MathCastle

AMC 10

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

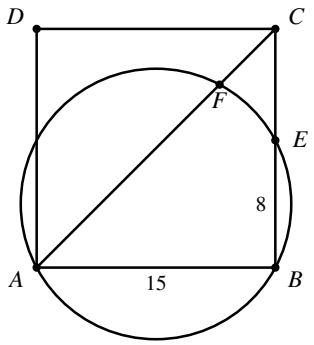
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MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Five married couples (each couple is one man and one woman) sit in a row of 10 chairs. Every person sits in a chair next to his or her own spouse, and no two women sit in neighbouring chairs. In how many different ways can the ten people be seated?

(A) 720 **(B)** 840 **(C)** 1440 **(D)** 1920 **(E)** 3840

2. Square $ABCD$ has side length 15. Point E lies on side $\overline{BC}$ with $BE = 8$. The circle passing through A, B and E meets the diagonal $\overline{AC}$ at A and at a second point F. What is AF?



(A) $\frac{7\sqrt{2}}{2}$ **(B)** $8\sqrt{2}$ **(C)** $\frac{23}{2}$ **(D)** $\frac{23\sqrt{2}}{2}$ **(E)** 17

3. A positive integer n has the property that when all of its positive divisors (including 1 and n) are multiplied together, the product is exactly $2^{30} \cdot 3^{15}$. What is n?

(A) 72 **(B)** 144 **(C)** 216 **(D)** 288 **(E)** 324

4. For a four-digit number N whose last digit is not 0, let R be the four-digit number formed by writing the digits of N in reverse order. (For example, if $N = 1234$ then $R = 4321$.) For how many such numbers N is $N - R = 3087$?

(A) 40 **(B)** 42 **(C)** 45 **(D)** 48 **(E)** 54

5. Three different real numbers form an arithmetic progression whose sum is 30. When each of the three numbers is squared, the three squares, taken in the same order, form a geometric progression. What is the largest of the three numbers?

(A) $10\sqrt{2}$ **(B)** 15 **(C)** $5\sqrt{2} + 10$ **(D)** 20 **(E)** $10 + 10\sqrt{2}$

6. In the coordinate plane, consider every circle that is tangent to the x-axis, tangent to the y-axis, and tangent to the line $3x + 4y = 12$. What is the sum of the radii of all such circles?

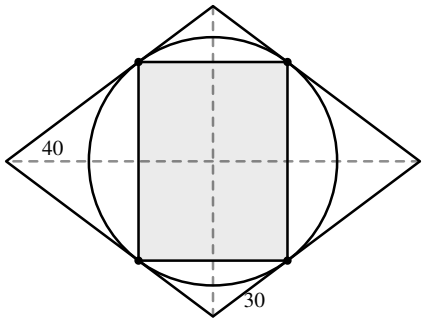
(A) 7 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

7. Six teams play a tournament in which every team plays every other team exactly once. There are no ties: each game is won by one of its two teams, each with probability $\frac{1}{2}$, independently of all other games. What is the probability that every team wins at least one game and also loses at least one game?

(A) $\frac{5}{8}$ **(B)** $\frac{169}{256}$ **(C)** $\frac{175}{256}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{7}{8}$

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8. A rhombus has diagonals of lengths 30 and 40. A circle is drawn inside the rhombus, touching all four of its sides. The four points where the circle touches the sides are the corners of a rectangle. What is the area of this rectangle?



(A) $\frac{3456}{25}$ **(B)** $\frac{5184}{25}$ **(C)** $\frac{6912}{25}$ **(D)** 288 **(E)** 300

9. The numbers $1, 2, 3, \ldots, 10$ are placed in the ten seats around a round table, one number per seat, so that every two numbers in neighbouring seats add up to a prime. Two placements that differ only by a rotation of the table count as the same, but a placement and its mirror image count as different. How many placements are there?

(A) 12 **(B)** 24 **(C)** 48 **(D)** 96 **(E)** 192

10. Two positive divisors of $1296 = 2^4 \cdot 3^4$ are chosen independently and at random, each divisor equally likely each time (so the two choices may be the same divisor). What is the probability that one of the two chosen divisors divides the other?

(A) $\frac{9}{25}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{17}{25}$ **(E)** $\frac{18}{25}$

11. Let $N = 7^7 + 7^{77} + 7^{777} + \cdots + 7^{7777777777}$, a sum of ten powers of 7 in which the k-th exponent is the k-digit number whose digits are all 7. What is the remainder when N is divided by 13?

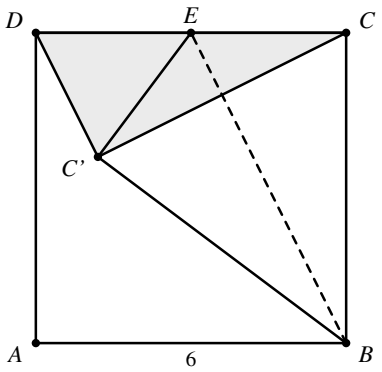
(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 11

12. The numbers $1, 2, 3, \ldots, 10$ are written on a board. A move consists of erasing any two numbers on the board and writing their average in their place. After nine moves only one number remains. Over all possible ways of making the moves, what is the largest possible value of the final number?

(A) $\frac{11}{2}$ **(B)** $\frac{17}{2}$ **(C)** 9 **(D)** $\frac{4609}{512}$ **(E)** $\frac{19}{2}$

13. Square $ABCD$ has side length 6, with its vertices labelled in order around it, and E is the midpoint of side $\overline{CD}$. The square is folded along the line BE, so that triangle BCE flips over onto the square and the corner C lands on a point C' inside the square. What is the area of triangle DCC'?

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(A) $\frac{36}{5}$ **(B)** 8 **(C)** 9 **(D)** $\frac{72}{5}$ **(E)** 18

14. Let $f(x) = x^2 + 6x + 7$. As x ranges over all real numbers, what is the smallest value that $f(f(x))$ can take?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

15. The number $2^{24} - 1$ equals 16,777,215. This is the largest value that a 24-bit binary counter can display (twenty-four 1s in binary). Counting 1 and the number itself, how many positive divisors does $2^{24} - 1$ have?

(A) 48 **(B)** 64 **(C)** 72 **(D)** 96 **(E)** 128

16. A bag holds 3 red and 3 blue marbles. They are drawn one at a time, at random and without replacement, until the bag is empty. Before each draw Ana guesses the colour of the marble about to be drawn. She always guesses the colour that has more marbles left in the bag; when the two colours are tied she guesses either colour with equal probability. What is the expected number of correct guesses Ana makes?

(A) $\frac{7}{2}$ **(B)** 4 **(C)** $\frac{41}{10}$ **(D)** $\frac{17}{4}$ **(E)** $\frac{9}{2}$

17. A cube has edge length 4. A sphere passes through all four vertices of the cube's bottom face and is tangent to all four edges of the cube's top face (each of those edges touches the sphere at exactly one point). What is the radius of the sphere?

