MathCastle

AMC 10

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Two real numbers x and y (not necessarily different) satisfy both $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. What is the value of $x + y$?

(A) -5 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 5

2. The number $10^6 = 1{,}000{,}000$ has 49 positive divisors. None of them is a multiple of 3, so each leaves a remainder of 1 or 2 when divided by 3. How many of the 49 divisors leave a remainder of 1 when divided by 3?

(A) 12 **(B)** 16 **(C)** 24 **(D)** 25 **(E)** 36

3. A convex polyhedron has 26 faces, each of which is either a triangle or a square. At every vertex of the polyhedron, exactly 4 faces meet. How many of the 26 faces are squares?

(A) 6 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 20

4. For each positive integer n, let L_n be the least common multiple of $1, 2, 3, \ldots, n$. For example, $L_5 = 60$ and $L_6 = 60$, but $L_7 = 420$. For how many integers n with $2 \le n \le 100$ is L_n greater than L_{n-1}?

(A) 25 **(B)** 30 **(C)** 34 **(D)** 35 **(E)** 50

5. A bag holds 4 red marbles and 4 blue marbles; all 8 marbles can be told apart. The marbles are split at random into 4 pairs, with every way of splitting the 8 marbles into 4 unordered pairs equally likely. What is the probability that every pair contains one red marble and one blue marble?

(A) $\frac{1}{16}$ **(B)** $\frac{4}{35}$ **(C)** $\frac{8}{35}$ **(D)** $\frac{16}{35}$ **(E)** $\frac{4}{7}$

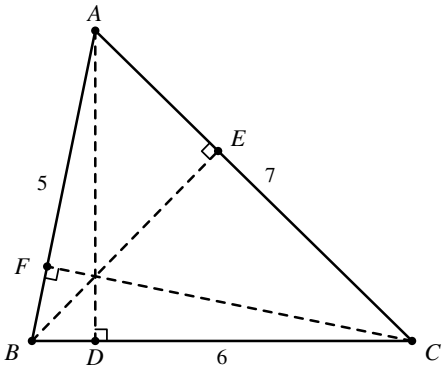
6. For a certain real number a, all three roots of the cubic equation $x^3 - 12x^2 + ax - 64 = 0$ are positive real numbers (a repeated root is counted as many times as it is repeated). What is a?

(A) 16 **(B)** 36 **(C)** 48 **(D)** 56 **(E)** 64

7. Consider the expression $n^5 - 5n^3 + 4n$. Some positive integers divide its value for every even integer n (positive, negative or zero). What is the largest positive integer d such that d divides $n^5 - 5n^3 + 4n$ for every even integer n?

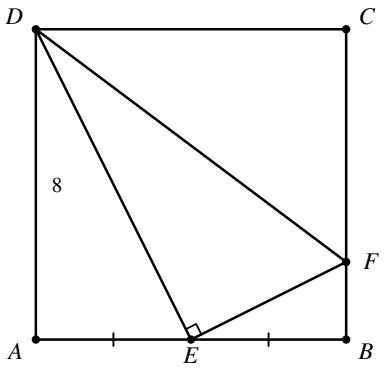
(A) 40 **(B)** 48 **(C)** 60 **(D)** 120 **(E)** 240

8. Triangle ABC has $AB = 5$, $BC = 6$ and $CA = 7$. Let D, E and F be the feet of the altitudes from A, B and C respectively. What is the perimeter of triangle DEF?



(A) $\frac{144}{35}$ **(B)** $\frac{36}{5}$ **(C)** $\frac{288}{35}$ **(D)** 9 **(E)** 18

9. Square $ABCD$ has side length 8, with its vertices labelled in order around the square. Point E is the midpoint of side $\overline{AB}$, and point F lies on side $\overline{BC}$ so that $\angle DEF = 90°$. What is the area of triangle DEF?



(A) 16 **(B)** 20 **(C)** 24 **(D)** 32 **(E)** 40

10. A regular 24-sided polygon is inscribed in a circle of radius R. How many of its diagonals have length strictly greater than R and strictly less than $R\sqrt{3}$?

(A) 36 **(B)** 72 **(C)** 96 **(D)** 120 **(E)** 144

11. How many subsets of $\{1, 2, 3, \ldots, 20\}$ have the property that no two different elements of the subset add up to a multiple of 5? The empty set and all one-element subsets have the property and should be counted.

(A) 3844 **(B)** 4805 **(C)** 5120 **(D)** 15376 **(E)** 59049

12. Ten points are marked on a circle. Four chords are to be drawn, each joining two of the marked points, so that no two of the four chords have any point in common: they do not cross each other, and they do not share an endpoint. In how many ways can the set of four chords be chosen?

(A) 630 **(B)** 1260 **(C)** 1890 **(D)** 2940 **(E)** 4725

13. Eight tennis players, ranked 1 (best) through 8, enter a knockout tournament. They are placed at random into the eight slots of a bracket, all 8! placements being equally likely. In the quarterfinals, slot 1 plays slot 2, slot 3 plays slot 4, slot 5 plays slot 6 and slot 7 plays slot 8. In the semifinals, the winner from slots 1–2 plays the winner from slots 3–4, and the winner from slots 5–6 plays the winner from slots 7–8; the two semifinal winners meet in the final. In every match the better-ranked player wins. What is the probability that the players ranked 2 and 3 play each other at some point in the tournament?

(A) $\frac{1}{7}$ **(B)** $\frac{4}{21}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{3}{7}$

14. Four of the twelve vertices of a regular dodecagon (a regular 12-sided polygon) are chosen, and they are joined in order around the dodecagon to form a convex quadrilateral. For how many of the $\binom{12}{4} = 495$ choices is the quadrilateral a trapezoid with *exactly* one pair of parallel sides?

(A) 120 **(B)** 135 **(C)** 150 **(D)** 165 **(E)** 180

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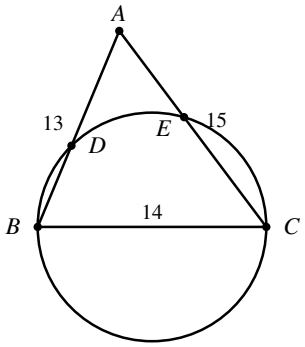
15. Ten houses stand along a straight road at the points $1, 2, 3, \ldots, 10$ (in kilometres from the start of the road). The house at point k has k residents. A bus stop is to be placed at a point x on the road so that the total walking distance of all the residents, $1 \cdot |x-1| + 2 \cdot |x-2| + 3 \cdot |x-3| + \cdots + 10 \cdot |x-10|$, is as small as possible. What is this smallest possible total?

