MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

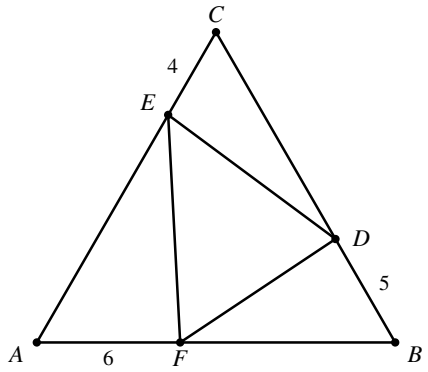
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1. A sequence is defined by $a_1 = 2$ and $a_{n+1} = a_n^2 + a_n + 1$ for every $n \geq 1$ (so $a_2 = 7$, $a_3 = 57$, and the terms grow very quickly). Find the remainder when a_6 is divided by 1000.

2. In triangle ABC, $\tan A = \dfrac{3}{4}$ and $\tan B = \dfrac{8}{15}$. The cosine of angle C can be written as $-\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

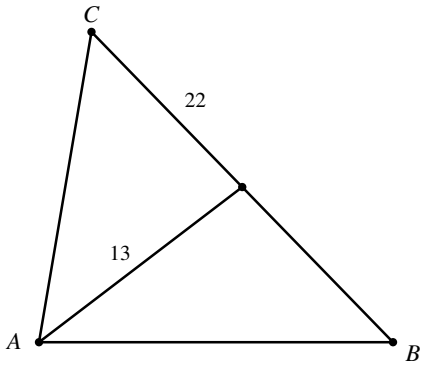
3. Positive real numbers a and b satisfy $a + \dfrac{1}{b} = \dfrac{5}{4}$ and $b + \dfrac{1}{a} = 5$. The product ab takes exactly two possible values; their sum is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Equilateral triangle ABC has side length 15. Point D lies on side $\overline{BC}$ with $BD = 5$, point E lies on side $\overline{CA}$ with $CE = 4$, and point F lies on side $\overline{AB}$ with $AF = 6$. The area of triangle DEF is $\dfrac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is not divisible by the square of any prime. Find $a + b + c$.



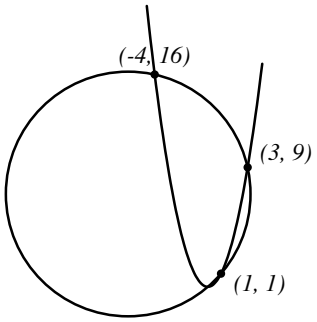
5. A row of 9 tiles contains four tiles labeled X, three labeled Y, and two labeled Z (tiles with the same label are not distinguishable from one another). In how many distinguishable arrangements of all nine tiles in a row is it true that no two X's are adjacent and no two Y's are adjacent? (Two Z's are allowed to be adjacent.)

6. In triangle ABC, $\cos A = \dfrac{1}{6}$, $BC = 22$, and the median from A to side $\overline{BC}$ has length 13. Find $AB \cdot AC$.



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7. A circle passes through the three points $(-4, 16)$, $(1, 1)$, and $(3, 9)$, which all lie on the parabola $y = x^2$. Find R^2, the square of the circle's radius.



8. Find the number of ordered pairs of positive integers (x, y) such that $x^2y + xy^2 + x + y = 2304$.

9. Find the number of ordered pairs of positive integers (a, b) with $1 \le a \le 50$ and $1 \le b \le 50$ such that $a^2 \equiv b \pmod 7$ and $b^2 \equiv a \pmod 7$.

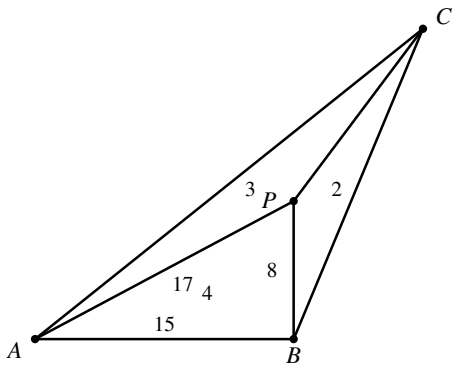
10. The 9 vertices of a regular 9-gon are each colored red, green, or blue so that no two vertices joined by a side of the 9-gon receive the same color. Two colorings are considered the same arrangement if one can be rotated (about the center of the 9-gon) to match the other; reflections are not identified with each other. Find the number of distinguishable colorings.

11. Real numbers w, x, y, z satisfy $w + x + y + z = 10$ and $w + 2x + 3y + 4z = 24$. Let S be the smallest possible value of $w^2 + x^2 + y^2 + z^2$ over all quadruples (w, x, y, z) satisfying both equations. If $S = m/n$ for relatively prime positive integers m and n, find $m + n$.

