MathCastle

AMC 10

Hard Edition

25 questions • 75 minutes

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 25-question multiple-choice test. Each question is followed by answers marked A, B, C, D and E. Exactly one of them is correct.
3. Mark one letter for each question.
4. Scoring: 6 points for each correct answer, 1.5 points for each question left blank, and 0 points for each wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators, phones or other devices.
6. Figures are not necessarily drawn to scale.
7. You have 75 minutes to finish the test.

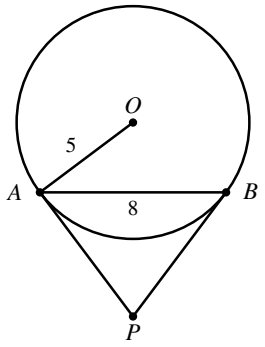
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. Four different real numbers are added in pairs, in all six possible ways, giving six sums. In some order, these six sums are 9, 13, 14, 16, 17 and 21. What is the sum of all possible values of the largest of the four numbers?

(A) 11 **(B)** 12 **(C)** 21 **(D)** 23 **(E)** 24

2. Point P lies outside a circle with centre O and radius 5. The two lines through P that are tangent to the circle touch it at A and B, and $AB = 8$. What is the radius of the circle inscribed in triangle PAB?



(A) 2 **(B)** $\frac{9}{4}$ **(C)** $\frac{8}{3}$ **(D)** 3 **(E)** $\frac{10}{3}$

3. For each positive integer n, let $k(n)$ be the largest integer whose cube is at most n; for example, $k(7) = 1$, $k(8) = 2$ and $k(30) = 3$. For how many integers n with $1 \le n \le 1000$ is n divisible by $k(n)$?

(A) 163 **(B)** 171 **(C)** 172 **(D)** 181 **(E)** 190

4. In how many ways can 12 be written as an ordered sum of odd positive integers? Order matters, so $1 + 3 + 1 + 7$, $7 + 1 + 3 + 1$ and $3 + 9$ are three different ways, and a sum may use any number of terms.

(A) 15 **(B)** 89 **(C)** 144 **(D)** 233 **(E)** 256

5. How many triples (a, b, c) of positive integers with $a < b < c$ form a geometric progression, meaning $\frac{b}{a} = \frac{c}{b}$, and satisfy $abc = 6^6 = 46656$? The common ratio does not have to be an integer.

(A) 11 **(B)** 12 **(C)** 13 **(D)** 24 **(E)** 25

6. Let $f(x) = x^2 - 3x + 1$. The equation $f(f(x)) = x$ has exactly four real solutions. What is the largest of them?

(A) $1 + \sqrt{2}$ **(B)** $1 + \sqrt{3}$ **(C)** 3 **(D)** $2 + \sqrt{2}$ **(E)** $2 + \sqrt{3}$

7. The numbers $1, 2, 3, \ldots, 10$ are placed around a circle in a random order, with all arrangements equally likely. Two numbers are neighbours if they are next to each other on the circle, so there are 10 pairs of neighbours. What is the expected number of pairs of neighbours whose two numbers differ by 1 or by 2?

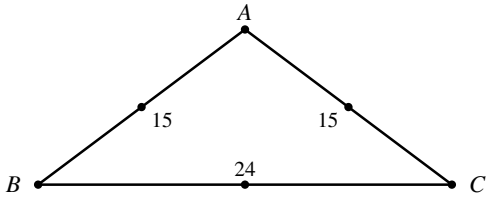
(A) $\frac{17}{9}$ **(B)** $\frac{10}{3}$ **(C)** $\frac{17}{5}$ **(D)** $\frac{34}{9}$ **(E)** 4

8. Each of the nine cells of a 3 by 3 grid is coloured black or white, independently and with probability $\frac{1}{2}$ each. What is the probability that no row and no column is entirely black?

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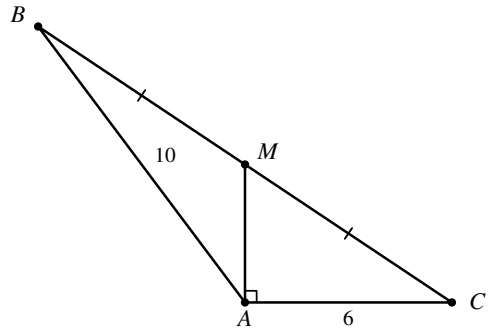
(A) $\frac{1}{4}$ **(B)** $\frac{117649}{262144}$ **(C)** $\frac{131}{256}$ **(D)** $\frac{265}{512}$ **(E)** $\frac{343}{512}$

9. Triangle ABC has $AB = AC = 15$ and $BC = 24$. What is the radius of the circle that passes through the midpoints of the three sides of the triangle?



(A) 4 **(B)** $\frac{9}{2}$ **(C)** 6 **(D)** $\frac{25}{4}$ **(E)** $\frac{25}{2}$

10. In triangle ABC, $AB = 10$ and $AC = 6$. The median from A (the segment from A to the midpoint M of $\overline{BC}$) is perpendicular to side $\overline{AC}$. What is BC?



(A) $2\sqrt{13}$ **(B)** $2\sqrt{34}$ **(C)** 12 **(D)** 14 **(E)** $4\sqrt{13}$

11. How many ordered pairs of integers (x, y) satisfy $x^2 + xy + y^2 = 91$? (Pairs such as $(9, 1)$ and $(1, 9)$ count as different.)

(A) 12 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 36

12. Let a, b and c be the three roots of the polynomial $x^3 - 3x^2 - x + 1$. What is the value of $\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a}$?

(A) -4 **(B)** -2 **(C)** $-\frac{1}{4}$ **(D)** 1 **(E)** 4

13. A ten-digit positive integer is written using only the digits 1 and 2 (each of its ten digits is a 1 or a 2; it need not use both). How many such ten-digit integers are divisible by 24?

(A) 43 **(B)** 44 **(C)** 86 **(D)** 128 **(E)** 256

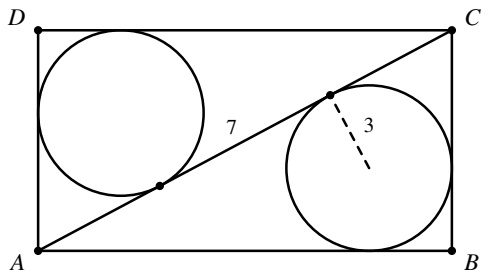
14. How many ordered pairs (a, b) of positive integers with $a \leq 100$ and $b \leq 100$ satisfy $a^2 + b^2 = 2(a + b) + 2ab$?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 32

15. Ana has 2 chips and Ben has 3 chips. They play a series of rounds. Ana wins each round with probability $\frac{2}{3}$ and Ben wins it with probability $\frac{1}{3}$, independently of all other rounds, and the loser of a round gives the winner one chip. They stop as soon as one player has all 5 chips. What is the probability that Ana ends up with all 5 chips?

