MathCastle

AMC 10

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

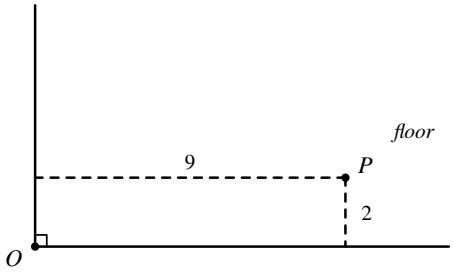
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A merchant's balance is a beam with a pan hanging from each end, but the pivot is not at the middle of the beam, so the left arm and the right arm have different lengths. With both pans empty the beam hangs level, and it balances exactly when (weight in the left pan) × (length of the left arm) equals (weight in the right pan) × (length of the right arm). A melon in the left pan is balanced exactly by 9 kg of standard weights in the right pan; the same melon in the right pan is balanced exactly by 16 kg of standard weights in the left pan. The merchant then fills two bags of rice, each sold as 10 kg. For the first bag he puts 10 kg of standard weights in the left pan and adds rice to the right pan until the beam balances; for the second bag he puts 10 kg of standard weights in the right pan and adds rice to the left pan until it balances. How many kilograms of rice do the two bags hold altogether?

(A) 20 **(B)** $\frac{125}{6}$ **(C)** 21 **(D)** $\frac{1685}{72}$ **(E)** 24

2. Two straight walls meet at a right angle at corner O. On the floor between them, point P is 2 m from one wall and 9 m from the other. Exactly two circles on the floor pass through P and are tangent to both walls. How far apart, in metres, are the centres of these two circles?



(A) $7\sqrt{2}$ **(B)** 12 **(C)** $10\sqrt{2}$ **(D)** $12\sqrt{2}$ **(E)** 22

3. Three different real numbers a, b and c satisfy $a^3 - 3a = b^3 - 3b = c^3 - 3c$. What is the value of $a^2 + b^2 + c^2$?

(A) -6 **(B)** 0 **(C)** 3 **(D)** 6 **(E)** 12

4. Each of the 10 students in a club secretly writes down the name of one of the other 9 students, chosen uniformly at random and independently of everyone else. Two students form a *mutual pair* if each of them wrote down the other's name. What is the expected number of mutual pairs?

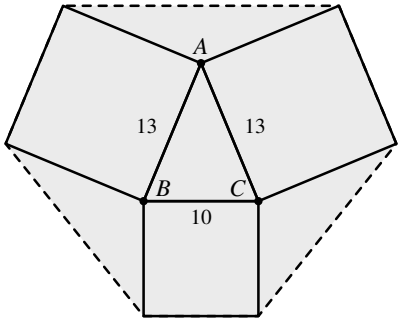
(A) $\frac{1}{9}$ **(B)** $\frac{10}{81}$ **(C)** $\frac{9}{20}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{5}{9}$

5. How many integers n with $1 \le n \le 100$ have a prime number of positive divisors? (For example, 4 qualifies, since it has the 3 divisors 1, 2, 4 and 3 is prime; 6 does not, since it has 4 divisors.)

(A) 25 **(B)** 29 **(C)** 30 **(D)** 31 **(E)** 32

6. Triangle ABC has side lengths $AB = AC = 13$ and $BC = 10$. On each side of the triangle a square is drawn outward, with that side as one of the square's sides. Each square has two corners that are not vertices of the triangle; these six corners are the vertices of a hexagon that surrounds the whole figure. What is the area of this hexagon?

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(A) 438 **(B)** 498 **(C)** 558 **(D)** 618 **(E)** 678

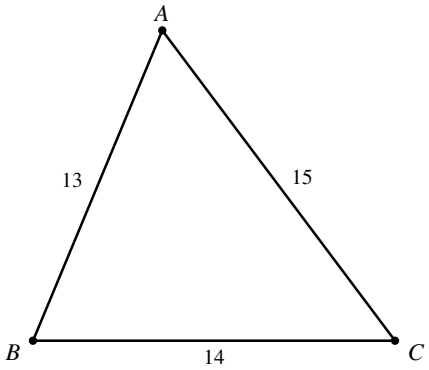
7. Each of the 9 cells of a 3×3 grid is coloured black or white, each colour with probability $\frac{1}{2}$, independently of the other cells. What is the probability that there is a path of black cells from some cell in the top row to some cell in the bottom row, where each step of the path moves to a cell sharing a side with the current cell?

(A) $\frac{169}{512}$ **(B)** $\frac{3}{8}$ **(C)** $\frac{197}{512}$ **(D)** $\frac{201}{512}$ **(E)** $\frac{1}{2}$

8. Call a positive integer N with at least two digits a *tripler* if moving its first (leftmost) digit to the end (rightmost place) produces the number $3N$. For example, moving the first digit of 1234 to the end would produce 2341. How many triplers are less than 10^{12}?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

9. Triangle ABC has side lengths $AB = 13$, $BC = 14$ and $CA = 15$. A point P is chosen uniformly at random inside the triangle. What is the probability that P is closer to A than it is to B, and also closer to A than it is to C?



(A) $\frac{279}{896}$ **(B)** $\frac{21}{64}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{323}{896}$ **(E)** $\frac{3}{8}$

10. Real numbers x, y and z satisfy $x + y + z = 6$ and $xy + yz + zx = 9$. What is the largest possible value of $x - z$?

(A) $\sqrt{6}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $2\sqrt{6}$

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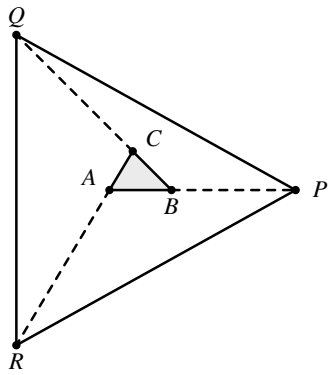
11. Let $S = \{1, 2, 3, 4, 5, 6\}$. How many ordered triples (A, B, C) of *nonempty* subsets of S have the property that each of the three sets is contained in the union of the other two? That is, $A \subseteq B \cup C$, $B \subseteq A \cup C$ and $C \subseteq A \cup B$.

(A) 15435 **(B)** 15436 **(C)** 15437 **(D)** 15624 **(E)** 15625

12. A fair coin is tossed repeatedly until it lands heads for the first time. Let N be the total number of tosses made, so $N = 1$ if the first toss is heads, $N = 2$ if the tosses go tails then heads, and so on. A game pays N^2 dollars. What is the expected payout, in dollars?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

13. Triangle ABC has area 5. Point P lies on ray AB beyond B with $BP = 2 \cdot AB$. Point Q lies on ray BC beyond C with $CQ = 3 \cdot BC$. Point R lies on ray CA beyond A with $AR = 4 \cdot CA$. What is the area of triangle PQR?



(A) 90 **(B)** 135 **(C)** 150 **(D)** 175 **(E)** 180

14. The number 30 is written as a sum of one or more positive integers that are all different from one another, and then those integers are multiplied together. For example, $30 = 5 + 10 + 15$ gives the product 750. What is the largest product that can be obtained this way?