(A) $\frac{\sqrt{41}}{2}$ **(B)** $2\sqrt{3}$ **(C)** $\frac{7}{2}$ **(D)** $2\sqrt{5}$ **(E)** $\sqrt{41}$

18. For a certain real number c, the equation $x^4 - 8x^3 + 14x^2 + 8x + c = 0$ has four different real roots, and these four roots form an arithmetic progression. What is c?

(A) -15 **(B)** -9 **(C)** -5 **(D)** 9 **(E)** 15

19. Split a four-digit number N into its first two digits and its last two digits, and read each pair as a number. Call these x and y, so that $N = 100x + y$, with $10 \le x \le 99$ and $0 \le y \le 99$ (the last pair may start with 0). For some four-digit numbers, $N = x^2 + y^2$. What is the sum of all four-digit numbers N with this property?

(A) 10066 **(B)** 10067 **(C)** 20066 **(D)** 20067 **(E)** 30067

20. The numbers $1, 2, 3, 4, 5, 6$ are arranged in a row, each used once. A descent is a place where a number

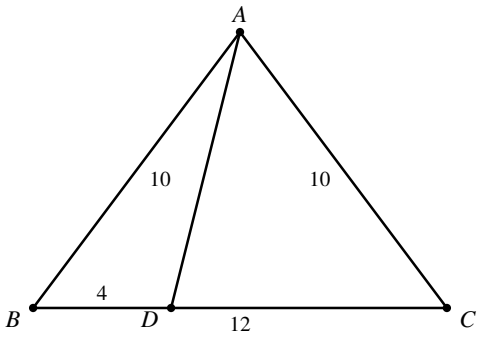
is immediately followed by a smaller number. For example, 2 5 1 3 6 4 has two descents (5 to 1, and 6 to 4). How many of the 720 arrangements have exactly one descent?

(A) 42 **(B)** 56 **(C)** 57 **(D)** 63 **(E)** 64

21. Real numbers x and y satisfy both $x^3 - 3xy^2 = 11$ and $3x^2y - y^3 = 2$. What is the largest possible value of x?

(A) $\frac{\sqrt{3}}{2} + 1$ **(B)** $\frac{1}{2} + \sqrt{3}$ **(C)** $\sqrt{5}$ **(D)** $\frac{11}{4}$ **(E)** $1 + 2\sqrt{3}$

22. In triangle ABC, $AB = AC = 10$ and $BC = 12$. Point D lies on side $\overline{BC}$ with $BD = 4$. What is the sum of the circumradius of triangle ABD and the circumradius of triangle ACD?



(A) $\frac{5\sqrt{17}}{4}$ **(B)** $\frac{25}{4}$ **(C)** $2\sqrt{17}$ **(D)** 10 **(E)** $\frac{5\sqrt{17}}{2}$

23. How many ordered pairs (a, b) of integers with $1 \le a \le 100$ and $1 \le b \le 100$ have the property that $a^2 + b^2$ is divisible by 13?

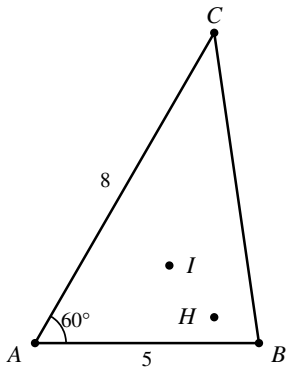
(A) 770 **(B)** 1442 **(C)** 1491 **(D)** 1540 **(E)** 1600

24. A regular icosahedron has 12 vertices and 30 edges. Each vertex is joined by edges to exactly 5 others, and each vertex has exactly one opposite vertex (the one farthest from it). A bug starts at a vertex S. Every minute it walks along an edge to one of the 5 neighbouring vertices, chosen uniformly at random, independently of its earlier moves. What is the expected number of minutes until the bug first reaches the vertex opposite S?

(A) 6 **(B)** 11 **(C)** 12 **(D)** 14 **(E)** 15

25. Triangle ABC has $AB = 5$, $AC = 8$ and $\angle BAC = 60°$. Let I be the center of the circle inscribed in the triangle, and let H be the orthocenter, the point where the three altitudes of the triangle meet. What is the distance IH?

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(A) 1 **(B)** $\frac{\sqrt{21}}{3}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** 3

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Answer Key — AMC 10 — Hard Edition Practice Contest 3

1. (A) 720

2. (D) $\frac{23\sqrt{2}}{2}$

3. (B) 144

4. (E) 54

5. (E) $10 + 10\sqrt{2}$

6. (E) 12

7. (C) $\frac{175}{256}$

8. (C) $\frac{6912}{25}$

9. (D) 96

10. (D) $\frac{17}{25}$

11. (B) 6

12. (D) $\frac{4609}{512}$

13. (A) $\frac{36}{5}$

14. (B) -1

15. (D) 96

16. (C) $\frac{41}{10}$

17. (A) $\frac{\sqrt{41}}{2}$

18. (A) -15

19. (A) 10066

20. (C) 57

21. (B) $\frac{1}{2} + \sqrt{3}$

22. (E) $\frac{5\sqrt{17}}{2}$

23. (C) 1491

24. (E) 15

25. (B) $\frac{\sqrt{21}}{3}$

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Solutions — AMC 10 — Hard Edition Practice Contest 3

1. Answer: (A) 720**. The idea.** Since every couple sits together, the row is made of five two-chair blocks, one per couple. Inside a block the man and woman are neighbours, which is allowed. The only danger is at the boundary between two blocks, so the question is really about the pattern of block orientations, which turns out to be very rigid.

Work it out. The five couples can be ordered as blocks in $5! = 120$ ways. Each block is either MW (man on the left) or WM. Two women are neighbours exactly when a block ending in W, an MW block, is immediately followed by a block starting with W, a WM block. So the orientations, read left to right, must never have MW followed directly by WM. Such a sequence is some number k of WM blocks followed by $5 - k$ MW blocks, with $k = 0, 1, \ldots, 5$: once an MW block appears, every later block is MW too. That gives 6 orientation patterns, and $120 \cdot 6 = 720$ seatings.

Check. With two couples the rule gives $2! \cdot 3 = 6$: in the order couple 1, couple 2 the patterns are WM WM, WM MW and MW MW, and only MW WM (which puts the two wives together in the middle) is excluded. A computer search over seatings of all five couples also finds 720. Ignoring the women's condition gives $5! \cdot 2^5 = 3840$; letting k run up to 6 by mistake gives $5! \cdot 7 = 840$; and doubling for whether the row starts with a man or a woman (already built into the patterns) gives 1440.

2. Answer: (D) $\frac{23\sqrt{2}}{2}$**. The idea.** The angle $\angle ABE$ is a corner of the square, so it is a right angle inscribed in the circle. That makes $\overline{AE}$ a diameter. Then F, also on the circle, sees the diameter at a right angle: $\angle AFE = 90°$. So F is simply the foot of the perpendicular from E to the diagonal, and AF is the shadow of $\overline{AE}$ on $\overline{AC}$.