12. Find the number of ways to partition the set $\{1, 2, \ldots, 9\}$ into three nonempty subsets, each containing an odd number of elements. (A partition is an unordered collection of subsets covering every element exactly once; the order in which the three subsets are listed does not matter.)

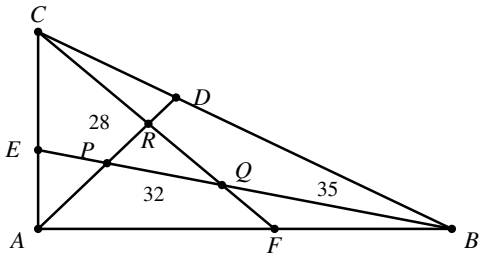
13. Point P lies inside triangle ABC so that $[PBC] : [PCA] : [PAB] = 2 : 3 : 4$, where $[XYZ]$ denotes the area of triangle XYZ. Given that $AB = 15$, $PA = 17$, and $PB = 8$, suppose $PC = \dfrac{m}{n}$ for relatively prime positive integers m and n. Find $m + n$.

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14. Each of the 16 unit cells of a 4×4 grid is independently colored black or white, each color with probability $1/2$. A 2×2 sub-square is any of the nine 2×2 blocks of cells aligned with the grid lines (sub-squares are allowed to overlap one another); such a sub-square is called monochromatic if all 4 of its cells share the same color. Let X be the number of monochromatic 2×2 sub-squares. If $E[X^2] = m/n$ for relatively prime positive integers m, n, find $m + n$.

15. Let D, E, F be points lying strictly between the endpoints of sides BC, CA, AB of triangle ABC, respectively. The lines AD, BE, CF are not all three concurrent, so they meet in pairs at three distinct points: P, the intersection of lines AD and BE; Q, the intersection of lines BE and CF; and R, the intersection of lines CF and AD. Suppose triangle ABC has area 105, triangle AFR has area 32, triangle BDP has area 35, and triangle CEQ has area 28. Find the area of triangle PQR.



Answer Key — AIME Practice Contest 9

1. 807

2. 121

3. 21

4. 42

5. 93

6. 288

7. 85

8. 4

9. 211

10. 58

11. 131

12. 820

13. 27

14. 29

15. 7

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Solutions — AIME Practice Contest 9

1. Answer: 807. **The idea.** We only need a_6 modulo 1000, and the rule $a_{n+1} = a_n^2 + a_n + 1$ only uses a_n through squaring and adding, both of which respect reduction mod 1000: if $a_n \equiv r \pmod{1000}$ then $a_{n+1} \equiv r^2 + r + 1 \pmod{1000}$. So we can replace every term by its last three digits as soon as we compute it, keeping the numbers small.

Work it out. $a_1 = 2$. $a_2 = 4 + 2 + 1 = 7$. $a_3 = 49 + 7 + 1 = 57$.

$a_4 = 57^2 + 57 + 1 = 3249 + 58 = 3307 \equiv 307 \pmod{1000}$.

$a_5 \equiv 307^2 + 307 + 1 = 94249 + 308 = 94557 \equiv 557 \pmod{1000}$.

$a_6 \equiv 557^2 + 557 + 1 = 310249 + 558 = 310807 \equiv \boxed{807} \pmod{1000}$.

Check. Computing without any reduction, the exact values are $a_1, \ldots, a_6 = 2,\ 7,\ 57,\ 3307,\ 10{,}939{,}557,\ 119{,}673{,}$ The last three digits of a_6 are indeed 807, confirming the shortcut of reducing at each step is valid.

2. Answer: 121. **The idea.** Since $A + B + C = 180°$, angle C is the supplement of $A + B$, so $\cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B$. Because $\tan A$ and $\tan B$ are both positive, A and B are both acute, so their sines and cosines are both positive and come straight from the two Pythagorean triples hiding in $\frac{3}{4}$ and $\frac{8}{15}$.

Work it out. From $\tan A = \dfrac{3}{4}$ (a 3-4-5 triangle), $\sin A = \dfrac{3}{5}$ and $\cos A = \dfrac{4}{5}$. From $\tan B = \dfrac{8}{15}$ (an 8-15-17 triangle), $\sin B = \dfrac{8}{17}$ and $\cos B = \dfrac{15}{17}$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \cdot \frac{15}{17} - \frac{3}{5} \cdot \frac{8}{17} = \frac{60}{85} - \frac{24}{85} = \frac{36}{85}.$$

So $\cos C = -\cos(A + B) = -\dfrac{36}{85}$. Since $85 = 5 \cdot 17$ and $36 = 2^2 \cdot 3^2$ share no common factor, $m + n = 36 + 85 = 121$.