(A) $\frac{8}{27}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{24}{31}$

16. In rectangle $ABCD$ the diagonal $\overline{AC}$ is drawn, splitting the rectangle into right triangles ABC and ACD. A circle is inscribed in each of these two triangles. Each circle has radius 3, and the two points where the circles touch the diagonal $\overline{AC}$ are 7 units apart. What is the area of the rectangle?



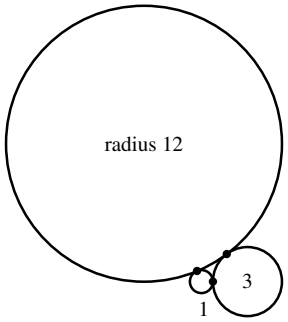
(A) 105 **(B)** 120 **(C)** 136 **(D)** 144 **(E)** 240

17. Lena writes down the 1000 numbers $2^1 - 1,\ 2^2 - 2,\ 2^3 - 3,\ \ldots,\ 2^{1000} - 1000$, that is, $2^n - n$ for every integer n from 1 to 1000. How many of the numbers she writes are multiples of 7?

(A) 47 **(B)** 48 **(C)** 94 **(D)** 141 **(E)** 142

18. Three circles with radii 1, 3 and 12 are each externally tangent to the other two, so there are exactly three points where two of the circles touch each other. What is the radius of the circle that passes through these three points of tangency?

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(A) $\frac{3}{2}$ **(B)** 2 **(C)** 3 **(D)** $\frac{16}{3}$ **(E)** $\frac{65}{8}$

19. A bag holds 6 red marbles and 4 blue marbles. Marbles are drawn from the bag one at a time, at random and without replacement, and the drawing stops as soon as all the marbles of one colour have been drawn. What is the expected number of marbles left in the bag when the drawing stops?

(A) $\frac{6}{5}$ **(B)** $\frac{58}{35}$ **(C)** $\frac{62}{35}$ **(D)** 2 **(E)** $\frac{12}{5}$

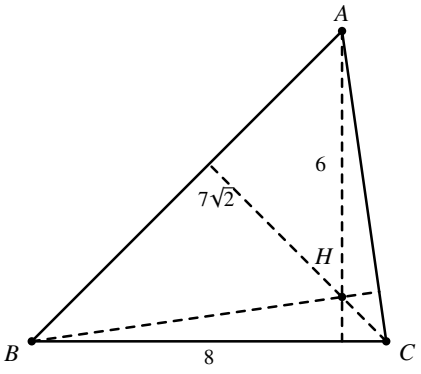
20. A motion sensor records each of 10 consecutive minutes as ON or OFF, so a record is a string of 10 symbols, and all 2^{10} strings are possible records. A switch-on is an OFF minute immediately followed by an ON minute. How many of the possible records contain exactly 3 switch-ons?

(A) 120 **(B)** 165 **(C)** 210 **(D)** 252 **(E)** 330

21. A polynomial $P(x)$ has integer coefficients, and $P(1) = 3$ and $P(3) = 11$. Among all such polynomials (of any degree), what is the smallest possible value of $|P(10)|$?

(A) 3 **(B)** 24 **(C)** 39 **(D)** 63 **(E)** 102

22. Triangle ABC is acute (all three of its angles are less than 90°), with $BC = 8$ and $AB = 7\sqrt{2}$. Its orthocentre H is the point where its three altitudes meet, and $AH = 6$. What is the area of triangle ABC?



(A) $\frac{476}{25}$ **(B)** 24 **(C)** 28 **(D)** 30 **(E)** 32

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23. Ten people sit around a round table. They are to be split into five pairs of partners for a card game; a split is determined only by who is partnered with whom (the pairs are not ordered or numbered). No two people who sit next to each other may be partners. In how many ways can the ten people be split into pairs?

(A) 263 **(B)** 291 **(C)** 293 **(D)** 295 **(E)** 420

24. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. How many of the numbers $F_1, F_2, \ldots, F_{2026}$ end in the two digits 00, that is, are multiples of 100?

(A) 6 **(B)** 13 **(C)** 27 **(D)** 40 **(E)** 67

25. An excircle of a triangle is a circle outside the triangle that is tangent to one side of the triangle and to the extensions of the other two sides beyond that side; every triangle has exactly three excircles. A certain triangle has excircles with radii 9, 18 and 21. What is the area of the triangle?

(A) 84 **(B)** 126 **(C)** 147 **(D)** 189 **(E)** 252

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Answer Key — AMC 10 — Hard Edition Practice Contest 1

1. (D) 23

2. (A) 2

3. (C) 172

4. (C) 144

5. (B) 12

6. (E) $2 + \sqrt{3}$

7. (D) $\frac{34}{9}$

8. (D) $\frac{265}{512}$

9. (D) $\frac{25}{4}$

10. (E) $4\sqrt{13}$

11. (D) 24

12. (A) -4

13. (A) 43

14. (A) 16

15. (E) $\frac{24}{31}$

16. (B) 120

17. (E) 142

18. (A) $\frac{3}{2}$

19. (C) $\frac{62}{35}$

20. (E) 330

21. (B) 24

22. (C) 28

23. (C) 293

24. (B) 13

25. (B) 126

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Solutions — AMC 10 — Hard Edition Practice Contest 1

1. Answer: (D) 23**. The idea.** Name the numbers $a < b < c < d$. Some of the six sums can be placed for certain: the two smallest and the two largest. The two middle sums, $a + d$ and $b + c$, cannot be told apart, so there are two cases to try, and both may work.

Work it out. Each number appears in three of the six sums, so $3(a + b + c + d) = 9 + 13 + 14 + 16 + 17 + 21 = 90$ and $a + b + c + d = 30$. The smallest sum is $a + b = 9$ and the next is $a + c = 13$; the largest is $c + d = 21$ and the next is $b + d = 17$. The remaining sums 14 and 16 are $a + d$ and $b + c$ in some order.