Work it out. Put $A = (0, 0)$, $B = (15, 0)$, $C = (15, 15)$, so $E = (15, 8)$ and the diagonal runs along the unit direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The projection of $\overrightarrow{AE} = (15, 8)$ onto that direction has length $\frac{15+8}{\sqrt{2}} = \frac{23}{\sqrt{2}} = \frac{23\sqrt{2}}{2}$. Since $\angle AFE = 90°$, this projection is exactly AF. (Without coordinates: the perpendicular from E to AC splits AE into pieces along AB and BE, each of which casts a shadow $\frac{1}{\sqrt{2}}$ times its length on the 45° diagonal, giving $\frac{15}{\sqrt{2}} + \frac{8}{\sqrt{2}}$.)

Check. $F = \left(\frac{23}{2}, \frac{23}{2}\right)$. The circle has centre at the midpoint of AE, $\left(\frac{15}{2}, 4\right)$, and radius $\frac{AE}{2} = \frac{17}{2}$; the distance from the centre to F is $\sqrt{4^2 + \left(\frac{15}{2}\right)^2} = \frac{17}{2}$, so F is on the circle. Power of the point C agrees: $CF \cdot CA = \frac{7\sqrt{2}}{2} \cdot 15\sqrt{2} = 105 = CE \cdot CB = 7 \cdot 15$. The choice $\frac{7\sqrt{2}}{2}$ is CF rather than AF, $\frac{23}{2}$ forgets the $\sqrt{2}$ (it is the x-coordinate of F), and 17 is the diameter AE itself.

3. Answer: (B) 144**. The idea.** Divisors come in pairs d and $\frac{n}{d}$ whose product is n. So the product of all τ divisors of n is $n^{\tau/2}$, where τ is the number of divisors. That turns the problem into a small equation in the exponents.

Work it out. Only the primes 2 and 3 can divide n, since every prime factor of n divides the product. Write $n = 2^a 3^b$, so $\tau = (a+1)(b+1)$. The product of the divisors is $n^{\tau/2} = 2^{a\tau/2} 3^{b\tau/2}$. Matching exponents, $a\tau = 60$ and $b\tau = 30$, so $a = 2b$ and then $b(2b+1)(b+1) = 30$. Trying $b = 1, 2, 3$ gives 6, 30, 84, and the left side increases with b, so $b = 2$ and $a = 4$. Hence $n = 2^4 \cdot 3^2 = 144$.

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Check. 144 has $5 \cdot 3 = 15$ divisors, so the product is $144^{15/2} = 12^{15} = 2^{30}3^{15}$. By contrast $72 = 2^3 3^2$ has 12 divisors and product $72^6 = 2^{18}3^{12}$, and $216 = 2^3 3^3$ gives $216^8 = 2^{24}3^{24}$.

4. Answer: (E) 54**. The idea.** Write $N = \overline{abcd}$. In $N - R$ the middle digits and the outer digits separate completely: $N - R = 999(a - d) + 90(b - c)$. The small range of the digit differences then forces both differences.

Work it out. $N-R = (1000a+100b+10c+d)-(1000d+100c+10b+a) = 999(a-d)+90(b-c)$. We need $999(a - d) + 90(b - c) = 3087$, with $-9 \le b - c \le 9$, so $90(b - c)$ lies between -810 and 810. Then $999(a - d)$ is between 2277 and 3897, so $a - d = 3$, which leaves $90(b - c) = 90$, i.e. $b - c = 1$. (Indeed $3087 = 2997 + 90$.) Now count. We need $d \ge 1$ (the last digit is not 0) and $a = d + 3 \le 9$, so $d \in \{1, \ldots, 6\}$: 6 choices. And $b = c + 1$ with $c \in \{0, \ldots, 8\}$: 9 choices. In total $6 \cdot 9 = 54$.

Check. With $N = 4321$: $R = 1234$ and $4321 - 1234 = 3087$, as required; so does $N = 9876$, since $9876 - 6789 = 3087$. A number such as 5431 has $b - c = 1$ but $a - d = 4$, and $5431 - 1345 = 4086$. Forgetting that c may be 0 (the pair $b = 1$, $c = 0$, as in 4321) leaves only 8 choices for c and gives $6 \cdot 8 = 48$.

5. Answer: (E) $10 + 10\sqrt{2}$**. The idea.** Write the progression symmetrically as $a - d$, a, $a + d$. The sum gives a at once. The geometric condition says the middle square is the geometric mean of the outer squares, and taking a square root of that equation produces a sign choice: the obvious sign forces $d = 0$, and the other sign is the answer.

Work it out. The sum is $3a = 30$, so $a = 10$. The squares $(a - d)^2$, a^2, $(a + d)^2$ form a geometric progression, so $a^4 = (a - d)^2(a + d)^2 = (a^2 - d^2)^2$. Hence $a^2 = a^2 - d^2$ or $a^2 = -(a^2 - d^2)$. The first gives $d = 0$, which is ruled out because the numbers are different. The second gives $d^2 = 2a^2 = 200$, so $d = \pm 10\sqrt{2}$. Either sign gives the same three numbers, $10 - 10\sqrt{2}$, 10 and $10 + 10\sqrt{2}$, and the largest is $10 + 10\sqrt{2}$.

Check. The squares are $100(3 - 2\sqrt{2})$, 100 and $100(3 + 2\sqrt{2})$. Since $(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$, the product of the outer squares is 100^2, the square of the middle one, so the ratio is constant $(3+2\sqrt{2})$. Note that the first number, $10 - 10\sqrt{2}$, is negative: the squares form a geometric progression even though the numbers themselves do not. The distractor 20 comes from trying $d = 10$, which gives squares $0, 100, 400$, not a geometric progression.

6. Answer: (E) 12**. The idea.** A circle tangent to both axes has its centre at $(\pm r, \pm r)$, where r is its radius. The three lines bound a 3-4-5 triangle, and circles tangent to all three lines of a triangle are its incircle and its three excircles, one for each side, so we should expect four circles, not just the ones in the first quadrant.

Work it out. The distance from (x, y) to the line is $\frac{|3x+4y-12|}{5}$, and it must equal r. Try each quadrant for the centre. Centre (r, r): $|7r - 12| = 5r$, so $r = 6$ or $r = 1$. Centre $(r, -r)$: $|-r - 12| = r + 12 = 5r$, so $r = 3$. Centre $(-r, r)$: $|r - 12| = 5r$, so $12 - r = 5r$ and $r = 2$ (the other sign gives a negative r). Centre $(-r, -r)$: $|-7r - 12| = 7r + 12 = 5r$ has no positive solution. So there are four circles, with radii $1, 2, 3, 6$, and the sum is 12.