(A) 6480 **(B)** 6720 **(C)** 7200 **(D)** 7560 **(E)** 8064

15. Several positive divisors of $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$ are chosen so that none of the chosen numbers divides a different chosen number. What is the largest possible number of divisors chosen?

(A) 12 **(B)** 15 **(C)** 20 **(D)** 23 **(E)** 30

16. Real numbers a and b satisfy $a^3 - 3a^2 + 5a = 1$ and $b^3 - 3b^2 + 5b = 5$. What is $a + b$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

17. Five fair six-sided dice are rolled. What is the probability that some number shows on exactly two of the dice? (For example, 3, 3, 5, 5, 5 and 2, 2, 1, 4, 6 both count, but 4, 4, 4, 4, 1 and 1, 2, 3, 4, 6 do not.)

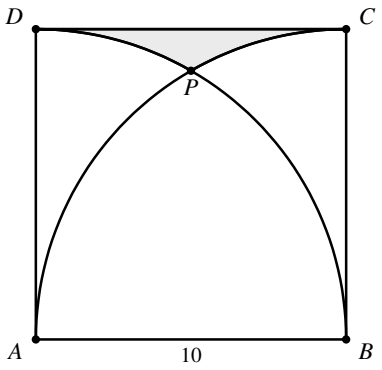
(A) $\frac{25}{54}$ **(B)** $\frac{25}{36}$ **(C)** $\frac{475}{648}$ **(D)** $\frac{175}{216}$ **(E)** $\frac{49}{54}$

18. A positive integer is a binary palindrome if its base-2 representation (written with no leading zeros)

reads the same forwards and backwards. For example, $5 = 101_2$ and $9 = 1001_2$ are binary palindromes, but $6 = 110_2$ is not. How many of the integers $1, 2, 3, \ldots, 1000$ are binary palindromes?

(A) 46 **(B)** 60 **(C)** 61 **(D)** 62 **(E)** 77

19. Square $ABCD$ has side length 10, with its vertices labelled in order around it. Inside the square, a quarter-circle arc of radius 10 centred at A runs from B to D, and a quarter-circle arc of radius 10 centred at B runs from A to C. The two arcs cross at a point P inside the square. What is the area of the curved triangle bounded by side $\overline{CD}$, the arc from D to P, and the arc from P to C (that is, the region of the square outside both quarter circles)?



(A) $-\frac{50\pi}{3} - 25\sqrt{3} + 100$ **(B)** $-25\sqrt{3} + \frac{50\pi}{3}$ **(C)** $100 - 25\pi$ **(D)** $-\frac{50\pi}{3} - \frac{25\sqrt{3}}{2} + 100$ **(E)** $100 - \frac{50\pi}{3}$

20. The ten letters $A, A, A, A, B, B, B, C, C, D$ (four A's, three B's, two C's and one D) are arranged in a row. Arrangements that differ only by swapping identical letters count as the same. In how many of the distinct arrangements does the first A come before the first B, the first B come before the first C, and the first C come before the D?

(A) 120 **(B)** 525 **(C)** 1260 **(D)** 1680 **(E)** 2520

21. Five people are at a party. For each of the 10 pairs of people, either the two know each other or they do not. Call a group of three of the people *uniform* if all three pairs in the group know each other, or no pair in the group knows each other. Of the $2^{10} = 1024$ possible patterns of who knows whom, how many have no uniform group of three?

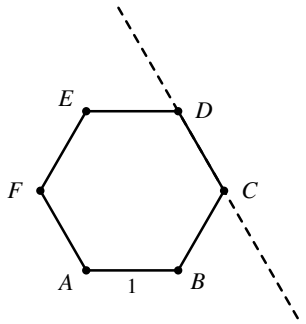
(A) 0 **(B)** 12 **(C)** 24 **(D)** 60 **(E)** 120

22. Positive real numbers a, b and c satisfy $a^2 + ab + b^2 = 25$, $b^2 + bc + c^2 = 49$ and $c^2 + ca + a^2 = 64$. What is the value of $ab + bc + ca$?

(A) 40 **(B)** 46 **(C)** 69 **(D)** $40\sqrt{3}$ **(E)** 80

23. Regular hexagon $ABCDEF$ has side length 1, with its vertices labelled in order around it. Exactly two circles pass through both A and B and are tangent to the line CD (the full line through C and D). What is the product of the radii of these two circles?

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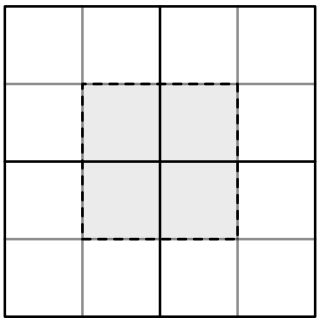


(A) $\frac{7}{3}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $\frac{11}{3}$ **(E)** 7

24. How many of the 1000 integers $0, 1, 2, \ldots, 999$ occur as the remainder when some perfect square k^2 (with k an integer) is divided by 1000? Equivalently, how many different endings of three digits (such as 000, 001 or 625) can a perfect square have?

(A) 153 **(B)** 159 **(C)** 189 **(D)** 315 **(E)** 501

25. Each of the 16 cells of a 4×4 grid is filled with one of the letters A, B, C, D. The grid must satisfy all of the following: every row contains each letter exactly once; every column contains each letter exactly once; each of the four 2×2 corner blocks (top-left, top-right, bottom-left, bottom-right) contains each letter exactly once; and the central 2×2 block (the four middle cells, in rows 2–3 and columns 2–3) also contains each letter exactly once. How many such grids are there? (Grids that differ by a rotation or reflection count as different.)



(A) 96 **(B)** 144 **(C)** 168 **(D)** 192 **(E)** 288

Answer Key — AMC 10 — Hard Edition Practice Contest 4

1. (B) $\frac{125}{6}$

2. (D) $12\sqrt{2}$

3. (D) 6

4. (E) $\frac{5}{9}$

5. (E) 32

6. (E) 678

7. (C) $\frac{197}{512}$

8. (D) 4

9. (B) $\frac{21}{64}$

10. (C) $2\sqrt{3}$

11. (A) 15435

12. (D) 6

13. (E) 180

14. (E) 8064

15. (A) 12

16. (B) 2

17. (C) $\frac{475}{648}$

18. (C) 61

19. (A) $-\frac{50\pi}{3} - 25\sqrt{3} + 100$

20. (D) 1680

21. (B) 12

22. (A) 40

23. (A) $\frac{7}{3}$

24. (B) 159

25. (C) 168

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Solutions — AMC 10 — Hard Edition Practice Contest 4

1. Answer: (B) $\frac{125}{6}$. The idea. Neither the arm lengths nor the melon's true weight is known, but only the *ratio* of the arms matters. The two melon weighings are two equations in the melon's weight and this ratio; dividing them eliminates the melon.