If $b + c = 16$: with $c - b = 13 - 9 = 4$ this gives $c = 10$, $b = 6$, then $a = 3$ and $d = 11$.

If $b + c = 14$: then $c = 9$, $b = 5$, $a = 4$ and $d = 12$.

Both are genuine solutions: $3, 6, 10, 11$ has pair sums $9, 13, 14, 16, 17, 21$, and so does $4, 5, 9, 12$. The largest number can be 11 or 12, and the answer is $11 + 12 = 23$.

Check. For $4, 5, 9, 12$: $4 + 5 = 9$, $4 + 9 = 13$, $5 + 9 = 14$, $4 + 12 = 16$, $5 + 12 = 17$, $9 + 12 = 21$. For $3, 6, 10, 11$: $9, 13, 14, 16, 17, 21$ again. Stopping after the first case gives the trap answers 11 or 12.

2. Answer: (A) 2**. The idea.** The right angles at the points of tangency make several similar right triangles along the line PO, which give all the lengths in triangle PAB. Then the inradius is the area divided by the semiperimeter.

Work it out. By symmetry PO is the perpendicular bisector of $\overline{AB}$; let it cross $\overline{AB}$ at M, so $AM = 4$ and $OM = \sqrt{25 - 16} = 3$. Since $\angle OAP = 90^\circ$ and $\overline{AM}$ is the altitude of right triangle OAP, we have $AM^2 = OM \cdot MP$, so $MP = \frac{16}{3}$. Then $PA = \sqrt{AM^2 + MP^2} = \sqrt{16 + \frac{256}{9}} = \frac{20}{3}$. Triangle PAB has area $\frac{1}{2} \cdot 8 \cdot \frac{16}{3} = \frac{64}{3}$ and semiperimeter $\frac{1}{2}\left(\frac{20}{3} + \frac{20}{3} + 8\right) = \frac{32}{3}$, so its inradius is $\frac{64/3}{32/3} = 2$.

Check. A second route: let X be the point where segment $\overline{PO}$ crosses the circle. The tangent–chord angle gives $\angle PAX = \angle XBA = \angle XAB$ (the last because $XA = XB$), so $\overline{AX}$ bisects angle PAB; X is also on the bisector PO of angle APB. So X is the incentre, and the inradius is its distance to $\overline{AB}$: $XM = OX - OM = 5 - 3 = 2$.

3. Answer: (C) 172**. The idea.** Group the numbers by the value of $k(n)$. For a fixed k, the numbers with $k(n) = k$ form the block from k^3 to $(k+1)^3 - 1$, and inside that block we just count multiples of k, which start at k^3 itself.

Work it out. For $1 \le k \le 9$ the block is $k^3 \le n \le (k + 1)^3 - 1 = k^3 + 3k^2 + 3k$. The multiples of k in it are $k^3 + mk$ for $m = 0, 1, \ldots, 3k + 3$, since $mk \le 3k^2 + 3k$ exactly when $m \le 3k + 3$. That is $3k + 4$ numbers per block, and the blocks for $k = 1, \ldots, 9$ cover 1 to 999:

$\sum_{k=1}^{9}(3k + 4) = 3 \cdot 45 + 36 = 171$.

Finally $n = 1000$ has $k(1000) = 10$, which divides it. The total is $171 + 1 = 172$.

Check. For $k = 1$ the block is $1, \ldots, 7$, all divisible by 1: $7 = 3 \cdot 1 + 4$. For $k = 2$ the block is $8, \ldots, 26$ and its even numbers are $8, 10, \ldots, 26$: ten of them, $3 \cdot 2 + 4 = 10$. Forgetting that the top of the block reaches $k^3 + 3k^2 + 3k$ (a multiple of k) gives $3k + 3$ per block and the trap 163; forgetting $n = 1000$ gives 171.

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4. Answer: (C) 144**. The idea.** Look at the last term. If it is 1, delete it. If it is 3 or more, shrink it by 2 (it stays odd and positive). Each operation is reversible, which gives a Fibonacci recursion.

Work it out. Let $c(n)$ be the number of ordered sums of odd positive integers equal to n. For $n \geq 3$: the sums ending in 1 correspond exactly to the sums of $n-1$ (delete the final 1, or append one). The sums ending in an odd number ≥ 3 correspond exactly to the sums of $n-2$ (subtract 2 from the last term, or add 2 back). So $c(n) = c(n-1) + c(n-2)$. Starting from $c(1) = 1$ (just 1) and $c(2) = 1$ (just $1+1$), the values are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$, so $c(12) = 144$.

Check. $c(4) = 3$: the sums are $1+1+1+1$, $1+3$ and $3+1$. That matches the list. The answer 15 would count unordered sums (partitions of 12 into odd parts), which the problem does not ask for.

5. Answer: (B) 12**. The idea.** In a geometric progression the middle term's square is the product of the outer two, so the product of all three is the cube of the middle term. That pins down b at once, and what is left is to count divisors.

Work it out. From $b^2 = ac$ we get $abc = b^3 = 6^6$, so $b = 6^2 = 36$ and $ac = 1296 = 2^4 \cdot 3^4$. Conversely, any positive integers $a < 36 < c$ with $ac = 1296$ give a valid triple, because then $b^2 = 1296 = ac$. The number 1296 has $(4+1)(4+1) = 25$ divisors. They pair up as $d \leftrightarrow \frac{1296}{d}$, with 36 paired with itself, so exactly $\frac{25-1}{2} = 12$ divisors are less than 36. Each of them is a possible a, and $c = \frac{1296}{a}$ is then determined. The answer is 12.

Check. The divisors below 36 are $1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27$: twelve of them. For example $a = 27$ gives $27, 36, 48$, with ratio $\frac{4}{3}$, and $27 \cdot 36 \cdot 48 = 46656$.

6. Answer: (E) $2+\sqrt{3}$**. The idea.** Any x with $f(x) = x$ also satisfies $f(f(x)) = f(x) = x$. So the quartic $f(f(x)) - x$ must contain the factor $f(x) - x$, and the quotient is only a quadratic.

Work it out. Write $y = f(x)$. Then $f(y) - f(x) = (y^2 - x^2) - 3(y - x) = (y - x)(y + x - 3)$, so $f(f(x)) - x = \big[f(y) - f(x)\big] + \big[f(x) - x\big] = (y - x)(y + x - 3) + (y - x) = (y - x)(y + x - 2)$.