Check. The triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 3)$ has area 6 and semiperimeter 6. Its inradius is $\frac{6}{6} = 1$, and its exradii are $\frac{6}{6-3} = 2$, $\frac{6}{6-4} = 3$ and $\frac{6}{6-5} = 6$, matching the four circles. Stopping in the first quadrant gives $1 + 6 = 7$; missing one of the side excircles gives 9 or 10; forgetting the incircle gives 11.

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7. Answer: (C) $\frac{175}{256}$**. The idea.** Count the bad outcomes: some team wins all its games or some team loses all its games. Two teams can never both win all their games (one of them lost their game against each other), and likewise two teams cannot both lose all their games. So inclusion–exclusion only has one overlap to worry about: an unbeaten team and a winless team at the same time.

Work it out. There are $\binom{6}{2} = 15$ games, so 2^{15} equally likely outcomes. A particular team wins all 5 of its games in 2^{10} outcomes (the other 10 games are free). At most one team can do this, so "some team is unbeaten" happens in $6 \cdot 2^{10}$ outcomes; "some team is winless" likewise happens in $6 \cdot 2^{10}$ outcomes. Both happen when some team X is unbeaten and a different team Y is winless: that fixes $5+5-1 = 9$ games (the game X vs Y is counted in both), leaving 2^6 outcomes, for each of the $6 \cdot 5 = 30$ ordered choices of X and Y. The bad outcomes number $2 \cdot 6 \cdot 2^{10} - 30 \cdot 2^6 = 12288 - 1920 = 10368$. The probability of a good outcome is $\frac{32768-10368}{32768} = \frac{22400}{32768} = \frac{175}{256}$.

Check. Forgetting the overlap gives $1 - \frac{12}{32} = \frac{5}{8}$. Treating "no unbeaten team" and "no winless team" as independent gives $\left(1 - \frac{6}{32}\right)^2 = \frac{169}{256}$, which is too small: the events are positively related. The exact count of 22400 good outcomes agrees with a computer check of all 2^{15} outcomes.

8. Answer: (C) $\frac{6912}{25}$**. The idea.** The diagonals cut the rhombus into four 15-20-25 right triangles meeting at the centre O, which is also the circle's centre. Each point of tangency is the foot of the altitude from O to the hypotenuse of one of these triangles, and similar triangles locate it exactly.

Work it out. Put O at the origin with the vertices at $(\pm 20, 0)$ and $(0, \pm 15)$. In the right triangle with legs $OP = 20$ (along the x-axis) and $OQ = 15$, hypotenuse $PQ = 25$, the altitude from O has length $\frac{15 \cdot 20}{25} = 12$ (this is the circle's radius). Its foot T splits the hypotenuse into $PT = \frac{20^2}{25} = 16$ and $TQ = \frac{15^2}{25} = 9$. Walking from $Q = (0, 15)$ toward $P = (20, 0)$ a fraction $\frac{9}{25}$ of the way gives $T = \left(\frac{9}{25} \cdot 20,\ 15 - \frac{9}{25} \cdot 15\right) = \left(\frac{36}{5}, \frac{48}{5}\right)$. By symmetry the four tangency points are $\left(\pm\frac{36}{5}, \pm\frac{48}{5}\right)$, a rectangle with sides $\frac{72}{5}$ and $\frac{96}{5}$, of area $\frac{72 \cdot 96}{25} = \frac{6912}{25}$.

Check. T is at distance $\sqrt{7.2^2 + 9.6^2} = \sqrt{144} = 12$ from O, the radius, and it lies on the side $3x + 4y = 60$: $21.6 + 38.4 = 60$. The trap 300 is the rectangle formed by the midpoints of the sides, which is a different rectangle because the rhombus is not a square.

9. Answer: (D) 96**. The idea.** Every neighbour sum is at least $1+2 = 3$, so it is an odd prime, and an odd sum needs one odd and one even number. So odds and evens alternate, and we need the ways to go round the ten numbers using only the allowed odd-even pairs. The allowed pairs (sums $3, 5, 7, 11, 13, 17, 19$) are: 1 with 2, 4, 6, 10; 3 with 2, 4, 8, 10; 5 with 2, 6, 8; 7 with 4, 6, 10; 9 with 2, 4, 8, 10. Two twin pairs make the count manageable: 4 and 10 have exactly the same partners $\{1, 3, 7, 9\}$, and 3 and 9 have exactly the same partners $\{2, 4, 8, 10\}$. So count the patterns with 4, 10 written as A, A' and 3, 9 as B, B', then multiply by $2 \cdot 2$ for the labellings.

Work it out. A circular order corresponds to a closed loop through all ten numbers; with 1 fixed in one seat, each loop can be read clockwise or counterclockwise, giving 2 placements. Split on the two neighbours of 6, which come from $\{1, 5, 7\}$.

Case 1-6-5. Then 7 must sit between A and A'. If 5's other neighbour is 2, then 8 sits between B and B', and the pieces 1-6-5-2, A-7-A', B-8-B' join in only one pattern (1 to an A, the other A to a B, the other B to 2). If 5's other neighbour is 8, the chain 1-6-5-8-B plus A-7-A' and the loose 2, B' close up in 2 patterns (1 next to 2, or 1 next to A). That is 3 patterns.

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Case 1-6-7. Then 5 sits between 2 and 8, giving chains A-7-6-1 and 2-5-8-B with A', B' loose; 1 goes next to 2 (1 pattern) or next to A' (2 patterns, as A' then meets B or B'). That is 3 patterns.

Case 5-6-7, with 7-A on the other side. If 5-2, then B-8-B' and 2's other neighbour is 1 or a B: 2 patterns. If 5-8-B, then the ends B (to 2 or A') and A (to 1 or B') give four combinations, and each closes up in exactly one way: 4 patterns. That is 6 patterns.

In all $3 + 3 + 6 = 12$ patterns, $12 \cdot 4 = 48$ loops, and $48 \cdot 2 = 96$ placements.

Check. One placement, clockwise: 1, 2, 3, 8, 5, 6, 7, 4, 9, 10, with sums 3, 5, 11, 13, 11, 13, 11, 13, 19, 11, all prime. A computer search of all $9! = 362880$ orders with 1 in a fixed seat finds exactly 96. Forgetting that mirror images are different gives 48; stopping at the 12 patterns forgets the twin relabellings.

10. Answer: (D) $\frac{17}{25}$**. The idea.** Write the divisors as $2^{a_1}3^{b_1}$ and $2^{a_2}3^{b_2}$ with all exponents from 0 to 4, chosen independently and uniformly. "d_1 divides d_2" means $a_1 \le a_2$ and $b_1 \le b_2$, and those two conditions are independent. The event "one divides the other" is the union of two such events, and they overlap exactly when the divisors are equal.