Work it out. Let the left arm have length a and the right arm length b, and let the melon weigh w kg. Melon on the left: $wa = 9b$, so $w = 9 \cdot \frac{b}{a}$. Melon on the right: $16a = wb$, so $w = 16 \cdot \frac{a}{b}$. Setting these equal, $\left(\frac{b}{a}\right)^2 = \frac{16}{9}$, so $\frac{b}{a} = \frac{4}{3}$ (and the melon weighs 12 kg). First bag, rice R_1 on the right: $10a = R_1 b$, so $R_1 = 10 \cdot \frac{a}{b} = \frac{15}{2}$. Second bag, rice R_2 on the left: $R_2 a = 10b$, so $R_2 = 10 \cdot \frac{b}{a} = \frac{40}{3}$. Altogether $\frac{15}{2} + \frac{40}{3} = \frac{45+80}{6} = \frac{125}{6}$ kg, about 20.83 kg.

Check. Plug back in with $a = 3$, $b = 4$: $12 \cdot 3 = 9 \cdot 4$ and $16 \cdot 3 = 12 \cdot 4$, so a 12 kg melon gives both readings; $10 \cdot 3 = 7.5 \cdot 4$ and $\frac{40}{3} \cdot 3 = 10 \cdot 4$, so both bags balance. In general the total is $10\left(k + \frac{1}{k}\right)$ with $k = \frac{b}{a}$, which is more than 20 whenever $k \neq 1$: alternating sides does not make the errors cancel, and the merchant always gives away extra rice. Forgetting the square root (using $\frac{b}{a} = \frac{16}{9}$) gives the distractor $\frac{45}{8} + \frac{160}{9} = \frac{1685}{72}$.

2. Answer: (D) $12\sqrt{2}$. The idea. A circle tangent to both walls has its centre equally far from them, so the centre sits on the corner's bisector at (r, r), where r is the radius. Passing through P gives one quadratic in r; its two roots are the two circles, and only the *difference* of the roots is needed.

Work it out. Put O at the origin with the walls along the axes, so $P = (2, 9)$. The centre (r, r) is at distance r from P: $(r-2)^2 + (r-9)^2 = r^2$, which simplifies to $r^2 - 22r + 85 = 0 = (r-5)(r-17)$. The centres are $(5, 5)$ and $(17, 17)$, and the distance between them is $\sqrt{12^2 + 12^2} = 12\sqrt{2}$.

Check. Without solving: the roots of $r^2 - 22r + 85$ differ by $\sqrt{22^2 - 4 \cdot 85} = \sqrt{144} = 12$, and two points on the line $y = x$ whose coordinates differ by 12 are $12\sqrt{2}$ apart. Also, $(5-2)^2 + (5-9)^2 = 25 = 5^2$ and $(17-2)^2 + (17-9)^2 = 289 = 17^2$, so both circles really pass through P.

3. Answer: (D) 6. The idea. The common value is unknown, so call it k. Then a, b and c are three different solutions of the same cubic equation $t^3 - 3t - k = 0$; a cubic has at most three roots, so they are *exactly* its roots, and Vieta's formulas give their symmetric sums without ever finding k.

Work it out. The cubic $t^3 + 0t^2 - 3t - k$ has roots a, b, c, so $a + b + c = 0$ (the t^2 coefficient is 0) and $ab + bc + ca = -3$. Therefore $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 0 + 6 = 6$.

Check. Such numbers exist: with $k = 0$ the roots are 0, $\sqrt{3}$, $-\sqrt{3}$, and $0 + 3 + 3 = 6$. A Vieta-free route confirms it for every k: from $a^3 - b^3 = 3(a - b)$ and $a \neq b$ we get $a^2 + ab + b^2 = 3$, and likewise $b^2 + bc + c^2 = 3$ and $c^2 + ca + a^2 = 3$. Subtracting the second from the first gives $(a - c)(a + b + c) = 0$, so $a + b + c = 0$; adding all three gives $2(a^2 + b^2 + c^2) + (ab + bc + ca) = 9$, and with $ab + bc + ca = -\frac{1}{2}(a^2 + b^2 + c^2)$ this is $\frac{3}{2}(a^2 + b^2 + c^2) = 9$. A sign slip ($ab + bc + ca = +3$) gives -6, which is impossible for a sum of squares.

4. Answer: (E) $\frac{5}{9}$. The idea. Counting outcomes with $0, 1, 2, \ldots$ mutual pairs is messy, but expected value is additive: the expected number of mutual pairs is the sum, over all possible pairs of students, of the probability that that particular pair is mutual.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

Work it out. There are $\binom{10}{2} = 45$ pairs of students. For a fixed pair $\{A, B\}$, student A writes B with probability $\frac{1}{9}$ and, independently, B writes A with probability $\frac{1}{9}$, so the pair is mutual with probability $\frac{1}{81}$. The expected number of mutual pairs is $45 \cdot \frac{1}{81} = \frac{5}{9}$.

Check. The same reasoning for a club of n gives $\binom{n}{2} \cdot \frac{1}{(n-1)^2} = \frac{n}{2(n-1)}$. For $n = 2$ this is 1, correct since the two students must write each other; for $n = 3$ it is $\frac{3}{4}$, and listing the $2^3 = 8$ equally likely outcomes confirms 6 of them contain exactly one mutual pair and 2 (the two cyclic ones) contain none. A common slip is to give each name probability $\frac{1}{10}$, forgetting that nobody writes their own name, which gives $45 \cdot \frac{1}{100} = \frac{9}{20}$; multiplying $\frac{1}{81}$ by the 10 students instead of the 45 pairs gives $\frac{10}{81}$.

5. Answer: (E) 32**. The idea.** If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, then n has $(e_1+1)(e_2+1)\cdots(e_r+1)$ divisors. Each factor is at least 2, so the product can be prime only when there is a single factor: n must be a prime power p^e, and then $e + 1$ must be prime.

Work it out. Go through the primes $q = e + 1$. $q = 2$: $n = p$ is prime, and there are 25 primes up to 100. $q = 3$: $n = p^2$, giving $4, 9, 25, 49$ ($11^2 = 121$ is too big), so 4 more. $q = 5$: $n = p^4$, giving 16 and 81, so 2 more. $q = 7$: $n = p^6$, giving only 64. $q = 11$: $2^{10} = 1024 > 100$, so none. The total is $25 + 4 + 2 + 1 = 32$.

Check. Prime powers that are *not* counted: $8 = 2^3$ and $27 = 3^3$ (four divisors), $32 = 2^5$ (six divisors); a quick look at 64 confirms its divisors $1, 2, 4, 8, 16, 32, 64$ number 7. The number 1 has one divisor, and 1 is not prime, so it is excluded. Stopping after the primes and prime squares gives 29; skipping the fourth powers gives 30; forgetting the sixth power 64 gives 31.

6. Answer: (E) 678**. The idea.** The hexagon splits into the triangle, the three squares, and three "gap" triangles, one at each vertex of ABC. Each gap triangle has two sides equal to two sides of ABC, and the angle between them is 180° minus the angle of ABC at that vertex (the four angles around the vertex are this angle, two right angles and the gap angle). Since $\sin(180^\circ - \theta) = \sin\theta$, every gap triangle has exactly the same area as ABC.