With $y = x^2 - 3x + 1$ this is $(x^2 - 4x + 1)(x^2 - 2x - 1)$. The first factor gives $x = 2 \pm \sqrt{3}$ (the fixed points) and the second gives $x = 1 \pm \sqrt{2}$ (a pair that f swaps). The largest of the four is $2 + \sqrt{3} \approx 3.73$.

Check. $f(2+\sqrt{3}) = (7 + 4\sqrt{3}) - (6 + 3\sqrt{3}) + 1 = 2 + \sqrt{3}$, a fixed point. And $f(1+\sqrt{2}) = (3 + 2\sqrt{2}) - (3 + 3\sqrt{2}) + 1 = 1 - \sqrt{2}$, so $f(f(1+\sqrt{2})) = 1 + \sqrt{2}$ by the symmetric computation. Since $1 + \sqrt{2} \approx 2.41$, $2 + \sqrt{3}$ is the largest.

7. Answer: (D) $\frac{34}{9}$**. The idea.** Use linearity of expectation over the *pairs of numbers* rather than over the positions: count the pairs $\{i, j\}$ that differ by 1 or 2, and multiply by the chance that one particular pair ends up side by side.

Work it out. The pairs that differ by 1 are $\{1,2\}, \ldots, \{9,10\}$: 9 pairs. The pairs that differ by 2 are $\{1,3\}, \ldots, \{8,10\}$: 8 pairs. That is 17 pairs. For a fixed pair $\{i, j\}$, look at where j sits relative to i: it is equally likely to be in any of the other 9 places, and 2 of them are next to i. So the pair are neighbours with probability $\frac{2}{9}$. The expected number of close neighbouring pairs is $17 \cdot \frac{2}{9} = \frac{34}{9}$.

Check. Each of the 10 neighbouring pairs is a uniformly random pair of numbers, one of $\binom{10}{2} = 45$, and 17 of those are close, so the expected count is $10 \cdot \frac{17}{45} = \frac{34}{9}$ again. In a row instead of a circle

the chance would be $\frac{2}{10}$, giving the trap $\frac{17}{5}$; forgetting that the neighbour may be on either side gives $\frac{17}{9}$.

8. Answer: (D) $\frac{265}{512}$. The idea. The row condition is easy to build in directly, because each row just avoids one pattern. Then correct for the columns with inclusion–exclusion: force a set of columns to be black and count how each row can still avoid being all black.

Work it out. There are $2^9 = 512$ equally likely grids. Suppose a chosen set of j columns is forced to be entirely black. Within each row, those j cells are black and the other $3 - j$ cells must not all be black, which leaves $2^{3-j} - 1$ choices per row, and so $(2^{3-j} - 1)^3$ grids with no black row. Inclusion–exclusion over the columns gives the number of grids with no black row and no black column:

$\binom{3}{0}7^3 - \binom{3}{1}3^3 + \binom{3}{2}1^3 - \binom{3}{3}0^3 = 343 - 81 + 3 - 0 = 265.$

The probability is $\frac{265}{512}$.

Check. The full two-way inclusion–exclusion, which forces i rows and j columns black and leaves $(3-i)(3-j)$ free cells, gives the same total: $\sum_{i,j}(-1)^{i+j}\binom{3}{i}\binom{3}{j}2^{(3-i)(3-j)} = 343-81+3+0 = 265$. Treating the six lines as independent would give $(7/8)^6$, which is wrong because the rows and columns share cells.

9. Answer: (D) $\frac{25}{4}$. The idea. The three midpoints form the midpoint triangle, whose sides are each half of a side of ABC and parallel to it. So the midpoint triangle is similar to ABC with ratio $\frac{1}{2}$, and the circle through the midpoints is its circumcircle: its radius is half the circumradius R of ABC. Find R from $R = \frac{abc}{4K}$.

Work it out. The altitude from A meets $\overline{BC}$ at its midpoint M, with $BM = 12$, so $AM = \sqrt{15^2 - 12^2} = 9$ and the area is $K = \frac{1}{2} \cdot 24 \cdot 9 = 108$. Then $R = \frac{15 \cdot 15 \cdot 24}{4 \cdot 108} = \frac{5400}{432} = \frac{25}{2}$. The circle through the midpoints has radius $\frac{1}{2} \cdot \frac{25}{2} = \frac{25}{4}$.

Check. Put M at the origin, $B = (-12, 0)$, $C = (12, 0)$, $A = (0, 9)$. The midpoints are $(0, 0)$, $\left(6, \frac{9}{2}\right)$ and $\left(-6, \frac{9}{2}\right)$. By symmetry the centre is $(0, k)$ with $k^2 = 36 + \left(\frac{9}{2} - k\right)^2$, so $9k = 36 + \frac{81}{4}$ and $k = \frac{25}{4}$; the radius is the distance from $(0, \frac{25}{4})$ to $(0, 0)$, which is $\frac{25}{4}$. The triangle is obtuse ($15^2 + 15^2 < 24^2$), so its circumcentre lies outside it, but the halving still holds. The trap $\frac{25}{2}$ is R itself; 4 is the inradius $\frac{108}{27}$, and $\frac{9}{2}$ is half the altitude.

10. Answer: (E) $4\sqrt{13}$. The idea. The right angle is at A between $\overline{AM}$ and $\overline{AC}$, not in the triangle itself, so no side is a hypotenuse yet. Use the fact that M is the midpoint of $\overline{BC}$: the line through M perpendicular to $\overline{AC}$ (the line AM) lies exactly halfway between B and C in the direction along $\overline{AC}$. So B sits as far on one side of line AM as C does on the other, which pins down the foot of the perpendicular from B to line AC.

Work it out. Let P be the foot of the perpendicular from B to line AC. Measured along line AC, the points B and C are at equal distances on opposite sides of line AM, and line AM meets line AC at A. Since C is 6 from A, the point P is 6 from A on the other side, so $AP = 6$ and $PC = 12$. In right triangle ABP, $BP = \sqrt{10^2 - 6^2} = 8$. Then in right triangle BPC, $BC = \sqrt{12^2 + 8^2} = \sqrt{208} = 4\sqrt{13}$.

Check. With $A = (0, 0)$ and $C = (6, 0)$, the point $B = (-6, 8)$ has $AB = 10$, and $M = (0, 4)$ lies on the y-axis, so $\overline{AM} \perp \overline{AC}$. Also $BC = \sqrt{12^2 + 8^2} = 4\sqrt{13}$. The trap $2\sqrt{13}$ is MC, half the

answer; $2\sqrt{34} = \sqrt{136}$ comes from wrongly putting the right angle at A of the triangle; 12 is PC alone; and 14 is $8 + 6$, adding legs instead of using Pythagoras.