Check. $\cos C < 0$ makes C obtuse, which is consistent: $A = \arctan\frac{3}{4} \approx 36.87°$ and $B = \arctan\frac{8}{15} \approx 28.07°$ sum to about 64.94°, well under 180°, leaving $C \approx 115.06°$ for a valid triangle. A fully independent numerical computation of $\cos\left(180° - \arctan\frac{3}{4} - \arctan\frac{8}{15}\right)$ gives $-0.42352941\ldots$, matching $-\frac{36}{85} = -0.42352941\ldots$ to high precision.

3. Answer: 21. **The idea.** Multiplying the two given equations produces ab, $\frac{1}{ab}$, and two constant 1's, so it turns the system into a single equation in $t = ab$ alone.

Work it out. Multiply: $\left(a + \dfrac{1}{b}\right)\left(b + \dfrac{1}{a}\right) = ab + 1 + 1 + \dfrac{1}{ab} = ab + \dfrac{1}{ab} + 2$. This equals $\dfrac{5}{4} \cdot 5 = \dfrac{25}{4}$, so with $t = ab$,

$$t + \frac{1}{t} = \frac{25}{4} - 2 = \frac{17}{4} \quad \Longrightarrow \quad 4t^2 - 17t + 4 = 0 \quad \Longrightarrow \quad (4t - 1)(t - 4) = 0,$$

giving $t = 4$ or $t = \dfrac{1}{4}$.

Both values actually occur: from $a + \frac{1}{b} = \frac{5}{4}$ and $ab = t$, multiplying by b gives $ab + 1 = \frac{5}{4}b$, so $b = \dfrac{4(t + 1)}{5}$. For $t = 4$: $b = 4$, $a = t/b = 1$; check $b + \frac{1}{a} = 4 + 1 = 5$. ✓ For $t = \frac{1}{4}$: $b = 1$,

$a = t/b = \frac{1}{4}$; check $b + \frac{1}{a} = 1 + 4 = 5$. ✓ So both $(a, b) = (1, 4)$ and $\left(\frac{1}{4}, 1\right)$ are genuine solutions, and ab really does take both values 4 and $\frac{1}{4}$.

The sum of the two values is $4 + \dfrac{1}{4} = \dfrac{17}{4}$, so $m + n = 17 + 4 = 21$.

Check. $(a, b) = (1, 4)$: $1 + \frac{1}{4} = \frac{5}{4}$ ✓, $4 + 1 = 5$ ✓. $(a, b) = \left(\frac{1}{4}, 1\right)$: $\frac{1}{4} + 1 = \frac{5}{4}$ ✓, $1 + 4 = 5$ ✓. A computer solve of the original system independently returns exactly these two pairs.

4. Answer: 42**. The idea.** Triangle DEF is what remains of $\triangle ABC$ after cutting off the three corner triangles at A, B, C. Each corner triangle has a 60° angle (an angle of the equilateral triangle), so its area is $\frac{1}{2} \cdot (\text{leg}_1)(\text{leg}_2) \sin 60°$. Subtracting the three corners from $[ABC]$ gives $[DEF]$.

Work it out. The side length is 15, so $[ABC] = \dfrac{\sqrt{3}}{4} \cdot 15^2 = \dfrac{225\sqrt{3}}{4}$.

At B: $BD = 5$, $BF = AB - AF = 15 - 6 = 9$, so $[BDF] = \dfrac{1}{2} \cdot 5 \cdot 9 \cdot \dfrac{\sqrt{3}}{2} = \dfrac{45\sqrt{3}}{4}$.

At C: $CE = 4$, $CD = BC - BD = 15 - 5 = 10$, so $[CED] = \dfrac{1}{2} \cdot 4 \cdot 10 \cdot \dfrac{\sqrt{3}}{2} = 10\sqrt{3} = \dfrac{40\sqrt{3}}{4}$.

At A: $AF = 6$, $AE = CA - CE = 15 - 4 = 11$, so $[AFE] = \dfrac{1}{2} \cdot 6 \cdot 11 \cdot \dfrac{\sqrt{3}}{2} = \dfrac{33\sqrt{3}}{2} = \dfrac{66\sqrt{3}}{4}$.

$$[DEF] = \frac{225\sqrt{3}}{4} - \frac{45\sqrt{3} + 40\sqrt{3} + 66\sqrt{3}}{4} = \frac{225\sqrt{3} - 151\sqrt{3}}{4} = \frac{74\sqrt{3}}{4} = \frac{37\sqrt{3}}{2}.$$

Since 37 and 2 are coprime and 3 is squarefree, $a + b + c = 37 + 3 + 2 = 42$.