(A) 112 **(B)** 113 **(C)** 125 **(D)** $\frac{275}{2}$ **(E)** 150

16. A circular necklace has 12 equally spaced beads, 6 black and 6 white. Two necklaces are considered the same if one can be rotated to look exactly like the other (flipping the necklace over is not allowed). How many different necklaces are there?

(A) 50 **(B)** 77 **(C)** 80 **(D)** 84 **(E)** 154

17. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. The circle with diameter $\overline{BC}$ crosses side $\overline{AB}$ at a point D other than B, and crosses side $\overline{AC}$ at a point E other than C. What is the length DE?



(A) $\frac{462}{65}$ **(B)** $\frac{15}{2}$ **(C)** $\frac{99}{13}$ **(D)** $\frac{1386}{169}$ **(E)** $\frac{56}{5}$

18. For how many integer bases $b \geq 2$ is the base-b representation of 2026 a three-digit palindrome, that is, of the form $\overline{aca}_b = a \cdot b^2 + c \cdot b + a$ with digits $1 \leq a \leq b-1$ and $0 \leq c \leq b-1$?

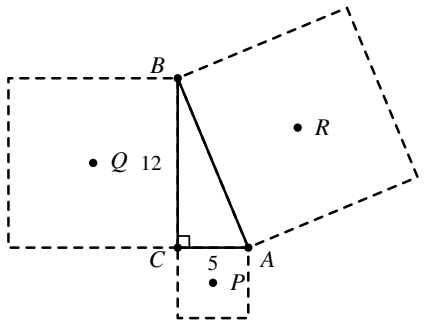
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

19. Seven children wear shirts numbered 1 through 7 and line up in a single queue, with positions numbered 1 (front) through 7 (back). The rule is that no child may stand more than two places behind the position matching their own number: the child in position i must have a shirt number of at least $i - 2$. (A child may stand any number of places ahead of their own number.) How many orderings of the seven children obey the rule?

(A) 162 **(B)** 243 **(C)** 324 **(D)** 432 **(E)** 486

20. Right triangle ABC has its right angle at C, with legs $CA = 5$ and $CB = 12$. On each of the three sides a square is drawn outward: the square has that side as one of its sides and lies on the opposite side of that side's line from the triangle. Let P, Q and R be the centres of the squares drawn on $\overline{CA}$, $\overline{CB}$ and $\overline{AB}$ respectively. What is the area of triangle PQR?

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(A) $\frac{169}{4}$ **(B)** $\frac{221}{4}$ **(C)** 60 **(D)** $\frac{289}{4}$ **(E)** $\frac{169}{2}$

21. For each positive integer n, consider the central binomial coefficient $\binom{2n}{n}$; for example $\binom{2}{1} = 2$, $\binom{4}{2} = 6$ and $\binom{6}{3} = 20$. For how many integers n with $1 \le n \le 100$ is $\binom{2n}{n}$ divisible by 8 but not by 16?

(A) 20 **(B)** 21 **(C)** 26 **(D)** 30 **(E)** 33

22. A fair six-sided die is rolled repeatedly. The rolling stops immediately after the first roll that is *not* larger than the roll just before it (the first roll never stops the game). For example, the rolls $2, 5, 6, 3$ end the game after 4 rolls, and $4, 4$ ends it after 2. What is the expected total number of rolls, counting the last one?

(A) $\left(\frac{7}{6}\right)^5$ **(B)** $\left(\frac{7}{6}\right)^6$ **(C)** $\left(\frac{7}{6}\right)^7$ **(D)** $\left(\frac{6}{5}\right)^6$ **(E)** $1 + \left(\frac{7}{6}\right)^6$

23. Square $ABCD$ has side length 4. A point P strictly inside the square is called *balanced* if the areas of the four triangles PAB, PBC, PCD and PDA are, in some order, in the ratio $1 : 2 : 3 : 4$. There are exactly eight balanced points, and they are the vertices of a convex octagon. What is the area of this octagon?

(A) $\frac{48}{25}$ **(B)** $\frac{16}{5}$ **(C)** $\frac{112}{25}$ **(D)** $\frac{128}{25}$ **(E)** $\frac{144}{25}$

24. Real numbers x, y and z satisfy $x + y + z = 0$ and $x^2 + y^2 + z^2 = 6$. What is the largest possible value of $x^5 + y^5 + z^5$?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 20 **(E)** 30

25. Five guests leave their hats in a box. At the end of the evening the hats are handed out in rounds. In each round, every guest still present draws one hat from the box, and all ways of matching the hats in the box to the guests present are equally likely. Every guest who draws their own hat keeps it and leaves; everyone else puts the hat they drew back in the box, and a new round begins with the remaining guests (whose own hats are exactly the hats in the box). This continues until everyone has left. Each time one guest takes one hat from the box counts as one draw. What is the expected total number of draws?

(A) 10 **(B)** 12 **(C)** $\frac{25}{2}$ **(D)** 15 **(E)** $\frac{35}{2}$

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Answer Key — AMC 10 — Hard Edition Practice Contest 2

1. (B) 1

2. (D) 25

3. (D) 18

4. (D) 35

5. (C) $\frac{8}{35}$

6. (C) 48

7. (E) 240

8. (C) $\frac{288}{35}$

9. (B) 20

10. (B) 72

11. (B) 4805

12. (A) 630

13. (D) $\frac{1}{3}$

14. (A) 120

15. (A) 112

16. (C) 80

17. (A) $\frac{462}{65}$

18. (A) 3

19. (E) 486

20. (D) $\frac{289}{4}$

21. (E) 33

22. (B) $\left(\frac{7}{6}\right)^6$

23. (C) $\frac{112}{25}$

24. (E) 30

25. (E) $\frac{35}{2}$

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Solutions — AMC 10 — Hard Edition Practice Contest 2

1. Answer: (B) 1**. The idea.** Both equations are symmetric, so rewrite them in terms of $s = x + y$ and $p = xy$. That gives a cubic in s with three roots, and the word *real* decides which one survives: real x, y need $s^2 \geq 4p$.

Work it out. From $x^2 + y^2 = s^2 - 2p = 7$ we get $p = \frac{s^2-7}{2}$. Then $x^3 + y^3 = s^3 - 3ps = s^3 - \frac{3s(s^2-7)}{2} = 10$. Multiply by 2: $2s^3 - 3s^3 + 21s = 20$, i.e. $s^3 - 21s + 20 = 0$, which factors as $(s-1)(s-4)(s+5) = 0$.

Now x and y are the roots of $t^2 - st + p = 0$, which are real exactly when $s^2 - 4p \geq 0$, i.e. $s^2 - 2(s^2 - 7) \geq 0$, i.e. $s^2 \leq 14$. That rules out $s = 4$ ($16 > 14$) and $s = -5$ ($25 > 14$). So $x + y = 1$.