11. Answer: (D) 24**. The idea.** Complete the square. Multiplying by 4 turns the mixed term into a square: $4x^2 + 4xy + y^2 = (2x + y)^2$. The equation becomes $(2x + y)^2 + 3y^2 = 364$, which bounds y and leaves only a short check.

Work it out. Let $u = 2x + y$, so $u^2 + 3y^2 = 364$. Then $|y| \le 11$, and u has the same parity as y (so that $x = \frac{u-y}{2}$ is an integer). For $y = 0, 1, \ldots, 11$ the value $364 - 3y^2$ is 364, 361, 352, 337, 316, 289, 256, 217, 172, 12
The perfect squares are $361 = 19^2$ ($y = 1$), $289 = 17^2$ ($y = 5$), $256 = 16^2$ ($y = 6$), $121 = 11^2$ ($y = 9$), $64 = 8^2$ ($y = 10$) and $1 = 1^2$ ($y = 11$), and in every case u has the same parity as y. Each of these six values of y gives two values $u = \pm\sqrt{364 - 3y^2}$ and hence two pairs; for instance $y = 1$ gives $x = 9$ or $x = -10$. That makes 12 pairs with $y > 0$. The map $(x, y) \mapsto (-x, -y)$ gives 12 more with $y < 0$, and $y = 0$ gives none (364 is not a square). The total is 24.

Check. $9^2 + 9 \cdot 1 + 1^2 = 91$, and $6^2 + 6 \cdot 5 + 5^2 = 36 + 30 + 25 = 91$. The form is symmetric under swapping x and y and under $(x, y) \mapsto (x + y, -y)$, so the solutions come in groups of 12 ($(9, 1)$, $(1, 9)$, $(10, -1)$, …). Here there are two such groups, the one through $(9, 1)$ and the one through $(6, 5)$, giving 24.

12. Answer: (A) -4**. The idea.** By Vieta, $a + b + c = 3$, so each pair sum is 3 minus the third root: $a + b = 3 - c$, and so on. The expression becomes $\sum \frac{1}{3-c}$, a sum of reciprocals of the numbers $3 - a$, $3 - b$, $3 - c$, and those numbers are the roots of a polynomial we can write down.

Work it out. Let $p(x) = x^3 - 3x^2 - x + 1$. The numbers $y = 3 - a$, $3 - b$, $3 - c$ are the roots of $p(3 - y) = 0$. Expanding: $(3 - y)^3 - 3(3 - y)^2 - (3 - y) + 1 = -y^3 + 6y^2 - 8y - 2$. So they are the roots of $y^3 - 6y^2 + 8y + 2$. For this cubic, the sum of the products of pairs of roots is 8 and the product of the roots is -2. The sum of the reciprocals is $\frac{y_1y_2 + y_2y_3 + y_3y_1}{y_1y_2y_3} = \frac{8}{-2} = -4$.

Check. Since $p(x) = (x - a)(x - b)(x - c)$, we have $\frac{p'(x)}{p(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$. At $x = 3$: $p(3) = 27 - 27 - 3 + 1 = -2$ and $p'(3) = 3 \cdot 9 - 6 \cdot 3 - 1 = 8$, so the sum is $\frac{8}{-2} = -4$ again. None of the pair sums is 0, because 3 is not a root of p.

13. Answer: (A) 43**. The idea.** $24 = 8 \cdot 3$ with $\gcd(8, 3) = 1$, so test the two divisibilities separately. Divisibility by 8 only looks at the last three digits, and divisibility by 3 only looks at the digit sum, so the number splits into a fixed tail and a free head.

Work it out. The last three digits form one of 111, 112, 121, 122, 211, 212, 221, 222. The odd ones are out; 122 and 222 end in 22, which is not a multiple of 4; and $212 = 8 \cdot 26 + 4$. Only $112 = 8 \cdot 14$ works, so every such number ends in 112.

The first seven digits are free. If k of them are 2's, the digit sum is $(7 + k) + (1 + 1 + 2) = 11 + k$, which must be a multiple of 3. So $k \equiv 1 \pmod 3$, i.e. $k = 1, 4$ or 7. The number of ways is $\binom{7}{1} + \binom{7}{4} + \binom{7}{7} = 7 + 35 + 1 = 43$.

Check. The smallest example is 2111111112 (one 2 in front, digit sum 12): indeed $2111111112 = 24 \cdot 87962963$. A common slip is to test only the last two digits for divisibility by 4 instead of the last three for 8; that lets in every number ending in 12 and gives 86. Dropping the divisibility-by-3 condition altogether leaves all $2^7 = 128$ numbers ending in 112.

14. Answer: (A) 16**. The idea.** Move the $2ab$ across: the left side becomes $(a-b)^2$. So the equation

says the square of the difference is twice the sum. That forces the difference to be even, and once the difference is named, the sum is determined, so the whole solution set is a one-parameter family.

Work it out. The equation is $(a-b)^2 = 2(a+b)$. The right side is even, so $a-b$ is even; write $a-b=2t$ for an integer t. Then $4t^2 = 2(a+b)$, so $a+b=2t^2$. Adding and subtracting, $a = t^2+t$ and $b = t^2-t$. Conversely every integer t gives a solution of the equation. We need $b = t(t-1) \geq 1$ and $a = t(t+1) \geq 1$, which rules out exactly $t=0$ and $t=1$ (and $t=-1$ gives $a=0$). For $t \geq 2$ the larger coordinate is $a = t^2+t \leq 100$, so $t \leq 9$ ($9\cdot 10 = 90$ but $10\cdot 11 = 110$). For $t \leq -2$ the roles swap: $b = t^2 + |t|$, so $|t| \leq 9$. That is $8+8=16$ pairs.

Check. $t=2$ gives $(6,2)$: $36+4=40$ and $2\cdot 8 + 2\cdot 12 = 16+24 = 40$. The largest is $t=9$: $(90,72)$, with $(a-b)^2 = 324 = 2\cdot 162$. The traps: 17 lets in $(0,0)$ from $t=0$; 18 lets in $(2,0)$ and $(0,2)$ from $t=\pm 1$, which are not positive; and 19 counts every t from -9 to 9, including $t=0$, which gives $(0,0)$; and 32 doubles every pair to account for swapping a and b, although the negative values of t already produce the swapped pairs.