Work it out. The altitude from A to BC is $\sqrt{13^2 - 5^2} = 12$, so $[ABC] = \frac{1}{2} \cdot 10 \cdot 12 = 60$. The squares have areas $169 + 169 + 100 = 438$. The hexagon's area is $438 + 60 + 3 \cdot 60 = 678$.

Check. A sine-free way to see the equal areas: rotate a gap triangle by 90° about its vertex; one of its sides lands along the extension of a side of ABC beyond that vertex, and the gap triangle becomes a triangle with the same height as ABC and an equal base, hence equal area. Coordinates confirm it: with $B = (-5, 0)$, $C = (5, 0)$, $A = (0, 12)$, the six outer corners are $(-5, -10)$, $(5, -10)$, $(17, 5)$, $(12, 17)$, $(-12, 17)$, $(-17, 5)$, and the shoelace formula gives area 678.

7. Answer: (C) $\frac{197}{512}$**. The idea.** Any such path must pass through the middle row, and it may snake sideways, so straight columns are not the only way. Look at the black cells of the middle row: they form one or two horizontal *segments* (runs of adjacent black cells). A path exists exactly when some segment has a black cell directly above one of its cells and a black cell directly below one of its cells: follow the path from its last visit to the top row to its first visit to the bottom row, and everything in between lies in one segment. So condition on the middle row.

Work it out. A segment of length L succeeds with probability $\left(1 - 2^{-L}\right)^2$ (not all L cells above white, not all L cells below white), and different segments use different columns, so they are independent. Over the 8 equally likely middle rows: no black cell, probability 0; one black cell

(3 rows), $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ each; two adjacent black cells (2 rows), $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$ each; the two corner cells, two separate segments, $1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$; all three black, $\left(\frac{7}{8}\right)^2 = \frac{49}{64}$. The probability is $\frac{1}{8}\left(\frac{3}{4} + \frac{9}{8} + \frac{7}{16} + \frac{49}{64}\right) = \frac{1}{8} \cdot \frac{48+72+28+49}{64} = \frac{197}{512}$.

Check. In counts: out of the 64 colourings of the top and bottom rows, the middle rows contribute $3 \cdot 16 + 2 \cdot 36 + 28 + 49 = 197$ good colourings in all, out of 512. The answer is less than $\frac{1}{2}$, as it should be: blocking the path only needs a chain of white cells from the left side to the right side in which *diagonal* steps are also allowed, which is easier to form. Counting only straight black columns gives $1 - \left(\frac{7}{8}\right)^3 = \frac{169}{512}$, and treating the two corner cells as if each succeeded separately (adding $\frac{1}{4} + \frac{1}{4}$ instead of using $\frac{7}{16}$) gives $\frac{201}{512}$.

8. Answer: (D) 4. **The idea.** Split N into its first digit d and the number M formed by the remaining k digits, so $N = d \cdot 10^k + M$ with $0 \le M < 10^k$. Moving d to the end gives $10M + d$. The tripler condition becomes one linear equation, and then divisibility by 7 and the size bound $M < 10^k$ pin everything down.

Work it out. $10M + d = 3(d \cdot 10^k + M)$ gives $7M = d(3 \cdot 10^k - 1)$. If $d = 7$ then $M = 3 \cdot 10^k - 1 \ge 10^k$, impossible. Otherwise 7 must divide $3 \cdot 10^k - 1$, i.e. $3 \cdot 10^k \equiv 1$, i.e. $10^k \equiv 5 \pmod 7$. The powers $10^1, 10^2, \ldots$ are $3, 2, 6, 4, 5, 1, \ldots \pmod 7$, repeating every 6, so $k \equiv 5 \pmod 6$: $k = 5$ or $k = 11$ for $N < 10^{12}$. Then $M = d \cdot \frac{3 \cdot 10^k - 1}{7}$, and $\frac{3 \cdot 10^k - 1}{7}$ is about $0.43 \cdot 10^k$, so $M < 10^k$ forces $d \le 2$. For $k = 5$: $\frac{299999}{7} = 42857$, giving $N = 142857$ and $N = 285714$. For $k = 11$: $\frac{3 \cdot 10^{11} - 1}{7} = 42857142857$, giving $N = 142857142857$ and 285714285714. That is 4 triplers.

Check. $3 \cdot 142857 = 428571$ and $3 \cdot 285714 = 857142$: each is its number with the first digit moved to the end. The 12-digit ones are these blocks written twice, and the rotation works block by block. The next length, $k = 17$, gives 18-digit numbers, beyond 10^{12}.

9. Answer: (B) $\frac{21}{64}$**. The idea.** The points closer to A than to B are those on A's side of the perpendicular bisector of AB; likewise for C. The two perpendicular bisectors meet at the circumcentre O, which lies inside the triangle because the triangle is acute ($13^2 + 14^2 > 15^2$). So the region is the quadrilateral $AMON$, where M and N are the midpoints of AB and AC, and it splits along AO into two right triangles.

Work it out. The triangle's area is 84 (split BC at the foot of the altitude from A into $5 + 9$; the altitude is 12). The circumradius is $R = \frac{abc}{4 \cdot \text{area}} = \frac{13 \cdot 14 \cdot 15}{336} = \frac{65}{8}$. Triangle AMO has a right angle at M, with $AM = \frac{13}{2}$ and hypotenuse $AO = R$, so $OM = \sqrt{\frac{4225}{64} - \frac{2704}{64}} = \frac{39}{8}$. Triangle ANO has $AN = \frac{15}{2}$ and $ON = \sqrt{\frac{4225}{64} - \frac{3600}{64}} = \frac{25}{8}$. The region has area $\frac{1}{2} \cdot \frac{13}{2} \cdot \frac{39}{8} + \frac{1}{2} \cdot \frac{15}{2} \cdot \frac{25}{8} = \frac{507+375}{32} = \frac{441}{16}$. The probability is $\frac{441/16}{84} = \frac{441}{1344} = \frac{21}{64}$.

Check. The same method gives the regions nearest B and nearest C: with $OP = \frac{33}{8}$ for the midpoint P of BC, they have areas $\frac{507}{32} + \frac{231}{16} = \frac{969}{32}$ and $\frac{375}{32} + \frac{231}{16} = \frac{837}{32}$, and $\frac{882+969+837}{32} = 84$, the whole triangle. The three regions are unequal ($\frac{882}{2688}$, $\frac{969}{2688}$, $\frac{837}{2688}$ of the triangle), which is why the naive guess $\frac{1}{3}$ fails, and the other two probabilities, $\frac{323}{896}$ for B and $\frac{279}{896}$ for C, are what one gets by clipping at the wrong vertex. In coordinates ($B = (0, 0)$, $C = (14, 0)$, $A = (5, 12)$), $O = \left(7, \frac{33}{8}\right)$, and the shoelace formula on A, $\left(\frac{5}{2}, 6\right)$, O, $\left(\frac{19}{2}, 6\right)$ gives $\frac{441}{16}$ again.