Check. With $s = 1$, $p = -3$: $x, y = \frac{1\pm\sqrt{13}}{2}$, both real. Then $x^2 + y^2 = 1 + 6 = 7$ and $x^3 + y^3 = 1 - 3(-3)(1) = 10$. With $s = 4$ we would need $p = \frac{9}{2}$, and $t^2 - 4t + \frac{9}{2}$ has discriminant $16 - 18 < 0$: those x, y are not real. Picking 4 (the only positive root besides 1), the rejected root -5, or its size 5 are the traps.

2. Answer: (D) 25**. The idea.** Every divisor has the form $2^a 5^b$ with $0 \leq a, b \leq 6$. Both 2 and 5 are 1 less than a multiple of 3, so modulo 3 each of them behaves like -1, and the remainder of $2^a 5^b$ depends only on whether $a + b$ is even or odd.

Work it out. Modulo 3, $2^a 5^b \equiv (-1)^a (-1)^b = (-1)^{a+b}$. So the remainder is 1 exactly when $a + b$ is even, that is, when a and b are both even or both odd. The exponents $0, 1, \ldots, 6$ include 4 even values $(0, 2, 4, 6)$ and 3 odd values $(1, 3, 5)$. Both even: $4 \cdot 4 = 16$ divisors; both odd: $3 \cdot 3 = 9$. Total $16 + 9 = 25$.

Check. The other $49 - 25 = 24$ divisors have $a + b$ odd: $4 \cdot 3 + 3 \cdot 4 = 24$. Small cases agree: $1 = 2^0 5^0$, 4, 10, 16, 25, 40 all leave remainder 1, while 2, 5, 8, 20, 50 leave remainder 2. Counting only the pairs with both exponents even gives the trap answer 16.

3. Answer: (D) 18**. The idea.** Count edges and vertices from the faces. Every edge borders two faces, and, since 4 faces meet at each vertex, 4 edges end at each vertex. Then Euler's formula $V - E + F = 2$ supplies the missing equation.

Work it out. Let there be t triangles and s squares, so $t + s = 26$. The faces have $3t + 4s$ sides in total, and each edge is counted twice: $E = \frac{3t+4s}{2}$. Each vertex is the end of exactly 4 edges, and each edge has 2 ends, so $4V = 2E$ and $V = \frac{3t+4s}{4}$. Euler: $\frac{3t+4s}{4} - \frac{3t+4s}{2} + 26 = 2$, so $\frac{3t+4s}{4} = 24$ and $3t + 4s = 96$. With $t + s = 26$: $3 \cdot 26 + s = 96$, so $s = 18$ and $t = 8$.

Check. Then $V = 24$ and $E = 48$, and $24 - 48 + 26 = 2$. Such a solid really exists: the rhombicuboctahedron, a cube whose 6 faces, 12 edges and 8 corners are replaced by $6 + 12 = 18$ squares and 8 triangles. The answer 8 counts the triangles instead.

4. Answer: (D) 35**. The idea.** L_n is bigger than L_{n-1} exactly when n does not already divide L_{n-1}. That happens precisely when n is a power of a single prime, so the question is to count prime powers up to 100.

Work it out. Since $L_n = \operatorname{lcm}(L_{n-1}, n)$, we have $L_n > L_{n-1}$ exactly when $n \nmid L_{n-1}$.

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If n has two different prime factors, write $n = uv$ with $u, v > 1$ and $\gcd(u, v) = 1$ (split off one prime's full power). Then $u, v < n$, so both divide L_{n-1}, and since they are coprime, $n = uv$ divides L_{n-1}: no growth.

If $n = p^k$ is a prime power, every number below n has fewer than k factors p (a multiple of p^k is at least $p^k = n$), so $p^k \nmid L_{n-1}$: L_n grows (by a factor p).

Prime powers from 2 to 100: the 25 primes; squares 4, 9, 25, 49; cubes 8, 27; fourth powers 16, 81; fifth power 32; sixth power 64. Total $25 + 4 + 2 + 2 + 1 + 1 = 35$.

Check. From $n = 2$ to 10: L grows at 2, 3, 4, 5, 7, 8, 9 (values 2, 6, 12, 60, 420, 840, 2520) and not at 6 or 10: seven prime powers, as predicted. Counting only the primes gives the trap answer 25, and forgetting one of the higher powers (typically $64 = 2^6$) gives 34.

5. Answer: (C) $\frac{8}{35}$**. The idea.** Count the pairings directly. A mixed pairing is the same thing as a matching of the 4 red marbles with the 4 blue marbles, and all pairings of 8 objects can be counted by repeatedly pairing off the lowest-labelled unpaired marble.

Work it out. Total pairings: the first marble has 7 possible partners, the next unpaired marble has 5, then 3, then 1, for $7 \cdot 5 \cdot 3 \cdot 1 = 105$. Mixed pairings: give each red marble a different blue partner, $4! = 24$ ways. The probability is $\frac{24}{105} = \frac{8}{35}$.

Check. Build the pairing one red marble at a time. The first red marble's partner is one of 7 marbles, 4 of them blue: $\frac{4}{7}$. Then the next unpaired red marble has 5 possible partners, 3 blue: $\frac{3}{5}$. Then $\frac{2}{3}$, and the last pair is forced. $\frac{4}{7} \cdot \frac{3}{5} \cdot \frac{2}{3} = \frac{24}{105} = \frac{8}{35}$. (Treating each pair as an independent fair coin, $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, is a common mistake.)

6. Answer: (C) 48**. The idea.** Vieta's formulas give the sum and the product of the roots, and these two numbers are exactly as far apart as the AM–GM inequality allows: the arithmetic mean equals the geometric mean. That forces all three roots to be equal.

Work it out. Let the roots be $r, s, t > 0$. By Vieta, $r + s + t = 12$ and $rst = 64$, and $a = rs + st + tr$. The arithmetic mean of the roots is $\frac{12}{3} = 4$ and their geometric mean is $\sqrt[3]{64} = 4$. For positive numbers the arithmetic mean is at least the geometric mean, with equality only when all the numbers are equal. So $r = s = t = 4$, and $a = 3 \cdot 4 \cdot 4 = 48$.

Check. $(x - 4)^3 = x^3 - 12x^2 + 48x - 64$, which has exactly the required form, with the triple root 4. Guessing unequal roots cannot work: 2, 4, 8 have product 64 but sum 14, and the value $a = 2 \cdot 4 + 4 \cdot 8 + 8 \cdot 2 = 56$ they would give is a trap. Some negative values of a (such as $a = -60$) do give three real roots, but then two of them are negative, since $rs + st + tr = a < 0$ is impossible for positive roots.