15. Answer: (E) $\frac{24}{31}$**. The idea.** Let p_k be Ana's chance of winning everything when she holds k chips. Conditioning on the next round gives a recursion whose successive differences shrink by a constant factor, so the p_k are partial sums of a geometric series.

Work it out. We have $p_0 = 0$, $p_5 = 1$ and $p_k = \frac{2}{3}p_{k+1} + \frac{1}{3}p_{k-1}$ for $1 \leq k \leq 4$. Multiply by 3 and rearrange: $2(p_{k+1} - p_k) = p_k - p_{k-1}$, so each difference is half the one before. Writing $p_1 - p_0 = t$, the five differences are $t, \frac{t}{2}, \frac{t}{4}, \frac{t}{8}, \frac{t}{16}$, and they add up to $p_5 - p_0 = 1$: $\frac{31}{16}t = 1$, $t = \frac{16}{31}$. Hence $p_2 = t + \frac{t}{2} = \frac{3}{2}\cdot\frac{16}{31} = \frac{24}{31}$.

Check. From Ben's side (he wins a round with probability $\frac{1}{3}$), his differences double each step: $s, 2s, 4s, 8s, 16s$ with $31s = 1$, and starting from 3 chips he takes everything with probability $s + 2s + 4s = \frac{7}{31}$. Indeed $\frac{24}{31} + \frac{7}{31} = 1$. If the game were fair the answer would be $\frac{2}{5}$, and $\frac{8}{27}$ is only the chance that Ana wins the first three rounds in a row.

16. Answer: (B) 120**. The idea.** Tangent lengths. The two tangent segments from a vertex to an incircle are equal, so where an incircle touches a side is determined by the side lengths. That turns both facts about the circles into simple equations in the rectangle's sides.

Work it out. Let $AB = CD = a$, $BC = AD = b$ and $AC = d$, so $a^2 + b^2 = d^2$. In triangle ABC the incircle touches AC at a point P with $AP = \frac{AB+AC-BC}{2} = \frac{a+d-b}{2}$. In triangle ACD it touches AC at Q with $AQ = \frac{AD+AC-CD}{2} = \frac{b+d-a}{2}$. So $PQ = |a-b| = 7$.

The inradius of a right triangle with legs a, b and hypotenuse d is $\frac{a+b-d}{2}$ (the tangent lengths from the right-angle vertex are both r), so $a+b = d+6$.

Now $(a+b)^2 + (a-b)^2 = 2(a^2+b^2) = 2d^2$ gives $(d+6)^2 + 49 = 2d^2$, i.e. $d^2 - 12d - 85 = 0 = (d-17)(d+5)$. So $d=17$, $a+b=23$, $|a-b|=7$: the sides are 15 and 8, and the area is 120.

Check. The 8-15-17 triangle has inradius $\frac{8+15-17}{2} = 3$, and the touch points are $\frac{15+17-8}{2} = 12$ and $\frac{8+17-15}{2} = 5$ from A, which are 7 apart. Also area $= ab = \frac{(a+b)^2-(a-b)^2}{4} = \frac{529-49}{4} = 120$.

17. Answer: (E) 142**. The idea.** Modulo 7, the powers of 2 repeat with period 3 (since $2^3 = 8 \equiv 1$), while n repeats with period 7. So whether $7 \mid 2^n - n$ depends only on n modulo $3\cdot 7 = 21$. Find the good residues in one block of 21, then count blocks.

Work it out. $2^n \bmod 7$ is $2, 4, 1$ according as $n \equiv 1, 2, 0 \pmod 3$. We need $n \equiv 2^n \pmod 7$:

if $n \equiv 1 \pmod 3$ we need $n \equiv 2 \pmod 7$, giving $n \equiv 16 \pmod{21}$;

if $n \equiv 2 \pmod 3$ we need $n \equiv 4 \pmod 7$, giving $n \equiv 11 \pmod{21}$;

if $n \equiv 0 \pmod 3$ we need $n \equiv 1 \pmod 7$, giving $n \equiv 15 \pmod{21}$.

(By the Chinese Remainder Theorem each pair of conditions has exactly one solution mod 21.) Since $1000 = 21 \cdot 47 + 13$, the numbers 1 to 987 form 47 full blocks with $3 \cdot 47 = 141$ solutions. The leftover numbers $988, \ldots, 1000$ have residues 1 to 13 mod 21, and only residue 11 (the number 998) is good. Total: 142.

Check. $2^{11} - 11 = 2037 = 7 \cdot 291$, $2^{15} - 15 = 32753 = 7 \cdot 4679$ and $2^{16} - 16 = 65520 = 7 \cdot 9360$, while no other $n \leq 21$ works. Forgetting the partial block at the end gives 141.

18. Answer: (A) $\frac{3}{2}$**. The idea.** Join the three centres to form a triangle. Each point of tangency lies on a side of this triangle, and from each centre the two tangency points on its sides are at the same distance (its radius). Equal tangent lengths from every vertex is exactly how the incircle touches a triangle, so the three points are the incircle's points of contact and the circle through them is the incircle.

Work it out. The centres O_1, O_2, O_3 (radii 1, 3, 12) are $1+3 = 4$, $1+12 = 13$ and $3+12 = 15$ apart. This triangle has semiperimeter $s = 16$, and its incircle touches the sides at distances $s - 15 = 1$ from O_1, $s - 13 = 3$ from O_2 and $s - 4 = 12$ from O_3: exactly the three tangency points. By Heron, the area is $\sqrt{16 \cdot 1 \cdot 3 \cdot 12} = \sqrt{576} = 24$, so the inradius is $\frac{24}{16} = \frac{3}{2}$.

Check. Coordinates: $O_1 = (0, 0)$, $O_2 = (4, 0)$, $O_3 = (-5, 12)$. The tangency points are $(1, 0)$, $\left(-\frac{5}{13}, \frac{12}{13}\right)$ and $\left(\frac{11}{5}, \frac{12}{5}\right)$, and each is at distance $\frac{3}{2}$ from $\left(1, \frac{3}{2}\right)$; for instance $\left(\frac{6}{5}\right)^2 + \left(\frac{9}{10}\right)^2 = \frac{225}{100}$. The distractor $\frac{65}{8}$ is the circumradius of the triangle of centres, a different circle.