Work it out. For one prime, $P(a_1 \le a_2) = \frac{15}{25} = \frac{3}{5}$: of the 25 pairs, 5 are ties and half of the other 20 have $a_1 < a_2$. By independence $P(d_1 \mid d_2) = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$, and likewise $P(d_2 \mid d_1) = \frac{9}{25}$. Both happen exactly when $d_1 = d_2$, which has probability $\frac{1}{25}$ (there are 25 divisors). So the answer is $\frac{9}{25} + \frac{9}{25} - \frac{1}{25} = \frac{17}{25}$.

Check. A tempting wrong method multiplies, prime by prime, the chance that the exponents are comparable; but exponents are always comparable, which would give probability 1. The failure is that d_1 may have the bigger power of 2 while d_2 has the bigger power of 3 (like 16 and 81). That happens with probability $2 \cdot \frac{10}{25} \cdot \frac{10}{25} = \frac{8}{25}$, and $1 - \frac{8}{25} = \frac{17}{25}$ agrees.

11. Answer: (B) 6**. The idea.** By Fermat's little theorem $7^{12} \equiv 1 \pmod{13}$, so each term depends only on its exponent modulo 12. To find a number modulo 12, find it modulo 4 (from its last two digits) and modulo 3 (from its digit sum). The exponents do not all behave alike, so the terms fall into a few classes.

Work it out. Let e_k be the exponent with k sevens. For $k = 1$, $e_1 = 7$. For $k \ge 2$, e_k ends in 77, and $77 \equiv 1 \pmod 4$; its digit sum is $7k \equiv k \pmod 3$. So $e_k \equiv 1 \pmod{12}$ when $k \equiv 1 \pmod 3$, $e_k \equiv 5 \pmod{12}$ when $k \equiv 2$, and $e_k \equiv 9 \pmod{12}$ when $k \equiv 0$. Now compute powers of 7 modulo 13: $7^2 \equiv 10$, $7^3 \equiv 5$, $7^4 \equiv 9$, $7^5 \equiv 11$, $7^6 \equiv 12 \equiv -1$, so $7^7 \equiv -7 \equiv 6$ and $7^9 \equiv -7^3 \equiv -5 \equiv 8$. The terms are therefore 6 (for $k = 1$), 11 (for $k = 2, 5, 8$), 8 (for $k = 3, 6, 9$) and 7 (for $k = 4, 7, 10$). Their sum is $6 + 3 \cdot 11 + 3 \cdot 8 + 3 \cdot 7 = 84 = 6 \cdot 13 + 6$, so the remainder is 6.

Check. Directly, $77 = 6 \cdot 12 + 5$, so $7^{77} \equiv 7^5 \equiv 11$, as claimed; and $777 = 64 \cdot 12 + 9$ gives $7^{777} \equiv 7^9 \equiv 8$. Treating every term like the first ($\equiv 6$) gives $60 \equiv 8$; applying the $k \ge 2$ rule to $k = 1$ as well (so the first term is $7^1 = 7$) gives $85 \equiv 7$.

12. Answer: (D) $\frac{4609}{512}$**. The idea.** Follow a single starting number. Each time the number carrying it is averaged, its share is halved. So the final number is $\sum w_k \cdot k$, where $w_k = 2^{-m_k}$ and m_k is how many averagings the number k takes part in; the weights always add up to 1, since a move turns two numbers carrying weights u and v into one number carrying $u + v$. Now bound how

much weight the big numbers can get, using the fact that a sum of a few powers of $\frac{1}{2}$ has only a few 1s in binary.

Work it out. Always average the two smallest numbers: 1 with 2, the result with 3, that result with 4, and so on, ending with 10. Then 10 gets weight $\frac{1}{2}$, 9 gets $\frac{1}{4}$, …, 2 gets $\frac{1}{512}$ and 1 gets $\frac{1}{512}$, and the final number is $\frac{10}{2}+\frac{9}{4}+\frac{8}{8}+\frac{7}{16}+\frac{6}{32}+\frac{5}{64}+\frac{4}{128}+\frac{3}{256}+\frac{2}{512}+\frac{1}{512}=\frac{4609}{512}$.

Nothing does better. Write W_j for the total weight on the numbers $j, j+1, \dots, 10$. Then $\sum w_k k = W_1 + W_2 + \cdots + W_{10}$, because the number k is counted in exactly k of the W_j. Here $W_1 = 1$. For $j \geq 2$, W_j is a sum of $t = 11 - j$ powers of $\frac{1}{2}$, and it is less than 1 because the number 1 has positive weight. Adding t powers of $\frac{1}{2}$ in binary gives at most t digits 1 (carrying only merges digits), and the largest binary number below 1 with at most t ones is $0.11\dots1 = 1 - 2^{-t}$. So $W_j \leq 1 - 2^{-(11-j)}$, and the final number is at most $1 + \sum_{t=1}^{9}(1-2^{-t}) = 10 - \left(1 - \frac{1}{512}\right) = \frac{4609}{512}$. The strategy above reaches every one of these bounds.

Check. Running the process: $1, 2 \to \frac{3}{2}$; with $3 \to \frac{9}{4}$; with $4 \to \frac{25}{8}$; with $5 \to \frac{65}{16}$; with $6 \to \frac{161}{32}$; with $7 \to \frac{385}{64}$; with $8 \to \frac{897}{128}$; with $9 \to \frac{2049}{256}$; with $10 \to \frac{4609}{512} \approx 9.002$. A computer search over every merge order agrees. The value $\frac{11}{2}$ is the average of $1, \dots, 10$, which a balanced pairing produces; it is far from the maximum.

13. Answer: (A) $\frac{36}{5}$. The idea. Folding keeps lengths, so $EC' = EC = 3 = ED$. That puts D, C and C' on the circle with centre E and diameter $\overline{DC}$, so the angle at C' in triangle DCC' is a right angle. Then only the two legs CC' and DC' are needed.

Work it out. The fold line BE is the perpendicular bisector of $\overline{CC'}$. Let F be the midpoint of $\overline{CC'}$; it is the foot of the perpendicular from C to BE. In right triangle BCE the legs are $BC = 6$ and $CE = 3$, and the hypotenuse is $BE = \sqrt{45} = 3\sqrt{5}$. So the altitude to the hypotenuse is $CF = \frac{6\cdot 3}{3\sqrt{5}} = \frac{6}{\sqrt{5}}$, and $CC' = 2\,CF = \frac{12}{\sqrt{5}}$. Since DC is a diameter, $\angle DC'C = 90°$, and $DC' = \sqrt{DC^2 - CC'^2} = \sqrt{36 - \frac{144}{5}} = \frac{6}{\sqrt{5}}$. The area is $\frac{1}{2}\cdot DC' \cdot CC' = \frac{1}{2}\cdot\frac{6}{\sqrt{5}}\cdot\frac{12}{\sqrt{5}} = \frac{36}{5}$.