7. Answer: (E) 240**. The idea.** Factor completely: $n^5 - 5n^3 + 4n = n(n^2 - 1)(n^2 - 4) = (n - 2)(n - 1)n(n + 1)(n + 2)$, a product of five consecutive integers, so $120 = 5!$ always divides it. For even n three of the five are even, which buys extra factors of 2.

Work it out. Write $n = 2m$. Then the product is $(2m - 2)(2m)(2m + 2) \cdot (2m - 1)(2m + 1) = 8 \cdot (m - 1)m(m + 1) \cdot (2m - 1)(2m + 1)$. Three consecutive integers $(m - 1)m(m + 1)$ are divisible by 6, so 48 divides the product; five consecutive integers include a multiple of 5, so 5 divides it too. Hence 240 always divides it. It cannot be more: $n = 4$ gives $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720 = 240 \cdot 3$ and $n = 6$ gives $4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 6720 = 240 \cdot 28$, and $\gcd(3, 28) = 1$, so no common divisor exceeds 240.

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Check. $n = 8$: $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 30240 = 240 \cdot 126$. $n = 10$: $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 = 95040 = 240 \cdot 396$. The answer 120 is right for all integers n (for $n = 3$ the value is exactly 120) but misses the extra factor 2 that even inputs force.

8. Answer: (C) $\frac{288}{35}$**. The idea.** Triangle AEF is similar to triangle ABC, scaled by $\cos A$: in right triangle ABE, $AE = AB\cos A$, and in right triangle ACF, $AF = AC\cos A$. So $EF = BC\cos A$, and likewise each side of DEF is the opposite side of ABC times the cosine of the angle it cuts off.

Work it out. By the law of cosines, $\cos A = \frac{7^2+5^2-6^2}{2\cdot 7\cdot 5} = \frac{38}{70} = \frac{19}{35}$, $\cos B = \frac{5^2+6^2-7^2}{2\cdot 5\cdot 6} = \frac{12}{60} = \frac{1}{5}$, $\cos C = \frac{6^2+7^2-5^2}{2\cdot 6\cdot 7} = \frac{60}{84} = \frac{5}{7}$. All are positive, so the triangle is acute and the feet lie on the sides. Then $EF = 6 \cdot \frac{19}{35} = \frac{114}{35}$, $FD = 7 \cdot \frac{1}{5} = \frac{49}{35}$, $DE = 5 \cdot \frac{5}{7} = \frac{125}{35}$, and the perimeter is $\frac{114+49+125}{35} = \frac{288}{35}$.

Check. Coordinates: $B = (0,0)$, $C = (6,0)$, $A = (1, 2\sqrt{6})$ gives $AB = 5$, $AC = 7$. The foot from A is $D = (1,0)$, and F, E come out to give the same three lengths. Also, the perimeter of the orthic triangle equals $\frac{2K}{R}$: here $K = 6\sqrt{6}$ (Heron) and $R = \frac{abc}{4K} = \frac{35}{4\sqrt{6}}$, so $\frac{2K}{R} = \frac{12\sqrt{6}\cdot 4\sqrt{6}}{35} = \frac{288}{35}$. The midpoint triangle has perimeter 9, a tempting wrong guess.

9. Answer: (B) 20**. The idea.** A right angle at E sitting on the side $\overline{AB}$ creates two similar right triangles, one on each side of E: the angles $\angle AED$ and $\angle BEF$ add up to 90°, so $\triangle DAE \sim \triangle EBF$. That fixes BF, and then the two legs DE and EF of the right triangle DEF give its area directly.

Work it out. Since $\angle AED + \angle DEF + \angle FEB = 180°$ and $\angle DEF = 90°$, we get $\angle AED + \angle FEB = 90°$. In right triangle DAE, $\angle AED + \angle ADE = 90°$, so $\angle ADE = \angle FEB$. Hence $\triangle DAE \sim \triangle EBF$ (right angles at A and B), with $\frac{BF}{BE} = \frac{AE}{AD}$, i.e. $BF = \frac{4\cdot 4}{8} = 2$.

Now $DE = \sqrt{8^2+4^2} = \sqrt{80}$ and $EF = \sqrt{4^2+2^2} = \sqrt{20}$. Triangle DEF has its right angle at E, so its area is $\frac{1}{2} \cdot \sqrt{80} \cdot \sqrt{20} = \frac{1}{2} \cdot \sqrt{1600} = \frac{1}{2} \cdot 40 = 20$.

Check. Subtract the three corner triangles from the square: $[DAE] = \frac{1}{2} \cdot 8 \cdot 4 = 16$, $[EBF] = \frac{1}{2} \cdot 4 \cdot 2 = 4$, $[FCD] = \frac{1}{2} \cdot 6 \cdot 8 = 24$. Then $64 - 16 - 4 - 24 = 20$. Forgetting the $\frac{1}{2}$ gives 40; taking $BF = 4$ (copying AE) would not make a right angle at all.

10. Answer: (B) 72**. The idea.** A chord's length depends only on the central angle it cuts off, and the two benchmark lengths are themselves chords: R is the chord of a 60° arc (it forms an equilateral triangle with the centre) and $R\sqrt{3}$ is the chord of a 120° arc (the side of the inscribed equilateral triangle). Since chord length grows with the central angle from 0° to 180°, the condition is just $60° < \theta < 120°$, where θ is the smaller central angle.

Work it out. Adjacent vertices are $\frac{360°}{24} = 15°$ apart, so a segment joining vertices k steps apart ($1 \le k \le 12$, measured the short way) has central angle $15k°$. We need $60 < 15k < 120$, i.e. $4 < k < 8$, so $k = 5, 6, 7$. For each $k < 12$ there are exactly 24 such segments (one starting at each vertex and going k steps clockwise; going counterclockwise gives the same segments again). So the count is $3 \cdot 24 = 72$.

Check. The endpoints really are excluded: $k = 4$ gives $2R\sin 30° = R$ exactly and $k = 8$ gives $2R\sin 60° = R\sqrt{3}$ exactly. Including one endpoint gives 96, both gives 120. Counting 6 qualifying partners from each of 24 vertices without halving gives 144; halving the per-k count (as if it were a diameter, $k = 12$, which gives only 12) gives 36. As a sanity check on the totals, $k = 2, \ldots, 11$ give 24 each and $k = 12$ gives 12, for $252 = \frac{24\cdot 21}{2}$ diagonals in all.

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11. Answer: (B) 4805**. The idea.** Whether two numbers add up to a multiple of 5 depends only on their remainders mod 5. Sort the twenty numbers into five remainder classes of four numbers each and decide class by class.

Work it out. Two different elements add to a multiple of 5 exactly when their remainders are 0 and 0, 1 and 4, or 2 and 3. Two numbers with remainder 1 are fine together (sum $\equiv 2$), and similarly within the classes 2, 3 and 4.