19. Answer: (C) $\frac{62}{35}$**. The idea.** Pretend all ten marbles are drawn in a random order, and ask of each single marble whether it is still in the bag at the moment the real drawing would stop. Then add up these chances (linearity of expectation): no need to find the distribution of the number left.

Work it out. A particular red marble is left in the bag exactly when all 4 blue marbles are drawn before it. (If so, the blues run out before it is drawn, and the reds cannot run out first, since it is still in the bag. If not, some blue is still in the bag when it is drawn, and the drawing was still going on at that moment.) Among this red marble and the 4 blue ones, each of the five is equally likely to come last, so the chance is $\frac{1}{5}$.

In the same way, a particular blue marble is left exactly when all 6 reds come before it, with chance $\frac{1}{7}$.

Expected number left: $6 \cdot \frac{1}{5} + 4 \cdot \frac{1}{7} = \frac{42+20}{35} = \frac{62}{35}$.

Check. Counting directly over the $\binom{10}{4} = 210$ equally likely colour patterns gives a total of 372 marbles left, and $\frac{372}{210} = \frac{62}{35}$. With one red and one blue marble the formula gives $\frac{1}{2} + \frac{1}{2} = 1$, and indeed exactly one marble is always left. Swapping the two chances $(6 \cdot \frac{1}{7} + 4 \cdot \frac{1}{5})$ gives the trap $\frac{58}{35}$.

20. Answer: (E) 330**. The idea.** Switch-ons and switch-offs alternate, but the record can start and end either way, which makes direct counting messy. Pad the record with an imaginary ON minute

in front and an OFF minute at the end. Now it starts ON and ends OFF, so its changes alternate off, on, off, ..., off: there is exactly one more switch-off than switch-on, and a record is determined by where its changes happen.

Work it out. The padded string has 12 symbols and 11 gaps between neighbours. The padding creates no switch-on (a switch-on needs an ON after an OFF, and the extra ON is first, the extra OFF last), so the record has exactly 3 switch-ons iff the padded string has 3 switch-ons, iff it has 3 switch-ons and 4 switch-offs, iff it changes in exactly 7 of its 11 gaps.

Conversely, choose any 7 of the 11 gaps. Starting from ON and flipping at those gaps gives a string whose last symbol is OFF (an odd number of flips), so removing the padding leaves a valid record with exactly 3 switch-ons. Hence the count is $\binom{11}{7} = 330$.

Check. For 3-minute records with exactly one switch-on the same argument predicts $\binom{4}{3} = 4$, and indeed the records are OFF OFF ON, OFF ON OFF, OFF ON ON and ON OFF ON.

21. Answer: (B) 24**. The idea.** For a polynomial with integer coefficients, $a-b$ divides $P(a)-P(b)$, because $a - b$ divides every $a^k - b^k$. Applying this to 10 and 1, and to 10 and 3, restricts $P(10)$ to one residue class mod 63; then show every value in that class really occurs.

Work it out. $9 \mid P(10) - P(1)$ gives $P(10) \equiv 3 \pmod 9$, and $7 \mid P(10) - P(3)$ gives $P(10) \equiv 11 \equiv 4 \pmod 7$. Going through $3, 12, 21, 30, 39$ (the numbers $\equiv 3$ mod 9), their remainders mod 7 are $3, 5, 0, 2, 4$, so $P(10) \equiv 39 \pmod{63}$. The values of this form closest to 0 are 39 and $39 - 63 = -24$, so $|P(10)| \geq 24$.

To reach -24: the line through $(1, 3)$ and $(3, 11)$ is $4x - 1$, and $P(x) = 4x - 1 - (x - 1)(x - 3) = -x^2 + 8x - 4$ has integer coefficients, $P(1) = 3$, $P(3) = 11$ and $P(10) = -100 + 80 - 4 = -24$. The minimum is 24.

Check. In general every valid P is $4x - 1 + (x - 1)(x - 3)Q(x)$ with Q an integer polynomial, so $P(10) = 39 + 63\,Q(10)$ can be any number $\equiv 39 \pmod{63}$. The trap answer 39 comes from forgetting that $P(10)$ may be negative.

22. Answer: (C) 28**. The idea.** The distance from a vertex to the orthocentre is twice the distance from the circumcentre O to the opposite side: $AH = 2 \cdot OM$, where M is the midpoint of $\overline{BC}$. With BC known, that pins down the circumcircle, and then A is found as a point on it.

Why $AH = 2 \cdot OM$. Let C' be the point of the circumcircle diametrically opposite C. Angles in a semicircle are right, so $C'B \perp BC$ and $C'A \perp AC$. The altitude AH is also perpendicular to BC, and the altitude BH is perpendicular to AC, so $AH \parallel C'B$ and $BH \parallel C'A$: $AHBC'$ is a parallelogram and $AH = C'B$. In triangle CBC' the points O and M are the midpoints of CC' and CB, so $C'B = 2 \cdot OM$.

Work it out. So $OM = 3$, and the circumradius is $R = OB = \sqrt{OM^2 + BM^2} = \sqrt{9 + 16} = 5$. Put M at the origin with $B = (-4, 0)$ and $C = (4, 0)$. The triangle is acute, so O is inside it, on the same side of BC as A: $O = (0, 3)$. Let $A = (x, h)$ with $h > 0$. Then

$x^2 + (h - 3)^2 = 25$ (on the circle) and $(x + 4)^2 + h^2 = 98$ (from $AB = 7\sqrt{2}$).

Subtracting, $8x + 6h + 7 = 73$, so $4x + 3h = 33$ and $x = \frac{33-3h}{4}$. Substituting into the first equation and clearing denominators gives $25h^2 - 294h + 833 = 0$, so $h = 7$ or $h = \frac{119}{25}$.

If $h = \frac{119}{25}$ then $x = \frac{117}{25} > 4$: the foot of the altitude from A lies beyond C, so the angle at C is obtuse, which is not allowed. So $h = 7$, $x = 3$, $A = (3, 7)$, and the area is $\frac{1}{2} \cdot 8 \cdot 7 = 28$.

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Check. With $A = (3, 7)$: $AB^2 = 7^2 + 7^2 = 98$. The altitude from A is the line $x = 3$; the altitude from B is perpendicular to AC, whose direction is $(1, -7)$, so it is the line through $(-4, 0)$ with slope $\frac{1}{7}$, which meets $x = 3$ at $(3, 1)$. So $H = (3, 1)$ and $AH = 6$. All three angles are acute: for instance $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-1)(-8) + 7 \cdot 0 = 8 > 0$. The obtuse triangle that also fits the other data has area $4 \cdot \frac{119}{25} = \frac{476}{25}$, a trap for anyone who forgets the word acute.