Check. Coordinates: $A = (0,0)$, $B = (6,0)$, $C = (6,6)$, $D = (0,6)$, $E = (3,6)$. Reflecting C in line BE gives $C' = \left(\frac{6}{5}, \frac{18}{5}\right)$; indeed $BC'^2 = \frac{576}{25} + \frac{324}{25} = 36$ and $EC'^2 = \frac{81}{25} + \frac{144}{25} = 9$. Its distance below side DC is $6 - \frac{18}{5} = \frac{12}{5}$, so the area is $\frac{1}{2}\cdot 6 \cdot \frac{12}{5} = \frac{36}{5}$. Forgetting the $\frac{1}{2}$ in the area formula gives $\frac{72}{5}$, and 18 is the area of the half-square BCD.

14. Answer: (B) -1. The idea. The tempting answer is the minimum of f itself, -2. But the outer f can only be fed values that the inner f actually produces, and the input that minimises f is not one of them.

Work it out. Complete the square: $f(x) = (x+3)^2 - 2$. So f takes every value $u \geq -2$, and nothing below -2. Now $f(f(x)) = f(u)$ with u ranging over $[-2, \infty)$. On this interval $u + 3 \geq 1 > 0$, so $f(u) = (u+3)^2 - 2$ is increasing there, and the smallest value is at the left end: $f(-2) = 1 - 2 = -1$. It is achieved when $f(x) = -2$, that is at $x = -3$.

Check. $f(-3) = 9 - 18 + 7 = -2$ and $f(-2) = 4 - 12 + 7 = -1$. To reach -2 the outer f would need input -3, but $f(x) = -3$ has no real solution since $f \geq -2$. So -1 is the minimum.

15. Answer: (D) 96. The idea. Never divide blindly. Use algebraic factorisations to split $2^{24} - 1$ into small pieces: difference of squares, and then (the step people miss) a sum of cubes for $2^{12}+1$.

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Work it out. Start with $2^{24}-1=(2^{12}-1)(2^{12}+1)$ and $2^{12}-1=(2^6-1)(2^6+1)=63\cdot 65=3^2\cdot 7\cdot 5\cdot 13$. For the other factor, $2^{12}+1=(2^4)^3+1^3=(2^4+1)(2^8-2^4+1)=17\cdot 241$. Here 241 is prime: it is not divisible by 2, 3, 5, 7, 11 or 13, and $17^2>241$. So $2^{24}-1=3^2\cdot 5\cdot 7\cdot 13\cdot 17\cdot 241$, and the number of divisors is $(2+1)(1+1)^5=3\cdot 32=96$.

Check. Multiply back: $63\cdot 65=4095=2^{12}-1$, $17\cdot 241=4097=2^{12}+1$ and $4095\cdot 4097=4096^2-1=16{,}777{,}215$. If you had treated 4097 as prime you would get $3\cdot 2^4=48$, which is the trap answer.

16. Answer: (C) $\frac{41}{10}$. The idea. Track the gap $g=|\text{red left}-\text{blue left}|$. It starts and ends at 0. On a draw made while the colours are unequal, a correct guess (the majority colour) shrinks the gap by 1 and a wrong guess grows it by 1. On a draw made from a tie, the gap goes from 0 to 1 whatever happens, and Ana is right with probability $\frac{1}{2}$. Balancing the gap turns the whole problem into counting ties.

Work it out. Let T be the number of draws made from a tie, and let c and w be the numbers of correct and wrong guesses on the other draws. The gap rises by $T+w$ in total and falls by c, and it ends where it began, so $c=T+w$. There are 6 draws, so $T+c+w=6$. Together these give $c=3$ exactly, on every order of draws! So the expected score is $3+\frac{1}{2}E[T]$. Think of the draws as a path from $(3,3)$ to $(0,0)$ in (red left, blue left), one step at a time; all $\binom{6}{3}=20$ paths are equally likely. A tie draw happens at each visit to $(3,3)$, $(2,2)$ or $(1,1)$. Every path starts at $(3,3)$. The number of paths through $(2,2)$ is $\binom{2}{1}\binom{4}{2}=12$, and the number through $(1,1)$ is $\binom{4}{2}\binom{2}{1}=12$. So $E[T]=\frac{20+12+12}{20}=\frac{11}{5}$, and the expected score is $3+\frac{11}{10}=\frac{41}{10}$.

Check. The order $RBRBRB$ is tied before draws 1, 3 and 5, so $T=3$; Ana is right on draws 2, 4, 6 and expects $3+\frac{3}{2}=\frac{9}{2}$. The order $RRRBBB$ has $T=1$: after a random first guess she guesses blue on draws 2 and 3 (wrong both times) and is right on draws 4, 5, 6, for $3+\frac{1}{2}$ in expectation. Both fit $3+\frac{T}{2}$. Assuming $T=3$ always gives the trap $\frac{9}{2}$; counting only the first draw as a tie gives $\frac{7}{2}$.

17. Answer: (A) $\frac{\sqrt{41}}{2}$. The idea. By symmetry the centre lies on the cube's vertical axis, at some unknown height h above the bottom face. A sphere tangent to a line is one whose centre is exactly R from that line. So write R^2 twice, once with a bottom vertex and once with a top edge, and set the two equal.

Work it out. Put the bottom face at height 0 and let the centre be O, at height h on the axis. A bottom vertex is $\sqrt{2^2+2^2}=2\sqrt{2}$ from the axis horizontally, so $R^2=8+h^2$. The nearest point of a top edge to O is the midpoint of that edge: it is 2 from the axis horizontally and $4-h$ above or below O. So $R^2=4+(4-h)^2$. Then $8+h^2=4+16-8h+h^2$, so $8h=12$ and $h=\frac{3}{2}$. Hence $R^2=8+\frac{9}{4}=\frac{41}{4}$ and $R=\frac{\sqrt{41}}{2}$.

Check. The top face is $4-\frac{3}{2}=\frac{5}{2}$ above O, and $\frac{5}{2}<R$, so the sphere cuts the plane of the top face in a circle of radius $\sqrt{\frac{41}{4}-\frac{25}{4}}=2$. That is exactly the circle inscribed in the 4×4 top face, touching each top edge at its midpoint, just as tangency demands. Forgetting to halve at the end gives $\sqrt{41}$; putting the centre in the bottom face gives $2\sqrt{5}$, its distance to a top edge; and the sphere through all eight vertices has radius $2\sqrt{3}$.

18. Answer: (A) -15. The idea. Four numbers in arithmetic progression can be written symmetrically as $m-3d,\ m-d,\ m+d,\ m+3d$. Vieta's formulas pin down the centre m. Shifting the

variable to the centre then turns the quartic into a biquadratic, which makes the step d easy to find.

Work it out. The sum of the roots is $4m = 8$, so $m = 2$. Set $y = x - 2$. The roots in y are $\pm d, \pm 3d$, so the polynomial must equal $(y^2 - d^2)(y^2 - 9d^2) = y^4 - 10d^2y^2 + 9d^4$. Expand in x and compare the x^2 coefficients. From $y^4 = (x-2)^4$ we get $24x^2$, and from $-10d^2y^2$ we get $-10d^2x^2$. So $24 - 10d^2 = 14$, which gives $d^2 = 1$. The roots are 2 ± 1 and 2 ± 3, that is $-1, 1, 3, 5$, and c is their product: $c = (-1)(1)(3)(5) = -15$.