Check. Placing $A = (0, 0)$, $B = (15, 0)$, $C = \left(\frac{15}{2}, \frac{15\sqrt{3}}{2}\right)$ and computing D, E, F from the given ratios, the shoelace formula for $\triangle DEF$ gives the same area $\frac{37\sqrt{3}}{2} \approx 32.04$, which is sensibly less than $[ABC] = \frac{225\sqrt{3}}{4} \approx 97.4$.

5. Answer: 93**. The idea.** First decide where the 3 Y's and 2 Z's go among themselves (ignoring the X's), then insert the 4 X's into the gaps of that row of 5 tiles. Putting at most one X into any single gap guarantees no two X's end up adjacent. But if two Y's already sit next to each other in the row of 5, the *only* way to keep them apart is to place an X into the gap between them — that gap's choice is forced, not free. So we split into cases by how the 3 Y's are grouped into blocks of consecutive Y's.

Work it out. There are $\binom{5}{3} = 10$ ways to arrange $YYYZZ$ in a row of 5. By the sizes of the maximal runs of Y's, these split into three shapes:

Shape 3 (one block YYY): arrangements $YYYZZ$, $ZYYYZ$, $ZZYYY$ — 3 of them. The block's 2 internal gaps must each get one X (forced), using 2 of the 4 X's. The row now has 3 units (YYY, Z, Z), hence 4 outer gaps for the remaining 2 X's: $\binom{4}{2} = 6$ ways. Subtotal: $3 \cdot 6 = 18$.

Shape 2 + 1 (blocks YY and Y separately): the other 6 arrangements ($YYZYZ$, $YYZZY$, $YZYYZ$, $YZZYY$, $ZYYZY$, $ZYZYY$). The one internal gap inside YY is forced to get an X, using 1 of the 4. The row has 4 units, hence 5 outer gaps for the remaining 3 X's: $\binom{5}{3} = 10$ ways. Subtotal: $6 \cdot 10 = 60$.

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Shape 1 + 1 + 1 (all Y's separated already): only $YZYZY$. No internal gaps are forced. The row has 5 units, hence 6 outer gaps for all 4 X's: $\binom{6}{4} = 15$ ways. Subtotal: $1 \cdot 15 = 15$.

Total: $18 + 60 + 15 = 93$.

Check. A direct computer enumeration of all $\binom{9}{4,3,2} = 1260$ distinguishable arrangements, filtering for no adjacent equal X's and no adjacent equal Y's, also gives 93. For a small sanity check by hand: the arrangement $XYXYXYXZZ$ has no two X's or Y's adjacent (and ends in the allowed ZZ), so it is correctly among the 93.

6. Answer: 288**. The idea.** Let $b = AC$ and $c = AB$. The Law of Cosines relates $b^2 + c^2$ and bc to BC, and the median-length formula relates $b^2 + c^2$ (again) to the median. Together they pin down bc without ever having to find b and c separately.

Work it out. The median-length formula for the median m_a from A states $m_a^2 = \dfrac{2b^2 + 2c^2 - BC^2}{4}$. With $m_a = 13$ and $BC = 22$: $4 \cdot 13^2 = 2(b^2+c^2) - 22^2$, so $2(b^2+c^2) = 4 \cdot 169 + 484 = 676 + 484 = 1160$, giving $b^2 + c^2 = 580$.

The Law of Cosines gives $BC^2 = b^2 + c^2 - 2bc\cos A$, so $484 = 580 - 2bc \cdot \dfrac{1}{6} = 580 - \dfrac{bc}{3}$. Then $\dfrac{bc}{3} = 580 - 484 = 96$, so $bc = 288$.

Hence $AB \cdot AC = bc = 288$.

Check. Solving $b^2 + c^2 = 580$ and $bc = 288$ directly gives $b = 16$, $c = 18$ (or the reverse): indeed $16^2 + 18^2 = 256 + 324 = 580$ and $16 \cdot 18 = 288$. With sides $AB = 18$, $AC = 16$, $BC = 22$, the Law of Cosines gives $\cos A = \dfrac{16^2 + 18^2 - 22^2}{2 \cdot 16 \cdot 18} = \dfrac{580 - 484}{576} = \dfrac{96}{576} = \dfrac{1}{6}$ ✓, and the median formula gives $m_a^2 = \dfrac{2(256 + 324) - 484}{4} = \dfrac{1160 - 484}{4} = \dfrac{676}{4} = 169$, so $m_a = 13$ ✓.