23. Answer: (C) 293**. The idea.** Without the rule, $2m$ people can be paired in $(2m - 1)(2m - 3) \cdots 3 \cdot 1$ ways. Use inclusion-exclusion over the forbidden pairs (the 10 pairs of neighbours). The work is in counting how many ways to choose k neighbour pairs that share no person, around a circle.

Work it out. Number the seats 1 to 10 and call the 10 neighbour pairs $e_1 = \{1, 2\}, e_2 = \{2, 3\}, \ldots, e_{10} = \{10, 1\}$. Let m_k be the number of ways to choose k of them with no person in two of them, i.e. no two consecutive around the circle. If e_1 is not chosen, we choose k non-consecutive pairs from the row $e_2, \ldots, e_{10}$ of 9: $\binom{10-k}{k}$ ways. If e_1 is chosen, e_2 and e_{10} are out, and we choose $k - 1$ non-consecutive pairs from the row $e_3, \ldots, e_9$ of 7: $\binom{9-k}{k-1}$ ways. This gives $m_0 = 1, m_1 = 10, m_2 = 28 + 7 = 35, m_3 = 35 + 15 = 50, m_4 = 15 + 10 = 25, m_5 = 1 + 1 = 2$.

If k given neighbour pairs are forced to be partners, the other $10 - 2k$ people pair up freely in $945, 105, 15, 3, 1, 1$ ways for $k = 0, 1, \ldots, 5$. By inclusion-exclusion the number of splits with no neighbours as partners is

$$945 - 10 \cdot 105 + 35 \cdot 15 - 50 \cdot 3 + 25 \cdot 1 - 2 \cdot 1 = 945 - 1050 + 525 - 150 + 25 - 2 = 293.$$

Check. The same method with 4 people at a table gives $3 - 4 \cdot 1 + 2 \cdot 1 = 1$, and indeed only the split into the two opposite pairs works; with 8 people it gives $105 - 8 \cdot 15 + 20 \cdot 3 - 16 \cdot 1 + 2 = 31$. A computer search over all 945 splits of ten people confirms 293. Dropping the last term gives the trap 295, and stopping after the pairs-of-pairs term gives 420.

24. Answer: (B) 13**. The idea.** For any modulus m, the Fibonacci numbers divisible by m are exactly those whose index is a multiple of the first such index $z(m)$. So we need $z(4)$ and $z(25)$, and then $100 \mid F_n$ iff n is a multiple of both.

Why the idea works. The identity $F_{a+b} = F_a F_{b+1} + F_{a-1} F_b$ (easily proved by induction on b) with $a = z = z(m)$ gives $F_{z+b} \equiv F_{z-1} F_b \pmod{m}$. Also F_{z-1} shares no prime factor with m: such a prime would divide F_z too, but consecutive Fibonacci numbers are coprime. So $m \mid F_{z+b}$ iff $m \mid F_b$, and the multiples of m sit exactly at the indices $z, 2z, 3z, \ldots$.

Work it out. Mod 4 the sequence runs $1, 1, 2, 3, 1, 0$, so $z(4) = 6$. Every multiple of 25 is a multiple of 5, and mod 5 the first zero is $F_5 = 5$, so $25 \mid F_n$ forces $5 \mid n$. Check the multiples of 5: $F_5 = 5$, $F_{10} = 55$, $F_{15} = 610$, $F_{20} = 6765$ are not multiples of 25, but $F_{25} = 75025 = 25 \cdot 3001$ is. So $z(25) = 25$.

Hence $100 \mid F_n$ iff $6 \mid n$ and $25 \mid n$, iff $150 \mid n$. Since $150 \cdot 13 = 1950 \le 2026 < 2100$, there are 13 such numbers.

Check. The last two digits of the Fibonacci numbers repeat with period 300, and a direct computation shows that within each period they are 00 exactly at $n \equiv 0$ and $n \equiv 150$, matching $150 \mid n$. Using 5 in place of 25 would give $30 \mid n$ and the wrong count 67.

25. Answer: (B) 126**. The idea.** Each exradius is the area divided by a simple length: $r_a = \frac{K}{s-a}$, where K is the area, s the semiperimeter and a the side the excircle touches. So the three exradii

give $s-a$, $s-b$, $s-c$ and (adding them) s, all as multiples of K. Heron's formula $K^2 = s(s-a)(s-b)(s-c)$ then contains K on both sides and can be solved for it.

Why $K = r_a(s-a)$. Let J be the centre of the excircle tangent to side $BC = a$ and to the extensions of $AB = c$ and $AC = b$; it lies at distance r_a from all three lines. Quadrilateral $ABJC$ is made of triangles JAB and JAC, and also of triangle ABC and triangle JBC. So $\frac{r_a c}{2} + \frac{r_a b}{2} = K + \frac{r_a a}{2}$, which gives $K = r_a \cdot \frac{b+c-a}{2} = r_a(s-a)$.

Work it out. So $s-a = \frac{K}{9}$, $s-b = \frac{K}{18}$ and $s-c = \frac{K}{21}$. Adding the three, $3s - 2s = s = K\left(\frac{1}{9} + \frac{1}{18} + \frac{1}{21}\right) = K \cdot \frac{14+7+6}{126} = \frac{3K}{14}$. Heron's formula becomes

$K^2 = \frac{3K}{14} \cdot \frac{K}{9} \cdot \frac{K}{18} \cdot \frac{K}{21} = \frac{K^4}{15876}$,

so $K^2 = 15876$ and $K = 126$.

Check. Then $s = 27$, $s-a = 14$, $s-b = 7$, $s-c = 6$, so the sides are 13, 20 and 21. Heron: $\sqrt{27 \cdot 14 \cdot 7 \cdot 6} = \sqrt{15876} = 126$, and the exradii are $\frac{126}{14} = 9$, $\frac{126}{7} = 18$, $\frac{126}{6} = 21$ as required. As a by-product the inradius is $\frac{126}{27} = \frac{14}{3}$, and indeed $\frac{1}{9} + \frac{1}{18} + \frac{1}{21} = \frac{3}{14}$ is its reciprocal: the reciprocals of the exradii always add up to the reciprocal of the inradius.