Check. Expand $(x+1)(x-1)(x-3)(x-5) = (x^2-1)(x^2-8x+15) = x^4 - 8x^3 + 14x^2 + 8x - 15$. It matches every given coefficient, including the x coefficient 8, which we never used, so it is a real consistency test.

19. Answer: (A) 10066**. The idea.** $100x + y = x^2 + y^2$ is a circle in disguise. Multiply by 4 and complete both squares to get a sum of two squares equal to a fixed number, then list the ways to write that number as a sum of two squares.

Work it out. Rewrite as $x^2 - 100x + y^2 - y = 0$. Multiplying by 4 gives $(2x-100)^2 + (2y-1)^2 = 10000 + 1 = 10001$. Now $10001 = 73 \cdot 137$, and both primes are 1 more than a multiple of 4: $73 = 8^2 + 3^2$ and $137 = 11^2 + 4^2$. Combining them gives the only two representations: $10001 = 100^2 + 1^2 = 76^2 + 65^2$. Since $2y - 1$ is odd and $2x - 100$ is even, the possibilities are: $2x - 100 = \pm 100$, $2y - 1 = \pm 1$, which gives $x \in \{0, 100\}$. Neither is a valid first pair. $2x - 100 = \pm 76$, $2y - 1 = \pm 65$, which gives $x \in \{12, 88\}$ and $y = 33$ (the value $y = -32$ is rejected). So $N = 1233$ or $N = 8833$, and the sum is 10066.

Check. $12^2 + 33^2 = 144 + 1089 = 1233$ and $88^2 + 33^2 = 7744 + 1089 = 8833$. Also, x and $100 - x$ always give the same $x^2 - 100x$, which explains why the two answers come as a pair $12, 88$ with the same y.

20. Answer: (C) 57**. The idea.** An arrangement with at most one descent is just two increasing runs placed one after the other. It is completely determined by which numbers go into the first run. So count the subsets, then throw away the ones that produce no descent at all.

Work it out. Choose any subset S of $\{1, \ldots, 6\}$. Write S in increasing order, then the rest in increasing order. This gives an arrangement with at most one descent, and every arrangement with at most one descent arises from exactly one S (namely, the numbers before its descent). So there are $2^6 = 64$ arrangements with at most one descent. The arrangement has no descent exactly when it is 1 2 3 4 5 6. That happens for 7 subsets: $S = \varnothing, \{1\}, \{1,2\}, \ldots, \{1,2,3,4,5,6\}$. So the answer is $64 - 7 = 57$.

Check. For three numbers the formula gives $2^3 - 4 = 4$, and indeed $132, 213, 231, 312$ are exactly the arrangements of $1, 2, 3$ with one descent. In general the count is $2^n - n - 1$ (an Eulerian number).

21. Answer: (B) $\frac{1}{2} + \sqrt{3}$**. The idea.** The two left-hand sides fit together: squaring and adding them gives a perfect cube, $(x^3 - 3xy^2)^2 + (3x^2y - y^3)^2 = (x^2 + y^2)^3$. So $x^2 + y^2$ is forced, and then the first equation becomes a cubic in x alone.

Work it out. Check the identity: the left side is $x^6 - 6x^4y^2 + 9x^2y^4 + 9x^4y^2 - 6x^2y^4 + y^6 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 = (x^2+y^2)^3$. With the given values, $(x^2+y^2)^3 = 121 + 4 = 125$, so $x^2 + y^2 = 5$. Substitute $y^2 = 5 - x^2$ into the first equation: $x^3 - 3x(5 - x^2) = 11$, that is $4x^3 - 15x - 11 = 0$. Here $x = -1$ is a root, and $4x^3 - 15x - 11 = (x+1)(4x^2 - 4x - 11)$, so $x = -1$ or $x = \frac{1 \pm 2\sqrt{3}}{2}$. The largest

candidate is $x = \frac{1+2\sqrt{3}}{2}$. It really occurs: then $x^2 = \frac{13+4\sqrt{3}}{4}$, so $y^2 = 5 - x^2 = \frac{7-4\sqrt{3}}{4} = \left(\frac{2-\sqrt{3}}{2}\right)^2$. The second equation reads $y(3x^2 - y^2) = 2$, and $3x^2 > y^2$, so y must be positive: $y = \frac{2-\sqrt{3}}{2}$. This pair satisfies $x^2 + y^2 = 5$ and the first equation (by construction) and the second one by the sign choice, since squaring and adding recovers $(3x^2y - y^3)^2 = 125 - 121 = 4$. So the largest value is $\frac{1+2\sqrt{3}}{2}$.

Check. The smallest root gives the neat solution $x = -1$, $y = -2$: $-1 - 3(-1)(4) = 11$ and $3(1)(-2) - (-8) = 2$. Numerically the answer is ≈ 2.232, just below the bound $\sqrt{5} \approx 2.236$ that $x^2 + y^2 = 5$ imposes; choosing $\sqrt{5}$ means stopping after the first step. (With complex numbers the identity says $|(x + yi)^3| = |11 + 2i|$, and the three solutions are 120° rotations of $(-1, -2)$.)

22. Answer: (E) $\frac{5\sqrt{17}}{2}$**. The idea.** By the extended law of sines, the circumradius of a triangle is any side divided by twice the sine of the opposite angle. In triangle ABD the side AB sits opposite $\angle ADB$, and in triangle ACD the side AC sits opposite $\angle ADC$. These two angles are supplementary, so they have the same sine, and $AB = AC$. So the two circumradii are equal, and we only need $\sin \angle ADB$.

Work it out. Let M be the midpoint of BC. Since $AB = AC$, $AM \perp BC$, $BM = 6$ and $AM = \sqrt{10^2 - 6^2} = 8$. Point D is 4 from B, so $DM = 2$ and $AD = \sqrt{8^2 + 2^2} = \sqrt{68} = 2\sqrt{17}$. (Stewart's theorem gives the same: $AD^2 = AB^2 - BD \cdot DC = 100 - 32 = 68$.) In right triangle AMD, $\sin \angle ADB = \frac{AM}{AD} = \frac{8}{2\sqrt{17}} = \frac{4}{\sqrt{17}}$. So each circumradius is $R = \frac{AB}{2 \sin \angle ADB} = \frac{10}{8/\sqrt{17}} = \frac{5\sqrt{17}}{4}$, and the sum is $\frac{5\sqrt{17}}{2}$.