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10. Answer: (C) $2\sqrt{3}$**. The idea.** Fix $s = x + z$. Then $y = 6 - s$ is determined, and so is xz, and the gap between two numbers with known sum and product is $\sqrt{(\text{sum})^2 - 4(\text{product})}$. This turns $(x - z)^2$ into a quadratic in s to maximise.

Work it out. With $s = x + z$ we have $y = 6 - s$ and $xz = 9 - y(x + z) = 9 - s(6 - s) = s^2 - 6s + 9$. Then $(x - z)^2 = s^2 - 4xz = s^2 - 4s^2 + 24s - 36 = -3s^2 + 24s - 36 = 12 - 3(s - 4)^2$. This is at most 12, reached at $s = 4$, so $x - z \le 2\sqrt{3}$. At $s = 4$: $y = 2$ and $xz = 1$ with $x - z = 2\sqrt{3}$, so $x = 2 + \sqrt{3}$, $z = 2 - \sqrt{3}$.

Check. $(2 + \sqrt{3}) + 2 + (2 - \sqrt{3}) = 6$ and $xy + yz + zx = 2(2 + \sqrt{3}) + 2(2 - \sqrt{3}) + (4 - 3) = 9$. The tempting answer 4 comes from noting that each variable separately lies in $[0, 4]$ (with $x = 4$ forcing $y = z = 1$), but $x = 4$ and $z = 0$ cannot happen together. Geometrically, $x^2 + y^2 + z^2 = 36 - 18 = 18$, so the point (x, y, z) lies on a circle of radius $\sqrt{6}$ around $(2, 2, 2)$, and $x - z$ is at most $\sqrt{6} \cdot \sqrt{2} = 2\sqrt{3}$.

11. Answer: (A) 15435**. The idea.** Decide the triple one element at a time: each element of S chooses which of A, B, C it belongs to, and the three containments say exactly that no element may belong to *exactly one* of the sets. Then remove the triples in which some set is empty, by inclusion–exclusion.

Work it out. An element in A only would break $A \subseteq B \cup C$, and similarly for B only and C only; every other membership pattern is fine. So each element has $8 - 3 = 5$ allowed patterns (in none, in exactly two of the sets in 3 ways, or in all three), and without the nonempty condition there are $5^6 = 15625$ triples. If $A = \varnothing$, the conditions force $B \subseteq C$ and $C \subseteq B$, so $B = C$, which can be any of $2^6 = 64$ subsets; the same count holds for $B = \varnothing$ or $C = \varnothing$. If two of the sets are empty, the third is contained in the empty set, so all three are empty: 1 triple for each of the 3 pairs, and 1 triple with all three empty. By inclusion–exclusion the triples with at least one empty set number $3 \cdot 64 - 3 \cdot 1 + 1 = 190$, leaving $15625 - 190 = 15435$.

Check. Per element, when $A = \varnothing$ the allowed patterns are "in none" and "in B and C", so 2^6 agrees. Try $S = \{1\}$: the 5 allowed patterns give triples $(\varnothing, \varnothing, \varnothing)$, three with one empty set, and $(\{1\}, \{1\}, \{1\})$; the formula gives $5 - 3 \cdot 2 + 3 - 1 = 1$, and indeed only $(\{1\}, \{1\}, \{1\})$ has all three sets nonempty. Dropping the final +1 gives 15436 and subtracting it instead gives 15437; removing only the all-empty triple gives 15624, and 15625 comes from ignoring the word nonempty.

12. Answer: (D) 6**. The idea.** $P(N = n) = \frac{1}{2^n}$, so the answer is the series $\sum_{n \ge 1} \frac{n^2}{2^n}$. It is not $(E[N])^2 = 4$: squaring rewards long runs extra. A clean way to sum it is to look at the first toss: if it is tails the game restarts, with one toss already spent.

Work it out. Let $E_1 = E[N]$ and $E_2 = E[N^2]$. With probability $\frac{1}{2}$ the first toss is heads and $N = 1$; otherwise $N = 1 + N'$ where N' has the same distribution as N. So $E_1 = \frac{1}{2} \cdot 1 + \frac{1}{2}(1 + E_1)$, giving $E_1 = 2$. Similarly $E_2 = \frac{1}{2} \cdot 1 + \frac{1}{2}E[(1 + N')^2] = \frac{1}{2} + \frac{1}{2}(1 + 2E_1 + E_2) = \frac{1}{2} + \frac{1}{2}(5 + E_2)$, so $\frac{1}{2}E_2 = 3$ and $E_2 = 6$. The expected payout is \$6.

Check. Shift and subtract on $S = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \cdots$: $S - \frac{S}{2} = \sum \frac{n^2 - (n-1)^2}{2^n} = \sum \frac{2n-1}{2^n} = 2E_1 - 1 = 3$, so $S = 6$. The partial sums 0.5, 1.5, 2.625, 3.625, 4.406, 4.969, 5.352, ... climb toward 6.

13. Answer: (E) 180**. The idea.** Triangle PQR is made of four pieces: triangle ABC in the middle and three corner triangles PBQ, QCR and RAP. Each corner triangle has an angle that is the supplement of an angle of ABC, and supplementary angles have the same sine. So the area of

each corner triangle is a simple multiple of $[ABC]$: multiply the ratios of the two sides around that angle.

Work it out. Since Q is on ray BC beyond C, segment BQ passes through C and $BQ = BC+CQ = 4 \cdot BC$. The angle PBQ is the supplement of angle ABC, so $[PBQ] = \frac{1}{2} \cdot BP \cdot BQ \cdot \sin\angle PBQ = \frac{1}{2} \cdot (2\,AB)(4\,BC) \sin\angle ABC = 8\,[ABC]$. In the same way, $CR = CA+AR = 5 \cdot CA$ gives $[QCR] = 3 \cdot 5\,[ABC] = 15\,[ABC]$, and $AP = AB + BP = 3 \cdot AB$ gives $[RAP] = 4 \cdot 3\,[ABC] = 12\,[ABC]$. The four pieces fill triangle PQR without overlapping, so $[PQR] = (1 + 8 + 15 + 12)[ABC] = 36 \cdot 5 = 180$. A common slip is to multiply the two added lengths ($2 \cdot 3$, $3 \cdot 4$, $4 \cdot 2$) instead of the full sides around each angle, which gives $27 \cdot 5 = 135$; another is to forget the middle triangle, which gives 175.

Check. Take $A = (0, 0)$, $B = (1, 0)$ and $C = (0, 10)$, so $[ABC] = 5$. Then $P = B + 2(B - A) = (3, 0)$, $Q = C + 3(C - B) = (-3, 40)$ and $R = A + 4(A - C) = (0, -40)$. The shoelace formula gives $\frac{1}{2}|3 \cdot 40 - 0 + (-3)(-40) - 0 + 0 - 3 \cdot (-40)| = \frac{1}{2} \cdot 360 = 180$.