Class 0 $(5, 10, 15, 20)$: at most one of them, $1 + 4 = 5$ ways.

Classes 1 and 4: we may use elements from at most one of the two classes. Either take none from both (1 way), or a nonempty set from class 1 only ($2^4 - 1 = 15$ ways), or a nonempty set from class 4 only (15 ways): 31 ways.

Classes 2 and 3: likewise 31 ways.

The three decisions are independent: $5 \cdot 31 \cdot 31 = 4805$.

Check. Counting the classes 1/4 as "$2^4 + 2^4 = 32$" counts the empty choice twice and gives $5 \cdot 32^2 = 5120$; forgetting that two multiples of 5 also clash gives $16 \cdot 31^2 = 15376$. A smaller case by hand: in $\{1, \ldots, 5\}$ the same method gives $2 \cdot 3 \cdot 3 = 18$, and listing confirms it (pick at most one of each of $\{1, 4\}$, $\{2, 3\}$, and include 5 or not).

12. Answer: (A) 630**. The idea.** First choose which 8 of the 10 points are used as endpoints. Whether chords cross depends only on the order of their endpoints around the circle, so for any 8 chosen points the number of ways to pair them up with no crossings is the same number M_8. Find M_8 by looking at the partner of one fixed point.

Work it out. Let M_{2k} be the number of ways to join $2k$ points on a circle by k non-crossing chords. Fix one point P. Its chord splits the remaining points into the two arcs on either side, no chord may cross it, so each arc must be paired up on its own and must contain an even number of points. This gives $M_0 = 1$, $M_2 = 1$, $M_4 = M_0M_2 + M_2M_0 = 2$, $M_6 = M_0M_4 + M_2M_2 + M_4M_0 = 5$ and $M_8 = M_0M_6 + M_2M_4 + M_4M_2 + M_6M_0 = 5 + 2 + 2 + 5 = 14$. The endpoints can be chosen in $\binom{10}{8} = 45$ ways, so the answer is $45 \cdot 14 = 630$.

Check. Ignoring the crossing condition, 8 points can be paired in $7 \cdot 5 \cdot 3 \cdot 1 = 105$ ways, giving $45 \cdot 105 = 4725$: far too many, since most of those pairings have crossings. With only 6 points and 3 chords the method gives $\binom{6}{6} \cdot 5 = 5$, and the five pictures are easy to draw for a regular hexagon: the two pairings that use every other side, and the three that use one long diagonal together with the two sides parallel to it.

13. Answer: (D) $\frac{1}{3}$**. The idea.** Put player 2 in a slot first and ask where player 3 lands. Player 2 loses only to player 1, so the only way the pair can miss each other is for player 1 to knock one of them out first, or for them never to be scheduled to meet. The surprising part: if 2 and 3 are in different halves of the bracket they can only meet in the final, and that is impossible, since player 1 is in one of those halves and wins it.

Work it out. Fix player 2's slot. Player 3 is equally likely to be in any of the other 7 slots.

Case 1: player 3 is player 2's quarterfinal opponent (probability $\frac{1}{7}$). They certainly meet.

Case 2: player 3 is in the same half but in the other quarterfinal (probability $\frac{2}{7}$). They meet in the semifinal if both win their quarterfinals. The only player who can beat either of them is player 1, so this fails exactly when player 1 is the quarterfinal opponent of 2 or of 3. Player 1 sits in one of the 6 remaining slots, 2 of which are bad, so this case contributes $\frac{2}{7} \cdot \frac{4}{6} = \frac{4}{21}$.

Case 3: player 3 is in the other half (probability $\frac{4}{7}$). Now they could only meet in the final, which requires each to win their half. But player 1 is in one of the two halves and wins it, so this never happens.

Total: $\frac{1}{7} + \frac{4}{21} = \frac{3}{21} + \frac{4}{21} = \frac{1}{3}$.

Check. Count placements: fix player 2 in slot 1. The $7 \cdot 6 = 42$ equally likely positions of players 3 and 1 give meetings for 6 pairs with 3 in slot 2 (any spot for 1), plus $2 \cdot 4 = 8$ pairs with 3 in slot 3 or 4 and 1 in slots 5–8: $14/42 = 1/3$. Ignoring player 1 in Case 2 gives the tempting $\frac{1}{7} + \frac{2}{7} = \frac{3}{7}$, and forgetting Case 1 leaves only $\frac{4}{21}$.

14. Answer: (A) 120**. The idea.** Number the vertices $0, 1, \ldots, 11$ around the dodecagon. The chord from i to j is perpendicular to the direction of angle $\frac{(i+j)\cdot 30^\circ}{2}$ (the bisector of the arc between them), so two chords are parallel exactly when their index sums $i + j$ agree mod 12. That sorts all 66 chords into 12 direction classes, and a quadrilateral has a pair of parallel opposite sides exactly when it is built from two chords of the same class. Two chords of one class never share a vertex (the partner of a vertex is forced) and never cross (parallel chords cannot cross), so any two of them are opposite sides of one of our quadrilaterals.

Work it out. For a class with sum s, the vertex i is paired with $s - i$. If s is odd no vertex is paired with itself, so the class has 6 chords; if s is even, the two vertices with $2i \equiv s$ are left out, leaving 5 chords. Pairs of parallel chords: $6\binom{6}{2} + 6\binom{5}{2} = 6 \cdot 15 + 6 \cdot 10 = 150$.

Each quadrilateral with exactly one parallel pair is counted once, but a quadrilateral with two parallel pairs, a parallelogram, is counted twice. A parallelogram inscribed in a circle is a rectangle, and its diagonals are diameters; there are 6 diameters, so $\binom{6}{2} = 15$ rectangles. Quadrilaterals with exactly one pair: $150 - 2 \cdot 15 = 120$.

Check. Subtracting the rectangles only once gives 135, the number of quadrilaterals with *at least* one parallel pair, not exactly one; forgetting to subtract them at all gives 150. A direct computer check of all 495 choices, testing each pair of opposite sides for parallelism with coordinates, also finds 120.

15. Answer: (A) 112**. The idea.** Think of the 55 residents as individuals. Moving the stop a little to the right shortens the walk of everyone to its right and lengthens the walk of everyone to its left, so the total goes down as long as more residents live to the right than to the left. The best spot is therefore a median of the 55 residents, not the midpoint of the houses.

Work it out. Between houses m and $m + 1$, the slope of the total is (residents to the left) − (residents to the right) $= (1 + \cdots + m) - (55 - (1 + \cdots + m))$. Since $1 + \cdots + 6 = 21 < 27.5$ and $1 + \cdots + 7 = 28 > 27.5$, the slope is $21 - 34 = -13$ just left of 7 and $28 - 27 = +1$ just right of 7. So the minimum is at $x = 7$.