7. Answer: 85**. The idea.** Write the circle as $x^2 + y^2 + Dx + Ey + F = 0$. Substituting $y = x^2$ turns each of the three given points into a root of the same quartic equation in x, which must therefore have a fourth root too (a circle and a parabola meet in at most 4 points). The quartic has no x^3 term, so the four roots sum to 0 — enough to pin down the missing fourth intersection, and from there, every coefficient.

Work it out. Substituting (x, x^2) into the circle's equation: $x^2 + x^4 + Dx + Ex^2 + F = 0$, i.e.

$x^4 + (1 + E)x^2 + Dx + F = 0$.

This quartic has roots $-4, 1, 3$, and a fourth root r. Since there is no x^3 term, the sum of all four roots is 0: $-4 + 1 + 3 + r = 0$, so $r = 0$.

Now match Vieta's formulas for the monic quartic with roots $-4, 1, 3, 0$:

Sum of pairwise products: $(-4)(1)+(-4)(3)+(-4)(0)+(1)(3)+(1)(0)+(3)(0) = -4-12+3 = -13$, which equals the x^2-coefficient $1 + E$, so $E = -14$.

Sum of triple products: $(-4)(1)(3)+(-4)(1)(0)+(-4)(3)(0)+(1)(3)(0) = -12$. For the quartic $x^4 + (1+E)x^2 + Dx + F$, the coefficient of x is D, and it equals $-$(sum of triple products) by Vieta, so $D = 12$.

Product of all four roots: $(-4)(1)(3)(0) = 0 = F$.

So $D = 12$, $E = -14$, $F = 0$, and $R^2 = \dfrac{D^2 + E^2 - 4F}{4} = \dfrac{144 + 196}{4} = \dfrac{340}{4} = 85$.

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Check. Finding the circumcenter directly, by intersecting the perpendicular bisectors of the segments joining the three given points, gives the center $(-6, 7)$, and the squared distance from this center to $(1, 1)$ is $(-6-1)^2 + (7-1)^2 = 49 + 36 = 85$, matching. The same center is also equidistant from $(-4, 16)$: $(-6+4)^2 + (7-16)^2 = 4 + 81 = 85$. ✓

8. Answer: 4. **The idea.** Factor the left side by grouping: $x^2y + xy^2 + x + y = xy(x+y) + (x+y) = (x+y)(xy+1)$. Writing $s = x + y$ and $p = xy$, the equation becomes $s(p+1) = 2304$, so s must be a divisor of 2304 and $p = \dfrac{2304}{s} - 1$. For an actual pair of positive integers with this sum and product, $x, y = \dfrac{s \mp \sqrt{s^2 - 4p}}{2}$, so $s^2 - 4p$ must be a perfect square with the same parity as s.

Work it out. Since $x, y \geq 1$, AM–GM gives $s^2 \geq 4p = 4\left(\frac{2304}{s} - 1\right)$, i.e. $s^3 + 4s \geq 9216$; this fails at $s = 20$ ($8000 + 80 = 8080 < 9216$) and holds from $s = 21$ on, so only divisors of $2304 = 2^8 \cdot 3^2$ with $s \geq 21$ can work: $24, 32, 36, 48, 64, 72, 96, 128, 144, 192, 256, 288, 384, 576, 768, 1152$. Computing $p = 2304/s - 1$ and the discriminant $s^2 - 4p$ for each of these: $s = 24$ gives $p = 95$ and discriminant $576 - 380 = 196 = 14^2$ (even, matching s's parity), giving $(x, y) = (5, 19)$; $s = 48$ gives $p = 47$ and discriminant $2304 - 188 = 2116 = 46^2$, giving $(x, y) = (1, 47)$. Every other divisor on the list gives a discriminant that is not a perfect square. Each unordered pair corresponds to 2 ordered pairs, for $2 + 2 = 4$ in all.

Check. Direct substitution: $(5, 19)$ gives $5 \cdot 19 \cdot 24 + 24 = 2280 + 24 = 2304$ ✓, and $(1, 47)$ gives $1 \cdot 47 \cdot 48 + 48 = 2256 + 48 = 2304$ ✓. A direct scan of every pair with $1 \leq x, y \leq 200$ confirms these 4 ordered pairs are the only solutions.

