3$ and $n \equiv 4$ (since $n+6 \equiv n+1$). That leaves 2 residues.

Mod 7: forbidden are $n \equiv 0$, 5 and 1, three different residues, so 4 remain.

So $1\cdot 1\cdot 2\cdot 4 = 8$ residues mod 210 work. The range 1 to 2100 is exactly 10 full blocks of 210, so the count is $8\cdot 10 = 80$.

**Check.** The conditions say $n \equiv 1 \pmod 2$, $n \equiv 2 \pmod 3$, $n \equiv 1$ or $2 \pmod 5$, and $n \equiv 2,3,4,6 \pmod 7$. The example $n = 11$ fits ($11 \equiv 1,2,1,4$), and $n = 13 \equiv 1 \pmod 3$ does not. The traps: counting only $n$ itself gives $\varphi(210)\cdot 10 = 480$; allowing even $n$ gives 160; forgetting that $n+6 \equiv n+1 \pmod 5$ leaves 3 residues mod 5 and gives 120.

**20. Answer: (E)** $\frac{275}{1152}$**. The idea.** Given $d$, the chance that $e = 1$ is $\frac{1}{\tau(d)}$, where $\tau(d)$ is the number of divisors of $d$. So we want the average of $\frac{1}{\tau(d)}$ over the divisors of 360. The number of divisors is multiplicative, and a random divisor of $360 = 2^3\cdot 3^2\cdot 5$ picks its three exponents independently. So the average splits into a product of three small averages.

**Work it out.** Write $d = 2^i 3^j 5^k$ with $i \in \{0,1,2,3\}$, $j \in \{0,1,2\}$ and $k \in \{0,1\}$, each chosen uniformly and independently. Then $\frac{1}{\tau(d)} = \frac{1}{i+1} \cdot \frac{1}{j+1} \cdot \frac{1}{k+1}$, and the average of a product of independent factors is the product of the averages:

$\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{25}{48}, \quad \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{11}{18}, \quad \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}.$

The probability is $\frac{25}{48} \cdot \frac{11}{18} \cdot \frac{3}{4} = \frac{825}{3456} = \frac{275}{1152}$.

**Check.** Try the same method on $12 = 2^2 \cdot 3$. Its divisors $1, 2, 3, 4, 6, 12$ have $1, 2, 2, 3, 4, 6$ divisors, so the average of $\frac{1}{\tau}$ is $\frac{1}{6}\left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}\right) = \frac{11}{24}$. The product formula gives $\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) \cdot \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{11}{18} \cdot \frac{3}{4} = \frac{11}{24}$. Answering $\frac{1}{24}$ ignores the first random choice; treating all 180 pairs $(d, e)$ as equally likely gives $\frac{24}{180} = \frac{2}{15}$; leaving out the exponent 0 in each average gives $\frac{65}{864}$.

**21. Answer: (E)** 214**. The idea.** Call the integer $\lfloor x \rfloor = n$. Then the equation says $x^2 = 10n - 9$, so each $n$ offers at most one candidate $x$, and the only question is whether that $x$ really has floor $n$.

**Work it out.** Since $x^2 \geq 0$, we need $10n - 9 \geq 0$, so $n \geq 1$. Then $x \geq n \geq 1 > 0$, which means $x = \sqrt{10n-9}$ (a negative root would have a negative floor). The condition $n \leq x < n+1$ becomes $n^2 \leq 10n - 9 < (n+1)^2$.

The left inequality is $n^2 - 10n + 9 = (n-1)(n-9) \leq 0$, so $1 \leq n \leq 9$. The right inequality is $n^2 - 8n + 10 > 0$, whose roots are $4 \pm \sqrt{6} \approx 1.55$ and $6.45$, so $n \leq 1$ or $n \geq 7$. Together: $n \in \{1, 7, 8, 9\}$.

The solutions are $x = 1, \sqrt{61}, \sqrt{71}, 9$, and the sum of their squares is $1 + 61 + 71 + 81 = 214$.

**Check.** $\sqrt{61} \approx 7.81$, $\sqrt{71} \approx 8.43$ and $\sqrt{81} = 9$ have floors 7, 8 and 9, as required. The near-miss $n = 6$ gives $\sqrt{51} \approx 7.14$, whose floor is 7, not 6. Missing the endpoint solution $x = 1$ gives 213, missing $x = 9$ gives 133, and keeping only the integer solutions gives 82.

**22. Answer: (E)** $\frac{49\sqrt{6}}{4}$**. The idea.** Triangle $ATC$ has its base $TC$ on line $BC$ and its apex at $A$, so it has the same height as triangle $ABC$. Everything comes down to finding $TC$, and the tangent-chord angle gives a pair of similar triangles that does exactly that.