The value there: houses 1 to 6 contribute $1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1 = 56$, house 7 contributes 0, and houses 8, 9, 10 contribute $8 \cdot 1 + 9 \cdot 2 + 10 \cdot 3 = 56$. Total 112.

Check. At $x = 8$ the total is $7 + 12 + 15 + 16 + 15 + 12 + 7 + 0 + 9 + 20 = 113 = 112 + 1$, matching the slope +1; at $x = 6$ it is $125 = 112 + 13$, matching the slope −13. The unweighted median $x = 5.5$ gives 137.5, far worse.

16. Answer: (C) 80**. The idea.** Dividing the $\binom{12}{6} = 924$ colourings of 12 fixed positions by the 12 rotations gives 77, which is wrong: symmetric necklaces (such as alternating black and white)

have fewer than 12 distinct rotations, so they are undercounted. The fix is to count pairs (rotation, colouring left unchanged by it) and divide by 12 (Burnside's lemma): every necklace contributes exactly 12 such pairs.

Work it out. A rotation by d positions splits the 12 positions into $g = \gcd(d, 12)$ cycles of length $12/g$, and a colouring is unchanged exactly when each cycle is one colour. With 6 black beads, this needs the cycle length $12/g$ to divide 6.

Identity ($g = 12$): $\binom{12}{6} = 924$.

Half-turn ($d = 6$, $g = 6$, cycles of length 2): pick 3 of 6 cycles to be black: $\binom{6}{3} = 20$.

Rotations by 4 and 8 ($g = 4$, cycles of length 3): $\binom{4}{2} = 6$ each, 12 in all.

Rotations by 2 and 10 ($g = 2$, cycles of length 6): $\binom{2}{1} = 2$ each, 4 in all.

All other rotations have cycle lengths 4 or 12, which do not divide 6: they fix nothing.

Total $924 + 20 + 12 + 4 = 960$, and $960/12 = 80$.

Check. The same method with 4 beads of each colour on 8 positions gives $(70+6+2+2)/8 = 10$, and listing those by hand confirms 10. Here $80 - 77 = 3$ comes precisely from the symmetric necklaces. (If flips were also allowed the count would drop to 50.)

17. Answer: (A) $\frac{462}{65}$. The idea. An angle in a semicircle is a right angle, so $\angle BDC = \angle BEC = 90°$: D and E are the feet of the altitudes from C and B. And since B, C, E, D lie on one circle, triangle ADE is a scaled copy of triangle ACB (flipped), so DE is just BC times a scale factor.

Work it out. Power of the point A with respect to the circle gives $AD \cdot AB = AE \cdot AC$, i.e. $\frac{AD}{AC} = \frac{AE}{AB}$; with the shared angle at A, triangles ADE and ACB are similar with ratio $k = \frac{AE}{AB}$, and $DE = k \cdot BC$.

Find AE from the two right triangles sharing the altitude BE: $BE^2 = 13^2 - AE^2 = 14^2 - (15 - AE)^2$. Expanding, $169 - AE^2 = 196 - 225 + 30AE - AE^2$, so $30AE = 198$ and $AE = \frac{33}{5}$.

So $k = \frac{33/5}{13} = \frac{33}{65}$ and $DE = \frac{33}{65} \cdot 14 = \frac{462}{65}$.

Check. Similarly, from the altitude CD: $15^2 - AD^2 = 14^2 - (13 - AD)^2$ gives $26AD = 198$, $AD = \frac{99}{13}$, and indeed $\frac{AD}{AC} = \frac{99}{13 \cdot 15} = \frac{33}{65}$, the same ratio. Coordinates $B = (0, 0)$, $C = (14, 0)$, $A = (5, 12)$ give $D = \left(\frac{350}{169}, \frac{840}{169}\right)$, $E = \left(\frac{224}{25}, \frac{168}{25}\right)$ and the same distance. Note $DE \neq \frac{1}{2}BC = 7$: DE is not a midline.

18. Answer: (A) 3. The idea. Three digits pins b into a short range, and the palindrome condition says "first digit = last digit". The first digit is $\lfloor 2026/b^2 \rfloor$, which is constant on short runs of b, and the last digit is $2026 \bmod b$. So for each possible first digit a we only need b in a small range with $b \mid 2026 - a$.

Work it out. Three digits means $b^2 \leq 2026 < b^3$, i.e. $13 \leq b \leq 45$. Group by $a = \lfloor 2026/b^2 \rfloor$ and test $b \mid 2026 - a$:

$a = 1$: $b = 32..45$, need $b \mid 2025 = 3^4 \cdot 5^2$: $b = 45$ works.

$a = 2$: $b = 27..31$, $2024 = 2^3 \cdot 11 \cdot 23$: none. $a = 3$: $b = 23..25$, $2023 = 7 \cdot 17^2$: none. $a = 4$: $b = 21, 22$, $2022 = 2 \cdot 3 \cdot 337$: none. $a = 5$: $b = 19, 20$, $2021 = 43 \cdot 47$: none. $a = 6$: $b = 18$, 2020: no. $a = 7$: $b = 16, 17$, $2019 = 3 \cdot 673$: none. $a = 8$: no b (no square lies in $(225, 253]$). $a = 9$: $b = 15$, 2017 is prime: no.

$a = 10$: $b = 14$, $2016 = 14 \cdot 144$: works. $a = 11$: $b = 13$, $2015 = 13 \cdot 155$: works.

So $b = 13, 14, 45$: three bases.

Check. $2026 = 11 \cdot 169 + 12 \cdot 13 + 11 = \overline{11\,12\,11}_{13}$, $2026 = 10 \cdot 196 + 4 \cdot 14 + 10 = \overline{10\,4\,10}_{14}$, $2026 = 1 \cdot 2025 + 0 \cdot 45 + 1 = \overline{101}_{45}$. (Counting the two-digit palindromes $\overline{11}_{2025}$ and $\overline{22}_{1012}$ too would give 5, but those are not three-digit.)

19. Answer: (E) 486**. The idea.** Filling the queue from the front is a mess, because early choices change what is left later in complicated ways. Fill it from the *back*: the back positions have the strictest requirements, and each time we move one place forward the pool of allowed numbers grows by exactly one while exactly one number has been used, so the number of choices stays constant.

Work it out. Position 7 needs a number ≥ 5: choose from $\{5, 6, 7\}$, 3 ways.

Position 6 needs a number ≥ 4: the pool $\{4, 5, 6, 7\}$ has 4 numbers, one already used (it was $\geq 5 \geq 4$), so 3 ways.