14. Answer: (E) 8064**. The idea.** Two local moves can only help, so an optimal set cannot allow either of them. First, a part equal to 1 is wasted: adding it to the largest part m instead turns a factor of $1 \cdot m$ into $m+1$. Second, if parts $x < y$ have $x+1$ and $y-1$ both unused and $x+1 < y-1$, replace them by $x+1$ and $y-1$: the sum is unchanged and $(x+1)(y-1) = xy + (y-x-1) > xy$. So the parts should be packed as tightly as possible: a run of consecutive integers with at most one value skipped.

Work it out. Without 1, the smallest possible parts are $2 + 3 + 4 + 5 + 6 + 7 = 27$, which leaves 3 to place. With at most one gap allowed, the candidates are few. Six parts: $\{2, 3, 4, 6, 7, 8\}$ (the run 2 to 8 minus 5, sum $35 - 5 = 30$), product $2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 = 8064$. Five parts: $\{4, 5, 6, 7, 8\}$, product 6720. Four parts: $\{6, 7, 8, 9\}$, product 3024. Seven or more different parts, none equal to 1, sum to at least $2 + \cdots + 8 = 35 > 30$. So the maximum is 8064. Traps: dumping all the extra 3 onto the top part, $\{2, 3, 4, 5, 6, 10\}$, gives only 7200; the lopsided $\{2, 3, 4, 5, 7, 9\}$ gives 7560 (the move $(5, 9) \to (6, 8)$ improves it); and using 1 as in $\{1, 2, 3, 4, 5, 6, 9\}$ gives 6480.

Check. Compare the top two six-part sets directly. $\{2, 3, 4, 6, 7, 8\}$ and $\{2, 3, 4, 5, 7, 9\}$ share $2, 3, 4, 7$; the rest give $6 \cdot 8 = 48$ versus $5 \cdot 9 = 45$, so $8064 = 7560 \cdot \frac{48}{45}$. A computer search over every set of different positive integers with sum 30 also finds 8064 as the largest product.

15. Answer: (A) 12**. The idea.** Give each divisor $2^a 3^b 5^c 7^d$ a level: $a + b + c + d$, its number of prime factors counted with repetition. If one divisor properly divides another, its level is strictly smaller, so all divisors on one level form an allowed set. For the upper bound, split all 60 divisors into chains, lists in which each number divides the next: an allowed set takes at most one number from each chain.

Work it out. *Lower bound.* Count level 4, i.e. $a + b + c + d = 4$ with $a \le 4$, $b \le 2$, $c, d \le 1$. If $c + d = 0$: $(a, b) = (4, 0), (3, 1), (2, 2)$, giving 3. If $c + d = 1$ (two ways): $a + b = 3$ with $(3, 0), (2, 1), (1, 2)$, giving 6. If $c + d = 2$: $a + b = 2$ with $(2, 0), (1, 1), (0, 2)$, giving 3. That is 12 divisors, none dividing another. (Levels 0 through 8 have $1, 4, 8, 11, 12, 11, 8, 4, 1$ divisors; level 4 is the largest.)

Upper bound. The divisors of $144 = 2^4 3^2$ split into three chains: $1, 2, 4, 8, 16, 48, 144$; then $3, 6, 12, 24, 72$; then $9, 18, 36$. The set $\{1, 5, 7, 35\}$ splits into the chain $1, 5, 35$ and the single 7. For a chain $c_1, \ldots, c_k$ of divisors of 144 (with $k \ge 3$), its products with $1, 5, 35$ form a $k \times 3$ grid that splits into three hook-shaped chains: $c_1, \ldots, c_k, 5c_k, 35c_k$; then $5c_1, \ldots, 5c_{k-1}, 35c_{k-1}$;

then $35c_1, \ldots, 35c_{k-2}$. That is $3 \cdot 3 = 9$ chains, and the multiples of 7 by each 144-chain give 3 more chains. All 60 divisors lie in these 12 chains, so at most 12 can be chosen. Traps: combining two neighbouring levels, such as levels 3 and 4 for 23 divisors, fails because, for instance, 2 times a level-3 divisor is on level 4; and the tempting "take all divisors above $\sqrt{5040}$" (half of 60, so 30) fails because, for instance, 72 divides 144.

Check. The chain counts add up: the grids have $3(7+5+3) = 45$ elements and the 7-chains have 15, total 60. A computer computation of the largest dividing-free set (via the chain-cover duality) also gives 12.

16. Answer: (B) 2**. The idea.** Neither cubic has a nice root, so do not try to solve them. Instead shift the variable by 1 to remove the square term. Both equations then say that the same odd, increasing function takes the values -2 and 2, which forces the two shifted values to be negatives of each other.

Work it out. Expand $(t-1)^3 + 2(t-1) = t^3 - 3t^2 + 3t - 1 + 2t - 2 = t^3 - 3t^2 + 5t - 3$. So with $g(u) = u^3 + 2u$, the equations say $g(a-1) = 1 - 3 = -2$ and $g(b-1) = 5 - 3 = 2$. The function g is odd, so $g(1-a) = 2 = g(b-1)$. It is also strictly increasing (it is a sum of two increasing functions), so it takes the value 2 only once. Hence $b - 1 = 1 - a$, and $a + b = 2$. This also shows a and b are each uniquely determined. A tempting wrong answer is 3, the sum of all three (complex) roots of either cubic by Vieta's formulas.

Check. Numerically, $a \approx 0.22908$ and $b \approx 1.77092$. Indeed $0.22908^3 - 3(0.22908)^2 + 5(0.22908) \approx 0.01202 - 0.15744 + 1.14542 = 1.00000$, and $a + b \approx 2.00000$.

17. Answer: (C) $\frac{475}{648}$**. The idea.** Sort the rolls by their pattern, meaning how many dice show each value that appears. The patterns with a value on exactly two dice are $2+1+1+1$, $2+2+1$ and $3+2$. It is easy to miss the last two. Count each pattern separately by choosing the values and then the dice.

Work it out. Pattern $2+1+1+1$: choose the paired value (6 ways), choose which two dice show it ($\binom{5}{2} = 10$), then give the other three dice distinct new values ($5 \cdot 4 \cdot 3 = 60$). That is 3600 rolls. Pattern $2+2+1$: choose the two paired values ($\binom{6}{2} = 15$) and the single value (4), then arrange them on the dice in $\frac{5!}{2!\,2!\,1!} = 30$ ways. That is 1800 rolls. Pattern $3+2$: choose the triple value (6) and the pair value (5), then choose which three dice show the triple ($\binom{5}{3} = 10$). That is 300 rolls. The total is 5700 of the $6^5 = 7776$ rolls, so the probability is $\frac{5700}{7776} = \frac{475}{648}$.

Check. Count the complement by pattern. All different: $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$. Pattern $3+1+1$: $6 \cdot 10 \cdot 20 = 1200$. Pattern $4+1$: $6 \cdot 5 \cdot 5 = 150$. Pattern 5: 6. That is 2076 rolls, and $2076 + 5700 = 7776$.

18. Answer: (C) 61**. The idea.** A palindrome is determined by its first half. A binary palindrome with L digits must begin with 1, so it also ends with 1. After that first digit, the next $\lceil L/2 \rceil - 1$ digits are free, and the second half is their mirror image. So there are $2^{\lceil L/2 \rceil - 1}$ palindromes of length L. The only subtlety is the cut-off at 1000, which falls inside the 10-digit numbers.