9. Answer: 211. **The idea.** Substituting $b \equiv a^2$ into $b^2 \equiv a \pmod 7$ gives $a^4 \equiv a$, i.e. $a(a^3 - 1) \equiv 0 \pmod 7$. Since 7 is prime, either $a \equiv 0$ or $a^3 \equiv 1 \pmod 7$. The nonzero residues mod 7 form a cyclic group of order 6 under multiplication, so the solutions of $t^3 \equiv 1$ form its unique subgroup of order $\gcd(3, 6) = 3$, which is $\{1, 2, 4\}$ (indeed $2^3 = 8 \equiv 1$ and $4^3 = 64 \equiv 1 \pmod 7$).

Work it out. So a mod 7 must be one of $0, 1, 2, 4$, and then $b \equiv a^2 \pmod 7$ is forced to a single residue: $0 \to 0$, $1 \to 1$, $2 \to 4$, $4 \to 2$. Among $1, \ldots, 50$ (which is $7 \cdot 7 + 1$ numbers), residue $1 \pmod 7$ occurs 8 times and every other residue occurs 7 times. The a-residues $0, 1, 2, 4$ occur $7, 8, 7, 7$ times, and their forced b-residues $0, 1, 4, 2$ occur $7, 8, 7, 7$ times respectively. So the count is $7 \cdot 7 + 8 \cdot 8 + 7 \cdot 7 + 7 \cdot 7 = 49 + 64 + 49 + 49 = 211$.

Check. A direct scan of all $50 \times 50 = 2500$ pairs, testing both congruences directly, confirms 211 pairs exactly.

10. Answer: 58. **The idea.** This is a job for Burnside's lemma: the number of colorings up to rotation is the average, over the 9 rotations of the 9-gon (including the identity), of the number of proper colorings fixed by that rotation.

Work it out. The identity fixes every proper 3-coloring of the 9-cycle; the chromatic polynomial of a cycle C_n with k colors is $(k-1)^n + (-1)^n(k-1)$, so with $n = 9, k = 3$ there are $2^9 - 2 = 510$ of them. A rotation by r steps fixes a coloring only if the coloring is invariant under shifting by r, which forces it to be built by repeating a block of length $g = \gcd(9, r)$ around the cycle; that repeating block must itself be a proper coloring of a g-cycle (with the extra requirement that it closes up properly, which the cycle chromatic polynomial already accounts for). For $g = 1$ (the six rotations $r = 1, 2, 4, 5, 7, 8$) this needs a proper coloring of a 1-cycle, impossible since a vertex

would be adjacent to itself with a repeated color — 0 fixed colorings. For $g = 3$ (the two rotations $r = 3, 6$) it needs a proper 3-coloring of a 3-cycle: $2^3 - 2 = 6$ of them, and each extends uniquely to the 9-gon. Averaging over all 9 rotations: $(510 + 6 \cdot 0 + 2 \cdot 6)/9 = 522/9 = 58$.

Check. Enumerating all $3^9 = 19683$ colorings directly, keeping the 510 proper ones, and grouping them into orbits under the 9 rotations by brute force (rather than trusting the Burnside count) gives exactly 58 orbits.

11. Answer: 131**. The idea.** Write $v = (w, x, y, z)$ and let $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 2, 3, 4)$, so the constraints say $a_1 \cdot v = 10$ and $a_2 \cdot v = 24$. Split $v = v_\parallel + v_\perp$ where $v_\parallel$ is a combination $\lambda a_1 + \mu a_2$ and $v_\perp$ is orthogonal to both a_1 and a_2. Since $a_1 \cdot v_\perp = a_2 \cdot v_\perp = 0$, the value of $v_\parallel$ is completely determined by the two constraints and does not depend on $v_\perp$ at all — but $|v|^2 = |v_\parallel|^2 + |v_\perp|^2 \geq |v_\parallel|^2$, with equality exactly when $v_\perp = 0$. So the minimum is achieved uniquely at $v = v_\parallel = \lambda a_1 + \mu a_2$, and it suffices to find λ, μ.

Work it out. Using $a_1 \cdot a_1 = 4$, $a_1 \cdot a_2 = 10$, $a_2 \cdot a_2 = 30$, the constraints become $4\lambda + 10\mu = 10$ and $10\lambda + 30\mu = 24$. Solving, $\mu = -1/5$ and $\lambda = 3$, so $v = 3(1, 1, 1, 1) - \frac{1}{5}(1, 2, 3, 4) = \left(\frac{14}{5}, \frac{13}{5}, \frac{12}{5}, \frac{11}{5}\right)$. One checks this indeed satisfies both original equations. Then $S = \frac{14^2+13^2+12^2+11^2}{25} = \frac{196+169+144+121}{25} = \frac{630}{25} = \frac{126}{5}$. Since $\gcd(126, 5) = 1$, $m + n = 126 + 5 = 131$.