Position 5 needs ≥ 3: 5 numbers, 2 used, 3 ways. Position 4 needs ≥ 2: 6 numbers, 3 used, 3 ways. Position 3 needs ≥ 1: all 7 numbers, 4 used, 3 ways.

Positions 2 and 1 have no restriction beyond what is left: 2 ways, then 1.

Total $3^5 \cdot 2 = 486$. The key point is that every number used further back is automatically inside each later (larger) pool, so the counts never depend on earlier choices.

Check. With 4 children the same argument gives $3 \cdot 3 \cdot 2 \cdot 1 = 18$; listing directly, the only bad orderings put child 1 in position 4, and there are $3! = 6$ of those, leaving $24 - 6 = 18$. Forgetting the 2 choices left for position 2 gives $3^5 = 243$.

20. Answer: (D) $\frac{289}{4}$**. The idea.** The centre of a square lies on the diagonals, which make 45° with the sides. So $\angle PCA = 45°$ and $\angle QCB = 45°$, with P and Q outside the triangle, and together with the right angle at C these make $45° + 90° + 45° = 180°$: the points P, C, Q are collinear. Also $\angle ACB = \angle ARB = 90°$ with C, R on opposite sides of AB, so A, C, B, R lie on the circle with diameter AB; since $RA = RB$, R is the midpoint of arc AB and CR bisects the right angle at C. A line at 45° to both legs inside the angle is perpendicular to line PQ, so CR is the height of triangle PQR on base PQ.

Work it out. The centre of a square is half a diagonal from each corner: $CP = \frac{5}{\sqrt{2}}$ and $CQ = \frac{12}{\sqrt{2}}$, so $PQ = \frac{17}{\sqrt{2}}$. For CR, use Ptolemy's theorem in the cyclic quadrilateral $ACBR$: $CR \cdot AB = CA \cdot RB + CB \cdot RA$, with $AB = 13$ and $RA = RB = \frac{13}{\sqrt{2}}$. So $13\,CR = (5 + 12) \cdot \frac{13}{\sqrt{2}}$ and $CR = \frac{17}{\sqrt{2}}$. The area is $\frac{1}{2} \cdot PQ \cdot CR = \frac{1}{2} \cdot \frac{17}{\sqrt{2}} \cdot \frac{17}{\sqrt{2}} = \frac{289}{4}$.

Check. Coordinates $C = (0, 0)$, $A = (5, 0)$, $B = (0, 12)$ give $P = \left(\frac{5}{2}, -\frac{5}{2}\right)$, $Q = (-6, 6)$ and $R = \left(\frac{17}{2}, \frac{17}{2}\right)$ (the midpoint $\left(\frac{5}{2}, 6\right)$ of AB moved $\frac{13}{2}$ away from C in the direction $\frac{1}{13}(12, 5)$). The shoelace formula gives $\frac{1}{2}\left|\frac{5}{2}(6 - \frac{17}{2}) - 6(\frac{17}{2} + \frac{5}{2}) + \frac{17}{2}(-\frac{5}{2} - 6)\right| = \frac{1}{2} \cdot \frac{289}{2} = \frac{289}{4}$. In general the area is $\frac{(a+b)^2}{4}$ for legs a, b. Taking $CR = \frac{13}{\sqrt{2}}$ (the distance from R to A, not to C) gives $\frac{221}{4}$, and using the hypotenuse 13 in place of $5 + 12$ for both lengths gives $\frac{169}{4}$.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

21. Answer: (E) 33**. The idea.** Count the factors of 2 with Legendre's formula in its binary form: the exponent of 2 in $m!$ is $m - s(m)$, where $s(m)$ is the number of 1s in the binary expansion of m. Doubling a number just appends a 0 in binary, so $s(2n) = s(n)$, and the exponent of 2 in $\binom{2n}{n}$ collapses to exactly $s(n)$.

Work it out. $v_2\binom{2n}{n} = v_2((2n)!) - 2v_2(n!) = (2n - s(2n)) - 2(n - s(n)) = 2s(n) - s(2n) = s(n)$.

So we need $s(n) = 3$: n has exactly three 1s in binary. Every $n \le 100$ is below $128 = 2^7$, so it has at most 7 binary digits; there are $\binom{7}{3} = 35$ such numbers below 128. Remove those above $100 = 1100100_2$: such a number needs 7 binary digits starting 11 (anything starting 10 is at most $1011000_2 = 88$), and the five with three 1s that start 11 are $97, 98, 100, 104, 112$. Only 104 and 112 exceed 100. Answer $35 - 2 = 33$.

Check. $n = 7 = 111_2$: $\binom{14}{7} = 3432 = 8 \cdot 429$ with 429 odd. $n = 3 = 11_2$: $\binom{6}{3} = 20 = 4 \cdot 5$, two factors of 2, as $s(3) = 2$ predicts. $n = 8 = 1000_2$: $\binom{16}{8} = 12870 = 2 \cdot 6435$, one factor.

22. Answer: (B) $\left(\frac{7}{6}\right)^6$**. The idea.** For a count N of rolls, $E[N] = P(N > 0) + P(N > 1) + P(N > 2) + \cdots$ (each roll is counted once for every k below it). And $N > k$ says exactly that the first k rolls are strictly increasing, which is easy: a strictly increasing run of length k is just a k-element subset of $\{1, \ldots, 6\}$ written in order.

Work it out. $P(N > k) = P(\text{first } k \text{ rolls strictly increase}) = \dfrac{\binom{6}{k}}{6^k}$ for $0 \le k \le 6$, and 0 for $k \ge 7$ (seven rolls cannot strictly increase). Therefore

$E[N] = \sum_{k=0}^{6} \binom{6}{k} \left(\frac{1}{6}\right)^k = \left(1 + \frac{1}{6}\right)^6 = \left(\frac{7}{6}\right)^6$ by the binomial theorem. That is $\frac{117649}{46656} \approx 2.52$.

Check. The terms $k = 0$ and $k = 1$ are both 1, correctly saying that the game always lasts at least 2 rolls. With a 2-sided "die" the formula gives $(3/2)^2 = 9/4$; directly, the game lasts 2 rolls unless the rolls are $1, 2$ (probability $\frac{1}{4}$), in which case it lasts 3: $2 + \frac{1}{4} = \frac{9}{4}$. Adding 1 for "the stopping roll" double-counts it and gives the distractor $1 + (7/6)^6$.

23. Answer: (C) $\frac{112}{25}$**. The idea.** Triangles on opposite sides of the square always share the square's area evenly: $[PAB] + [PCD] = \frac{1}{2} \cdot 4 \cdot (\text{distance from } AB \text{ to } CD) = 8$, and likewise $[PBC] + [PDA] = 8$. So the four areas must split into two opposite pairs with equal sums, which pins down which areas can face each other, and each area tells us a distance from P to a side.