**Work it out.** Because $AB < AC$, the tangent at $A$ meets line $BC$ beyond $B$, so $T$, $B$, $C$ lie in that order. The angle between the tangent $TA$ and the chord $AB$ equals the inscribed angle on the other side, so $\angle TAB = \angle ACB = \angle TCA$. Triangles $TAB$ and $TCA$ also share the angle at $T$, so they are similar, with

$\frac{TA}{TC} = \frac{TB}{TA} = \frac{AB}{CA} = \frac{5}{7}$.

Hence $TB = \frac{5}{7}TA$ and $TC = \frac{7}{5}TA$, so $TC - TB = \left(\frac{7}{5} - \frac{5}{7}\right)TA = \frac{24}{35}TA$. But $TC - TB = BC = 6$, so $TA = \frac{35}{4}$, $TB = \frac{25}{4}$ and $TC = \frac{49}{4}$.

Now the height. By Heron's formula with $s = 9$, $[ABC] = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$, so the height from $A$ to line $BC$ is $\frac{2 \cdot 6\sqrt{6}}{6} = 2\sqrt{6}$. Therefore $[ATC] = \frac{1}{2} \cdot \frac{49}{4} \cdot 2\sqrt{6} = \frac{49\sqrt{6}}{4}$.

**Check.** Power of the point $T$: $TA^2 = \frac{1225}{16}$ and $TB \cdot TC = \frac{25}{4} \cdot \frac{49}{4} = \frac{1225}{16}$, as it must be. In coordinates $B = (0,0)$, $C = (6,0)$, $A = (1, 2\sqrt{6})$, the circumcentre is $\left(3, \frac{5\sqrt{6}}{12}\right)$, and the tangent at $A$ meets the $x$-axis at $\left(-\frac{25}{4}, 0\right)$, so $TB = \frac{25}{4}$ indeed. Using base $TB$ instead of $TC$ gives $\frac{25\sqrt{6}}{4}$, using $TA$ as if it were the base gives $\frac{35\sqrt{6}}{4}$, and taking the side $CA = 7$ as the base gives $7\sqrt{6}$.

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**23. Answer: (D)** 271**. The idea.** A number below $3^{10}$ that uses only digits 0 and 1 is a sum of distinct powers $3^0, 3^1, \ldots, 3^9$. So it is determined by choosing which of the ten places hold a 1, and there are $2^{10} - 1$ positive ones. Modulo 4, we have $3 \equiv -1$, so $3^i \equiv 1$ for even $i$ and $\equiv -1$ for odd $i$. Divisibility by 4 therefore becomes a condition on how many 1s sit in even places versus odd places.

**Work it out.** Let $E$ be the number of 1s in the five even places $(3^0, 3^2, \ldots, 3^8)$ and $O$ the number in the five odd places. Then $n \equiv E - O \pmod 4$, and $E - O$ lies between $-5$ and 5. So we need $E - O \in \{-4, 0, 4\}$.

$E = O$: $\sum_{k=0}^{5} \binom{5}{k}^2 = \binom{10}{5} = 252$ choices, including the empty choice $n = 0$.

$E - O = 4$: $(E, O) = (4, 0)$ or $(5, 1)$, which gives $5 \cdot 1 + 1 \cdot 5 = 10$ choices. By symmetry, $E - O = -4$ also gives 10.

Removing $n = 0$, the count is $252 + 10 + 10 - 1 = 271$.

**Check.** Small cases: $4 = 11_3$ has one 1 in each type of place, and 4 is divisible by 4. $12 = 110_3$ and $28 = 1001_3$ are also multiples of 4. The number $81 + 27 + 9 + 1 = 118 = 11101_3$ has $E = 3$, $O = 1$, so $E - O = 2$, and indeed $118 \equiv 2 \pmod 4$. Forgetting the $E - O = \pm 4$ cases gives 251, and forgetting to drop 0 gives 272.

**24. Answer: (E)** 296**. The idea.** Count the ordered lists that really are the sides of a convex pentagon, then group them. Grouping by the starting rod is clean because 5 is prime. Grouping by direction needs care: a pattern that reads the same backwards must not be halved.

**Work it out.** A convex pentagon with sides $a, b, c, d, e$ (in that order) exists exactly when each side is shorter than the sum of the other four. With perimeter 20, that means every side is less than 10, so at most 9.