Work it out. Since $2^9 = 512 \le 1000 < 1024 = 2^{10}$, the numbers up to 1000 have at most 10 binary digits. The counts for lengths $L = 1, 2, \ldots, 9$ are $1, 1, 2, 2, 4, 4, 8, 8, 16$, which total 46. There are $2^4 = 16$ palindromes of length 10, and all of them lie between 512 and 1023. The largest ones start 11111, and the mirror forces the whole number to be $1111111111_2 = 1023 > 1000$.

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The next largest is $1111001111_2 = 975 \le 1000$. So only 1023 is too big, and 15 of the 16 count. The total is $46 + 15 = 61$.

Check. Small cases agree. Up to 7 the binary palindromes are $1, 3, 5, 7$, and the formula gives $1 + 1 + 2 = 4$. Forgetting that 1023 is too big gives 62, and stopping at 9 digits gives 46.

19. Answer: (A) $-\frac{50\pi}{3} - 25\sqrt{3} + 100$**. The idea.** The region is the square minus the union of the two quarter disks, and that union is found by inclusion–exclusion. The hard part is the overlap of the two quarter disks. Because $PA = PB = AB = 10$, triangle ABP is equilateral. The overlap is two 60° sectors that share that triangle.

Work it out. Each quarter disk has area $\frac{1}{4}\pi \cdot 100 = 25\pi$. Their overlap is the set of points within 10 of both A and B, and it is bounded by $\overline{AB}$ and the two arcs through P. It is the union of the sector at A between rays AB and AP and the sector at B between rays BA and BP, each 60° with area $\frac{100\pi}{6}$. These two sectors overlap exactly in triangle ABP, which has area $25\sqrt{3}$. So the overlap is $\frac{100\pi}{3} - 25\sqrt{3}$. The union of the quarter disks is $50\pi - \frac{100\pi}{3} + 25\sqrt{3} = \frac{50\pi}{3} + 25\sqrt{3}$. The curved triangle has area $100 - 25\sqrt{3} - \frac{50\pi}{3}$.

Check. Numerically, $100 - 43.30 - 52.36 \approx 4.34$. This is plausible, since the region is a thin sliver below $\overline{CD}$ whose tip P sits at height $5\sqrt{3} \approx 8.66$. A rough triangle estimate is $\frac{1}{2} \cdot 10 \cdot 1.34 = 6.7$, and the region is smaller because its two sides curve inward.

20. Answer: (D) 1680**. The idea.** Peel off one letter type at a time. The condition says the very first letter is an A. Once the A's are ignored, the first remaining letter is a B. Once the B's are also ignored, the first remaining letter is a C. For a random arrangement these three events are independent, because where the A's sit tells you nothing about the relative order of the other letters, and so on. The letters are not equally numerous, so the answer is not simply $\frac{1}{24}$ of the total.

Work it out. There are $\frac{10!}{4!\,3!\,2!\,1!} = 12600$ arrangements. Build a random one in stages. First choose the order of the C's and D (a random arrangement of C, C, D). Then insert the B's at random. Then insert the A's at random. The first letter overall is A with probability $\frac{4}{10}$. Among the six non-A letters, the first is a B with probability $\frac{3}{6}$. Among C, C, D, the first is a C with probability $\frac{2}{3}$. So the probability is $\frac{4}{10} \cdot \frac{3}{6} \cdot \frac{2}{3} = \frac{2}{15}$, and the count is $12600 \cdot \frac{2}{15} = 1680$.

Check. The same method on a small case with letters A, B, B asks for the first A before the first B. It gives $3 \cdot \frac{1}{3} = 1$, and indeed only ABB works. Requiring the reverse order D, C, B, A would give $12600 \cdot \frac{1}{10} \cdot \frac{2}{9} \cdot \frac{3}{7} = 120$, which shows how much the letter counts matter. The naive $12600/24 = 525$ is wrong.

21. Answer: (B) 12**. The idea.** Look at one person, say X, and the four others. If X knows at least three of them, look at those three. If any two of them know each other, those two together with X form a group where everyone knows everyone. If no two of them know each other, those three form a group where nobody knows anybody. Either way there is a uniform group. The same argument works if X fails to know at least three of the others. So in a good pattern, every person knows exactly two of the other four.

Work it out. Draw the five people as points and join two points when the people know each other. Every point has exactly two lines, so following lines from any point you go around a closed loop. A loop needs at least three points, and five points cannot split into two loops (that would

need a loop of two points). So the lines form a single loop through all five people: a pentagon. Conversely, a pentagon works. The pairs who know each other are the pentagon's sides, and no three sides form a triangle. The pairs who do not know each other are the diagonals, which form a five-pointed star, and that is again a single loop through all five with no triangle. So we count the ways to arrange five labelled people in a loop. Fix person 1; the other four can follow around the loop in $4! = 24$ orders, and each loop is counted twice (once in each direction). The answer is $\frac{24}{2} = 12$.

Check. Forgetting that a loop read backwards is the same loop gives 24. Thinking every pattern must contain a uniform group gives 0; that is true for six people, but the pentagon shows it fails for five. A computer check of all 1024 patterns finds exactly 12 with no uniform group, one for each pentagon.

22. Answer: (A) 40**. The idea.** The expression $a^2 + ab + b^2$ is the law of cosines for an angle of 120°, because $a^2 + b^2 - 2ab\cos 120° = a^2 + ab + b^2$. So the three equations describe three segments of lengths a, b, c from one point, at 120° to each other, whose far ends form a triangle with sides 5, 7, 8. Computing that triangle's area in two ways gives $ab + bc + ca$ directly, without finding a, b, c.

Work it out. From a point O draw three rays with 120° between each two of them, and mark X, Y, Z on them with $OX = a$, $OY = b$ and $OZ = c$. By the law of cosines, $XY^2 = a^2 + ab + b^2 = 25$, $YZ^2 = 49$ and $ZX^2 = 64$, so triangle XYZ has sides 5, 7, 8. The three angles at O add to 360° and each is less than 180°, so O is inside triangle XYZ and the triangles OXY, OYZ, OZX fill it. Each has area $\frac{1}{2} \cdot (\text{two sides}) \cdot \sin 120°$, so $[XYZ] = \frac{\sqrt{3}}{4}(ab + bc + ca)$. By Heron's formula with $s = 10$, $[XYZ] = \sqrt{10 \cdot 5 \cdot 3 \cdot 2} = 10\sqrt{3}$. Therefore $ab + bc + ca = \frac{4 \cdot 10\sqrt{3}}{\sqrt{3}} = 40$. Adding the three equations gives only $2(a^2 + b^2 + c^2) + (ab + bc + ca) = 138$, which cannot separate the two sums; guessing they are equal gives the trap 46.

Check. Solving numerically gives $a \approx 3.52180$, $b \approx 2.20113$ and $c \approx 5.63489$. Then $a^2 + ab + b^2 \approx 12.4031 + 7.7519 + 4.8450 = 25.000$, and $ab + bc + ca \approx 7.7519 + 12.4031 + 19.8450 = 40.000$.