Work it out. The four areas add to 16 and are in ratio $1 : 2 : 3 : 4$, so they are $1.6,\ 3.2,\ 4.8,\ 6.4$. The only ways to split them into two pairs each summing to 8 are $\{1.6, 6.4\}$ and $\{3.2, 4.8\}$. A triangle with base 4 and area K has height $K/2$, so in one direction P is at distances 0.8 and 3.2 from the two opposite sides, and in the other direction at distances 1.6 and 2.4.

Put $A = (0, 0)$, $B = (4, 0)$, $C = (4, 4)$, $D = (0, 4)$. Then either $x \in \{0.8, 3.2\}$ and $y \in \{1.6, 2.4\}$, or $x \in \{1.6, 2.4\}$ and $y \in \{0.8, 3.2\}$: eight points. They all lie in the square $0.8 \le x, y \le 3.2$ of side 2.4, two on each side of it, and the octagon is that square with its four corners cut off. Each cut corner, e.g. at $(0.8, 0.8)$ with neighbours $(0.8, 1.6)$ and $(1.6, 0.8)$, is a right triangle with legs 0.8 and area 0.32.

$$\text{Area} = 2.4^2 - 4(0.32) = 5.76 - 1.28 = 4.48 = \frac{112}{25}.$$

Check. Take $P = (0.8, 1.6)$: $[PAB] = 2(1.6) = 3.2$, $[PBC] = 2(3.2) = 6.4$, $[PCD] = 2(2.4) = 4.8$, $[PDA] = 2(0.8) = 1.6$, in ratio $2 : 4 : 3 : 1$. The octagon's sides alternate between length 0.8 (on the square's sides) and $0.8\sqrt{2}$ (the cuts), so it is not regular, but it is convex. Using only one

family of four points gives a 2.4×0.8 rectangle, area $\frac{48}{25}$; forgetting to halve the corner triangles gives $\frac{16}{5}$; the uncut square is $\frac{144}{25}$.

24. Answer: (E) 30**. The idea.** The two conditions fix two of the three elementary symmetric functions, so x, y, z are the roots of a cubic with a single free coefficient $q = xyz$. Power sums of the roots are then polynomials in q (here $x^5 + y^5 + z^5$ turns out to be $15q$), and the real question becomes: how large can q be while all three roots stay *real*?

Work it out. $xy + yz + zx = \frac{(x+y+z)^2-(x^2+y^2+z^2)}{2} = -3$, so x, y, z are the roots of $t^3 - 3t - q$ with $q = xyz$. Each root satisfies $t^3 = 3t + q$; multiplying by t^{k-3} and summing over the roots gives $p_k = 3p_{k-2} + q\,p_{k-3}$ for the power sums $p_k = x^k + y^k + z^k$. From $p_0 = 3$, $p_1 = 0$, $p_2 = 6$: $p_3 = 3p_1 + qp_0 = 3q$, $p_4 = 3p_2 + qp_1 = 18$, $p_5 = 3p_3 + qp_2 = 9q + 6q = 15q$.

Now the real-root condition. The graph of $g(t) = t^3 - 3t$ has a local maximum $g(-1) = 2$ and a local minimum $g(1) = -2$, and $t^3 - 3t = q$ has three real roots (counted with multiplicity) exactly when the horizontal line at height q meets the graph three times: $-2 \le q \le 2$. So $p_5 = 15q \le 30$, with equality at $q = 2$, where the roots are $2, -1, -1$.

Check. $(x, y, z) = (2, -1, -1)$: sum 0, squares $4 + 1 + 1 = 6$, fifth powers $32 - 1 - 1 = 30$. The AM–GM bound $|xyz| \le (\sqrt{2})^3 = 2\sqrt{2}$ is not attainable here (it would need $|x| = |y| = |z|$, impossible with sum 0), so it cannot give the maximum. Note also that 18 is p_4, not p_5.

25. Answer: (E) $\frac{35}{2}$**. The idea.** Let T_k be the expected number of draws still to come when k guests remain. A round with k guests costs k draws and sends F guests home, where F is the number of fixed points of a random permutation of k: $P(F = j) = \binom{k}{j} D_{k-j}/k!$, with derangement numbers $D_0, \ldots, D_5 = 1, 0, 1, 2, 9, 44$. Since $F = 0$ leaves us where we started, T_k appears on both sides of its equation; solve for it.

Work it out. $T_1 = 1$ (the lone guest draws their own hat).

$k = 2$: $F = 0$ or 2, each with probability $\frac{1}{2}$: $T_2 = 2 + \frac{1}{2}T_2$, so $T_2 = 4$.

$k = 3$: $P(F = 0, 1, 3) = \frac{2}{6}, \frac{3}{6}, \frac{1}{6}$: $T_3 = 3 + \frac{1}{3}T_3 + \frac{1}{2}T_2$, so $\frac{2}{3}T_3 = 5$, $T_3 = \frac{15}{2}$.

$k = 4$: $P(F = 0, 1, 2, 4) = \frac{9}{24}, \frac{8}{24}, \frac{6}{24}, \frac{1}{24}$: $T_4 = 4 + \frac{3}{8}T_4 + \frac{1}{3} \cdot \frac{15}{2} + \frac{1}{4} \cdot 4$, so $\frac{5}{8}T_4 = \frac{15}{2}$, $T_4 = 12$.

$k = 5$: $P(F = 0, 1, 2, 3, 5) = \frac{44}{120}, \frac{45}{120}, \frac{20}{120}, \frac{10}{120}, \frac{1}{120}$: $T_5 = 5 + \frac{11}{30}T_5 + \frac{3}{8} \cdot 12 + \frac{1}{6} \cdot \frac{15}{2} + \frac{1}{12} \cdot 4 = \frac{133}{12} + \frac{11}{30}T_5$, so $\frac{19}{30}T_5 = \frac{133}{12}$ and $T_5 = \frac{35}{2}$.

Check. The values $4, \frac{15}{2}, 12, \frac{35}{2}$ fit $T_k = \frac{k(k+2)}{2}$ for $k \ge 2$, and this can be proved: for $k \ge 2$ a random permutation has $E[F] = 1$ and $E[F^2] = 2$, and $F = k-1$ is impossible, so if $T_m = \frac{m(m+2)}{2}$ for $2 \le m < k$ (and $T_0 = 0$), then $T_k = k + E\left[\frac{(k-F)(k-F+2)}{2}\right] = k + \frac{k^2-2k+2+2k-2}{2} = \frac{k(k+2)}{2}$. Guessing "one guest leaves per round" gives $5+4+3+2+1 = 15$ instead; the right count is larger because rounds with no one leaving are common.