Ordered lists of five positive integers with sum 20: $\binom{19}{4} = 3876$. Lists with a side of at least 10: if that side is $a$, set $a' = a - 9 \geq 1$; the new list sums to 11, giving $\binom{10}{4} = 210$ lists. There are 5 positions for the long side, and two long sides are impossible. So $3876 - 5 \cdot 210 = 2826$ lists are valid.

Starting rods first. If a list equals one of its nontrivial cyclic shifts then, since 5 is prime, repeating that shift reaches every position, so all entries are equal: only $(4, 4, 4, 4, 4)$. Every other list has 5 different shifts, so there are $1 + \frac{2825}{5} = 566$ cyclic classes.

Now direction. Reading every list of a class backwards gives a class. Call a class symmetric if this gives the same class; the other classes pair off. In a symmetric class, some list is a shift of its own reverse, which means the loop has a mirror line. For five rods, a mirror line passes through the middle of one rod and the opposite corner, so starting at that rod the list has the form $(a, b, c, c, b)$. The starting rod is unique unless the loop has two mirror lines, and two mirror lines combine into a rotation, which forces $(4, 4, 4, 4, 4)$. So symmetric classes correspond one-to-one to lists of the form $(a, b, c, c, b)$.

Count those: $a = 20 - 2(b + c)$ must be even and between 1 and 9, so $a \in \{2, 4, 6, 8\}$ with $b + c = 9, 8, 7, 6$, which gives $8 + 7 + 6 + 5 = 26$ symmetric classes. The other $566 - 26 = 540$ classes pair up into 270 patterns. The total is $26 + 270 = 296$.

**Check.** Burnside's lemma averages the number of lists fixed by each of the 10 symmetries of the loop: the identity fixes 2826, each of the 4 rotations fixes only $(4, 4, 4, 4, 4)$, and each of the 5 reflections fixes 26. That gives $\frac{2826+4+130}{10} = 296$ again. Halving 566 blindly gives 283; forgetting the symmetric patterns altogether gives 270; counting only $(4, 4, 4, 4, 4)$ as symmetric gives 271;

and treating $b$ and $c$ in $(a, b, c, c, b)$ as interchangeable finds only 14 symmetric classes, giving $276 + 14 = 290$.

**25. Answer: (D)** 64**. The idea.** Lay two balanced colourings on top of each other and mark the cells where they differ. Every cell has an odd number of black neighbours in each colouring, so it has an even number of marked neighbours. Call a set of cells even if every cell of the grid has an even number of neighbours in the set. Conversely, switching the colours of an even set in a balanced colouring keeps it balanced. So once we have one balanced colouring, the balanced colourings correspond exactly to the even sets, and we only need to count even sets.

**Work it out.** One balanced colouring: reading the rows from top to bottom, colour black the cells in columns $3, 4$ of row 1; columns $1, 6$ of rows 2 and 3; columns $3, 4$ of row 4; nothing in row 5; and columns $1, 2, 5, 6$ of row 6. Checking all 36 cells shows that each has 1 or 3 black neighbours.

Now count even sets. Choose which cells of row 1 are in the set. A cell of row 1 already knows its left and right neighbours, so the cell directly below it is forced (in the set exactly when the count so far is odd). So row 1 forces row 2, row 2 forces row 3 (each cell of row 2 now knows three neighbours), and so on down to row 6. That gives at most $2^6$ even sets, one for each top row, and a top row succeeds exactly when the forced row 6 also passes its own test.

The forcing is compatible with combining: if top rows $R$ and $S$ both succeed, so does the top row made of cells in exactly one of $R$, $S$ (mark cells in exactly one of the two forced grids; every neighbour count is a difference of two even numbers). So it is enough to test the six top rows with a single cell. A single cell in column 1 forces the main diagonal $(1, 1), (2, 2), \ldots, (6, 6)$. A cell in column 2 forces the two diagonals next to it, and a cell in column 3 forces the tilted rectangle with corners $(1, 3)$, $(3, 1)$, $(6, 4)$, $(4, 6)$, a path that bounces off all four walls. Columns $4, 5, 6$ are mirror images. In each picture every cell of the grid has 0 or 2 neighbours in the set, including the bottom row. So all 64 top rows succeed, and there are 64 balanced colourings.

**Check.** Each of these single-cell pictures is symmetric across the main diagonal. That is why the square works: the rule forcing each row from the row above is the same as the rule forcing each column from the column to its left, so the bottom row satisfies its test just as the rightmost column satisfies the test for row 1. The balanced colouring really is needed: a $5 \times 5$ grid has 32 even sets but no balanced colouring at all, so its answer is 0. Assuming the colouring is unique gives 1; letting the first row and the first column both be free gives $2^{12} = 4096$; guessing that the bottom row adds one more parity condition gives 32.

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