23. Answer: (A) $\frac{7}{3}$**. The idea.** Extend side AB until it meets line CD at a point X. The power of X gives the tangent length: if the circle touches line CD at T, then $XT^2 = XA \cdot XB$. That puts T at one of two spots, one on each side of X, which explains the two circles. Each radius then comes from one right triangle, and the product simplifies nicely.

Work it out. The interior angles are 120°, so lines AB and CD meet beyond $\overline{BC}$ to form an equilateral triangle XBC with side 1. Hence $XB = 1$, $XA = 2$ and $XT = \sqrt{2}$. Now use coordinates with X at the origin and line CD as the x-axis, with $C = (1, 0)$ and $D = (2, 0)$. Line XA makes a 60° angle with it, so $B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $A = \left(1, \sqrt{3}\right)$. A circle tangent to the x-axis at $T = (t, 0)$ has centre (t, r). Passing through B and A requires $\left(t - \frac{1}{2}\right)^2 + \left(r - \frac{\sqrt{3}}{2}\right)^2 = r^2$ and $(t - 1)^2 + (r - \sqrt{3})^2 = r^2$. Expanding gives $r = \frac{t^2 - t + 1}{\sqrt{3}}$ and $r = \frac{t^2 - 2t + 4}{2\sqrt{3}}$. Setting these equal gives $t^2 = 2$, which confirms $t = \pm\sqrt{2}$. Then $r = \frac{3 \mp \sqrt{2}}{\sqrt{3}}$, and the product is $\frac{(3 - \sqrt{2})(3 + \sqrt{2})}{3} = \frac{9 - 2}{3} = \frac{7}{3}$.

Check. The radii are about 0.917 and 2.549. Both exceed $\frac{1}{2}$, as a circle through a chord of length 1 must. For $t = \sqrt{2}$, the centre $(\sqrt{2}, 0.917)$ is at distance $\sqrt{(\sqrt{2} - 1)^2 + (0.917 - 1.732)^2} \approx \sqrt{0.172 + 0.664} \approx 0.914$ from A, which matches the radius 0.917 to rounding.

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24. Answer: (B) 159**. The idea.** Split $1000 = 8 \cdot 125$. By the Chinese Remainder Theorem, a remainder r modulo 1000 is a square remainder exactly when it is a square remainder modulo 8 and modulo 125: if $r \equiv k_1^2 \pmod 8$ and $r \equiv k_2^2 \pmod{125}$, pick k with $k \equiv k_1 \pmod 8$ and $k \equiv k_2 \pmod{125}$, and then $k^2 \equiv r \pmod{1000}$. So the answer is (number of squares mod 8) $\times$ (number of squares mod 125). The care is needed modulo 125, which is a prime power, not a prime.

Work it out. Modulo 8, the squares of $0, 1, \ldots, 7$ are $0, 1, 4, 1, 0, 1, 4, 1$: there are 3 square remainders. Modulo 125, first take k not divisible by 5; there are 100 such remainders. If $k^2 \equiv j^2$ with both not divisible by 5, then $125 \mid (k - j)(k + j)$. The factors $k - j$ and $k + j$ cannot both be multiples of 5, since then 5 would divide their sum $2k$. So 125 divides one of them, and $j \equiv \pm k$. Each value is therefore hit by exactly two of the 100 remainders, giving 50 values, none divisible by 5. Next take $k = 5m$. Then $k^2 = 25m^2$, and modulo 125 this depends only on $m^2 \bmod 5 \in \{0, 1, 4\}$, giving 0, 25 and 100. That is $50 + 3 = 53$ square remainders modulo 125, and $3 \cdot 53 = 159$ modulo 1000. Forgetting 25 and 100 gives 153; using the prime rule $\frac{p+1}{2}$ for 125 as if it were prime gives $3 \cdot 63 = 189$.

Check. The same method modulo $10 = 2 \cdot 5$ predicts $2 \cdot 3 = 6$ possible last digits, and indeed squares end only in $0, 1, 4, 5, 6, 9$. Modulo 1000, a direct computer check of $k^2 \bmod 1000$ for $k = 0, 1, \ldots, 999$ finds exactly 159 different values.

25. Answer: (C) 168**. The idea.** Renaming the letters does not affect any condition, so fix the top-left block and multiply by $4! = 24$. With that block fixed, the top-right block has only two independent binary choices, and so does the bottom-left block. The bottom-right block is then forced. So everything reduces to 16 cases described by four yes/no switches, and the conditions become simple logical constraints on the switches.

Work it out. Write the letters as $1, 2, 3, 4$ and fix the top-left block with rows 1 2 and 3 4. Row 1 must finish with 3, 4 in some order, and row 2 with 1, 2 in some order. Let $x = 1$ mean row 1 ends 4 3, and $y = 1$ mean row 2 ends 2 1. Similarly, column 1 must continue with 2, 4 and column 2 with 1, 3. Let $u = 1$ mean column 1 continues 4, 2, and $v = 1$ mean column 2 continues 3, 1. Each bottom-right cell must avoid the two letters to its left in the bottom-left block and the two letters above it in the top-right block. Checking the 16 cases shows the bottom-right block is consistent exactly when $x = y$ or $u = v$. (If $x \neq y$ and $u \neq v$, some bottom-right cell has no letter left.) That gives $16 - 4 = 12$ completions and $24 \cdot 12 = 288$ grids without the centre condition. Now add the centre block. Its cells are the fixed 4, the entry 1 or 2 to its right (it is 2 exactly when $y = 1$), the entry 1 or 3 below it (it is 3 exactly when $v = 1$), and the diagonal cell in row 3, column 3. The diagonal cell is in the same column as the right entry and the same row as the lower entry, so it automatically differs from both. Where is the bottom-right block's 4? If $x = 0$, row 1 has its 4 in column 4, so the block's 4 must go in column 3; if $x = 1$, column 3 already has its 4 in row 1, so the block's 4 goes in column 4. In the same way, $u = 0$ puts column 1's 4 in row 4, which forces the block's 4 into row 3, while $u = 1$ forces it into row 4. So the diagonal cell is 4 exactly when $x = u = 0$. The centre avoids a repeat exactly when (i) the right and lower entries differ, meaning $y = 1$ or $v = 1$, and (ii) the diagonal cell is not 4, meaning $x = 1$ or $u = 1$. Now count switch settings satisfying (i), (ii) and consistency. If $x = y = 1$, then u and v are free: 4 settings. If $x = y = 0$, then (i) and (ii) force $u = v = 1$: 1 setting. If $x \neq y$, consistency needs $u = v$, and (i) or (ii) then forces $u = v = 1$: 2 settings. That is 7 settings in all, so the answer is $24 \cdot 7 = 168$.

Check. A computer search over all 24^4 ways to choose the four rows finds 288 grids satisfying the row, column and corner conditions, and 168 of them also have a valid centre block. The 288

agrees with the well-known count of 4×4 sudoku grids.