Check. Any other valid quadruple is $v + t\,d$ for a direction d with $a_1 \cdot d = a_2 \cdot d = 0$ (e.g. $d = (1, -2, 1, 0)$ or $d = (0, 1, -2, 1)$); substituting shows $|v + td|^2 = \frac{126}{5} + t^2|d|^2 + 2t(v \cdot d)$, and $v \cdot d = 0$ for every such d (as it must be, since v is the orthogonal-decomposition minimizer), so the value only increases as t moves away from 0.

12. Answer: 820**. The idea.** Three positive odd integers summing to 9 can only be, as an unordered triple of sizes, $\{1, 1, 7\}$, $\{1, 3, 5\}$, or $\{3, 3, 3\}$ (any other combination either uses an even number or exceeds 9). Count partitions of each shape separately.

Work it out. Shape $\{1, 1, 7\}$: choose the 7-element block in $\binom{9}{7} = 36$ ways; the other two elements automatically form two singleton blocks, with no further choice since singletons can't be confused with each other once their contents are fixed — 36 partitions. Shape $\{1, 3, 5\}$: choose the singleton in 9 ways, then the size-3 block from the remaining 8 elements in $\binom{8}{3} = 56$ ways (the last 5 elements form the remaining block) — $9 \cdot 56 = 504$ partitions. Shape $\{3, 3, 3\}$: the number of ways to split 9 labeled elements into three unordered blocks of size 3 is $\dfrac{9!}{3!\,3!\,3!\,3!} = \dfrac{362880}{6 \cdot 6 \cdot 6 \cdot 6} = 280$ (divide by 3! at the end since the three size-3 blocks are themselves unordered). Adding: $36+504+280 = 820$.

Check. Enumerating all sequences that assign each of the 9 elements to one of 3 labeled boxes, keeping those that use all three boxes with every box of odd size, and collapsing labelings that produce the same unordered partition, gives 820 distinct partitions directly.

13. Answer: 27**. The idea.** The area ratios say P has barycentric weights $(2, 3, 4)$ relative to (A, B, C) — precisely $P = \dfrac{2A + 3B + 4C}{9}$, since each vertex's weight is proportional to the area of the sub-triangle *not* containing it. For $P = \dfrac{w_A A + w_B B + w_C C}{W}$ (with $W = w_A + w_B + w_C$), a direct expansion gives the weighted "parallel-axis" identity $W \cdot PX^2 = w_A \cdot XA^2 + w_B \cdot XB^2 + w_C \cdot XC^2 - \dfrac{1}{W}\left(w_A w_B\, AB^2 + w_B w_C\, BC^2 + w_C w_A\, CA^2\right)$ for any point X; taking $X = A, B, C$ in turn

(one term vanishes each time) writes PA^2, PB^2, PC^2 as linear combinations of $a^2 = BC^2$, $b^2 = CA^2$, $c^2 = AB^2$.

Work it out. With $w_A = 2, w_B = 3, w_C = 4, W = 9$ and $K = \frac{1}{9}(6c^2 + 12a^2 + 8b^2)$ the shared correction term: $9PA^2 = 3c^2 + 4b^2 - K$, $9PB^2 = 2c^2 + 4a^2 - K$, $9PC^2 = 2b^2 + 3a^2 - K$. With $c = 15$: subtracting the first two equations cancels K and gives $4b^2 - 4a^2 = 9(PA^2 - PB^2) - c^2 = 9(289-64) - 225 = 1800$, so $b^2 - a^2 = 450$. Substituting K into the PA^2 equation and simplifying gives a second linear relation, $7b^2 - 3a^2 = 4671$. Solving the two linear equations: $a^2 = \frac{1521}{4}$ (so $a = \frac{39}{2}$) and $b^2 = \frac{3321}{4}$ — both consistent with the triangle inequality against $c = 15$. Finally $9PC^2 = 2b^2 + 3a^2 - K = \frac{5625}{4}$, so $PC^2 = \frac{625}{4}$ and $PC = \frac{25}{2}$. Since $\gcd(25, 2) = 1$, $m + n = 27$.

Check. Placing $A = (0, 0)$, $B = (15, 0)$ and solving for C from $BC = \frac{39}{2}$, $CA = \frac{9\sqrt{41}}{2}$ gives $C = \left(\frac{45}{2}, 18\right)$. Then $P = \dfrac{2A + 3B + 4C}{9} = (15, 8)$, and indeed $PA = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ ✓, $PB = \sqrt{0^2 + 8^2} = 8$ ✓, and $PC = \sqrt{\left(\frac{45}{2} - 15\right)^2 + (18 - 8)^2} = \sqrt{7.5^2 + 10^2} = \sqrt{156.25} = \frac{25}{2}$ ✓— every given value matches exactly.