Check. Compute one radius another way: triangle ABD has sides 10, 4, $2\sqrt{17}$ and area $\frac{1}{2} \cdot 4 \cdot 8 = 16$, so $R = \frac{abc}{4K} = \frac{10 \cdot 4 \cdot 2\sqrt{17}}{64} = \frac{5\sqrt{17}}{4}$. It matches. The sum is about 10.31. Stopping after one radius gives $\frac{5\sqrt{17}}{4}$, and the circumradius of the whole triangle ABC is only $\frac{25}{4}$. If D were the midpoint, both radii would be $\frac{10}{2 \cdot 1} = 5$ for a sum of 10; moving D off the midpoint makes $\sin \angle ADB$ smaller and the radii larger, as found.

23. Answer: (C) 1491**. The idea.** Modulo 13, the number -1 is a perfect square: $5^2 = 25 \equiv -1$. So $a^2 + b^2 \equiv 0$ factors as $(b - 5a)(b + 5a) \equiv 0$. Because 13 is prime, this means $b \equiv 5a$ or $b \equiv -5a = 8a \pmod{13}$. Then count carefully, because 100 is not a multiple of 13.

Work it out. If $13 \mid a$ then $13 \mid b^2$, so $13 \mid b$. There are 7 multiples of 13 up to 100, which gives $7 \cdot 7 = 49$ pairs. Otherwise $a \not\equiv 0$, and b must be $\equiv 5a$ or $\equiv 8a$. These two residues are different and nonzero. Since $100 = 7 \cdot 13 + 9$, the residues 1 to 9 each occur 8 times among $1, \ldots, 100$, and the residues $10, 11, 12, 0$ occur 7 times. The map $r \mapsto 5r \bmod 13$ sends $1, 2, \ldots, 12$ to $5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8$. Pairing the counts, $n(r)\, n(5r)$ is 64 for $r = 1, 3, 4, 6, 7, 8, 9, 56$ for $r = 2, 5, 11, 12$, and 49 for $r = 10$. The total is $7 \cdot 64 + 4 \cdot 56 + 49 = 721$. For the case $b \equiv 8a$: since $5 \cdot 8 = 40 \equiv 1 \pmod{13}$, the condition $b \equiv 8a$ is the same as $a \equiv 5b$. So these pairs are exactly the pairs of the first case with a and b swapped, and there are also 721 of them. The two cases never overlap, because $5a \equiv 8a$ would force $13 \mid 3a$. Total: $49 + 721 + 721 = 1491$.

Check. If all 13 residues were equally common, the fraction of good pairs would be $\frac{1+2 \cdot 12}{169} = \frac{25}{169}$, and $10000 \cdot \frac{25}{169} \approx 1479$. Our exact count is close to this, and slightly higher, because the over-represented residues 1 to 9 often pair with each other. A direct computer count over all 10000 pairs also gives 1491.

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24. Answer: (E) 15**. The idea.** Only the bug's distance from S matters, not which vertex it is on. Group the vertices by distance from S into layers and write one first-step equation per layer.

Work it out. The layers have sizes $1, 5, 5, 1$: S itself, its 5 neighbours (a pentagon around S), the 5 neighbours of the opposite vertex T, and T. A vertex in layer 1 is joined to S, to 2 vertices of its own pentagon and to 2 in layer 2 (its degree is $1+2+2=5$). By symmetry, a vertex in layer 2 is joined to 2 in layer 1, 2 in layer 2 and to T. Let E_k be the expected time to reach T from layer k. Then $E_0 = 1 + E_1$, $E_1 = 1 + \frac{1}{5}E_0 + \frac{2}{5}E_1 + \frac{2}{5}E_2$, $E_2 = 1 + \frac{2}{5}E_1 + \frac{2}{5}E_2$. The last equation gives $3E_2 = 5 + 2E_1$. Substituting $E_0 = 1 + E_1$ into the middle one gives $5E_1 = 6 + E_1 + 2E_1 + 2E_2$, so $E_1 = 3 + E_2$. Then $3E_2 = 5 + 6 + 2E_2$, so $E_2 = 11$, $E_1 = 14$ and $E_0 = 15$.

Check. From layer 2 the bug hits T with probability $\frac{1}{5}$ each step. It takes 11 steps on average rather than 5, because it often drifts back to layer 1. The edge count works out too: between layers 1 and 2 there are $5 \cdot 2 = 10$ edges counted from either side, and in total $5 + 5 + 10 + 5 + 5 = 30$ edges.

25. Answer: (B) $\frac{\sqrt{21}}{3}$**. The idea.** A 60° angle at A is special: the incenter I, the orthocenter H and the circumcenter O all see BC at the same angle, 120°. So B, C, O, I, H all lie on one circle. Better still, $AO = AH$ here, so the reflection in the angle bisector AI swaps O and H. That gives $IH = IO$, and IO comes from Euler's formula.

Work it out. By the law of cosines, $BC^2 = 25 + 64 - 40 = 49$, so $BC = 7$. The circumradius is $R = \frac{BC}{2\sin 60^\circ} = \frac{7}{\sqrt{3}}$. The area is $\frac{1}{2} \cdot 5 \cdot 8 \sin 60^\circ = 10\sqrt{3}$ and the semiperimeter is 10, so the inradius is $r = \sqrt{3}$. Angles: $\angle BIC = 90^\circ + \frac{A}{2} = 120^\circ$, $\angle BHC = 180^\circ - A = 120^\circ$ and $\angle BOC = 2A = 120^\circ$. All three points lie on the same side of BC as A, so B, C, O, I, H are concyclic. Now reflect in the angle bisector AI. The rays AO and AH are mirror images, because $\angle BAH = \angle CAO = 90^\circ - B$. They also have equal lengths: $AO = R$, and $AH = 2R\cos A = R$ when $A = 60^\circ$. So the reflection in line AI swaps O and H. The point I lies on that line, so $IO = IH$. (The concyclicity explains why this is natural: the centre of the circle through B, I, C is the midpoint of arc BC, which also lies on AI.) Euler's formula gives $IO^2 = R^2 - 2Rr = \frac{49}{3} - 2 \cdot \frac{7}{\sqrt{3}} \cdot \sqrt{3} = \frac{49}{3} - 14 = \frac{7}{3}$. So $IH = IO = \sqrt{\frac{7}{3}} = \frac{\sqrt{21}}{3}$.

Check. Coordinates: $A = (0,0)$, $B = (5,0)$, $C = (4, 4\sqrt{3})$. The incenter is $\frac{7A+8B+5C}{20} = (3, \sqrt{3})$. The altitude from C is the line $x = 4$. The altitude from B is perpendicular to AC (direction $(1, \sqrt{3})$), so it is $x + \sqrt{3}y = 5$, which meets $x = 4$ at $y = \frac{1}{\sqrt{3}}$. So $H = (4, \frac{\sqrt{3}}{3})$ and $IH^2 = 1 + \left(\sqrt{3} - \frac{\sqrt{3}}{3}\right)^2 = 1 + \frac{4}{3} = \frac{7}{3}$. Both routes give $IH = \frac{\sqrt{21}}{3}$.

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