14. Answer: 29. **The idea.** Label the nine 2×2 sub-squares by the row and column (0, 1, 2 each) of their top-left cell, and let I_s be the indicator that sub-square s is monochromatic. Then $X = \sum_s I_s$, so $X^2 = \sum_s I_s + 2\sum_{s<t} I_s I_t$, and by linearity $E[X^2] = \sum_s E[I_s] + 2\sum_{s<t} E[I_s I_t]$. Each term $E[I_s I_t]$ only depends on how many cells sub-squares s, t share.

Work it out. A single sub-square is monochromatic with probability $2 \cdot (1/2)^4 = 1/8$, contributing $9 \cdot \frac{1}{8} = \frac{9}{8}$ to the sum. For a pair of sub-squares, if they share k cells, the two squares are simultaneously monochromatic exactly when all $8 - k$ distinct cells among them share one common color, which has probability $2 \cdot (1/2)^{8-k}$ (if $k \geq 1$, since shared cells force both squares to agree) or, when $k = 0$, the independent product $(1/8)(1/8) = 2^{-6}$ — which happens to equal the $k = 1$ value too. Checking all $\binom{9}{2} = 36$ pairs of the nine sub-squares by their relative offset: 16 pairs share 0 cells (probability $1/64$ each), 8 pairs share exactly 1 cell, diagonally (probability $1/64$ each), and 12 pairs share exactly 2 cells, edge-adjacent (probability $1/32$ each). So $2\sum_{s<t} E[I_s I_t] = 2\left[16 \cdot \frac{1}{64} + 8 \cdot \frac{1}{64} + 12 \cdot \frac{1}{32}\right] = 2\left[\frac{24}{64} + \frac{24}{64}\right] = 2 \cdot \frac{48}{64} = \frac{3}{2}$. Total: $E[X^2] = \frac{9}{8} + \frac{3}{2} = \frac{9}{8} + \frac{12}{8} = \frac{21}{8}$. Since $\gcd(21, 8) = 1$, $m + n = 29$.

Check. Looping over all $2^{16} = 65536$ colorings, computing X exactly for each by checking all nine sub-squares, and averaging X^2 gives exactly $21/8$, confirming the count-based derivation.

15. Answer: 7. **The idea.** Place $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$ (area ratios are unaffected by this affine choice). Let $x = BD/DC$, $y = CE/EA$, $z = AF/FB$, so $D = \frac{B + xC}{x+1}$, $E = \frac{C + yA}{y+1}$, $F = \frac{A + zB}{z+1}$. Writing out the three cevian lines and intersecting them pairwise (to get P, Q, R) and computing the resulting triangle areas as exact rational functions of x, y, z gives, as fractions of $[ABC]$: $[AFR] = \frac{xz^2}{xz^2+xz+z^2+2z+1}$, $[BDP] = \frac{x^2 y}{(x+1)(xy+x+1)}$, and $[CEQ] = \frac{y^2 z}{y^2 z+y^2+yz+2y+1}$. (These are not quoted from memory — they come from direct coordinate computation, the same way Routh's theorem itself is proved.) The given data says these equal $32/105$, $35/105$, $28/105$.

Work it out. Solving the three equations simultaneously (eliminating one variable at a time leaves a polynomial in x alone whose every coefficient besides the forced root is positive, so no other positive root exists) gives the unique positive solution $x = 2$, $y = 3/2$, $z = 4/3$ — meaning $BD : DC = 2 : 1$, $CE : EA = 3 : 2$, $AF : FB = 4 : 3$. Note $xyz = 4 \neq 1$, consistent with the cevians not being concurrent (Ceva's theorem says concurrency would need $xyz = 1$).

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Now finish with the same coordinate method used above: placing the real triangle as $A = (0, 0)$, $B = (21, 0)$, $C = (0, 10)$ (which has area 105), the section-formula points D, E, F for these ratios and the pairwise intersections P, Q, R can be computed exactly, and they reproduce $[ABC] = 105$, $[AFR] = 32$, $[BDP] = 35$, $[CEQ] = 28$ exactly as given, with $[PQR] = 7$.

Check. The three given areas plus $[PQR] = 7$ leave $105 - (32 + 35 + 28 + 7) = 3$ for the three remaining quadrilateral regions combined, and computing those three quadrilateral areas directly from the same coordinates gives exactly 3 in total — the seven pieces exactly tile triangle ABC, confirming the whole configuration is consistent.

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