MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

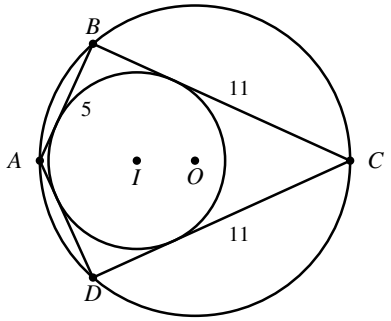
MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. For how many integers n with $1 \le n \le 1000$ does n have an odd number of digits when written in base 2 *and* an odd number of digits when written in base 3?

2. Twenty identical coins are handed out to five children, Ann, Ben, Cal, Dee and Eli, so that every child gets at least one coin and no two children get the same number of coins. In how many ways can this be done?

3. A sequence starts with $a_1 = 2026$. For each $n \ge 1$, the next term a_{n+1} is obtained by adding to a_n the largest divisor of a_n that is smaller than a_n. (For example, the largest divisor of 12 that is smaller than 12 is 6.) Find the remainder when a_{100} is divided by 1000.

4. Quadrilateral $ABCD$ is convex, can be inscribed in a circle, and also has a circle tangent to all four of its sides. Its sides satisfy $AB = 5$ and $BC = CD = 11$. Let O be the center of the circle through A, B, C, D, and let I be the center of the circle tangent to the four sides. Then $OI^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



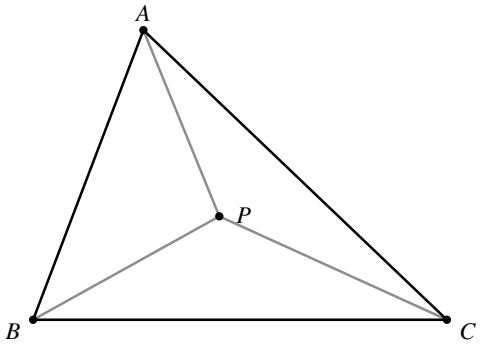
5. For each positive integer n let $f(n) = \left\lfloor \dfrac{2n}{\lfloor \sqrt{n} \rfloor} \right\rfloor$, where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. Find the remainder when $f(1) + f(2) + \cdots + f(2025)$ is divided by 1000.

6. A line meets the parabola $y = x^2$ at two points, and both of these points have integer coordinates. The line and the parabola together bound a region. Exactly 275 points with integer coordinates lie strictly inside this region (not on the line and not on the parabola). Find the area of the region.

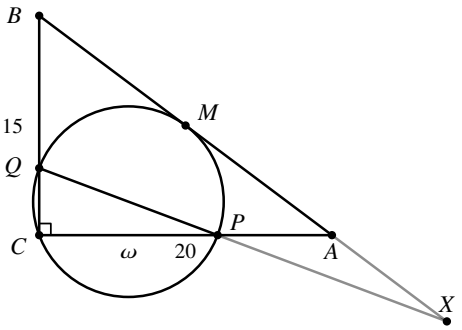
7. Let $z_1, z_2, \ldots, z_{250}$ be the 250 complex numbers z with $z^{250} = 1$. The sum
$$\sum_{k=1}^{250} \frac{1}{1 + z_k + z_k^2}$$
is a rational number $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. A point P is chosen uniformly at random inside triangle ABC. The segments PA, PB, PC cut the triangle into three smaller triangles PBC, PCA and PAB. List their three areas from largest to smallest. The probability that the largest area is at most twice the middle one *and* the middle one is at most twice the smallest is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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9. Right triangle ABC has its right angle at C, with $CA = 20$ and $CB = 15$. Let M be the midpoint of the hypotenuse $\overline{AB}$. Circle ω passes through C and is tangent to line AB at M. Circle ω meets segment $\overline{CA}$ again at P and segment $\overline{CB}$ again at Q, and line PQ meets line AB at X. Then $XM = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



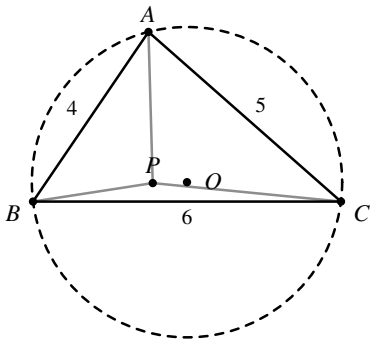
10. For each positive integer n, let $f(n)$ be the number of ordered pairs (a, b) of positive integers with $a + b = n$ such that ab is divisible by n. For example, $f(4) = 1$ because of the pair $(2, 2)$, and $f(6) = 0$. Find $f(1) + f(2) + f(3) + \cdots + f(500)$.

11. Eleven people, numbered 1 through 11, sit in eleven chairs numbered 1 through 11, person i in chair i. They all stand up and sit down again, one person per chair, so that for every person the difference between the new chair number and the old one is *not* a multiple of 3. (A difference of 0 is a multiple of 3, so everyone must move.) Let N be the number of possible new seatings. Find the remainder when N is divided by 1000.

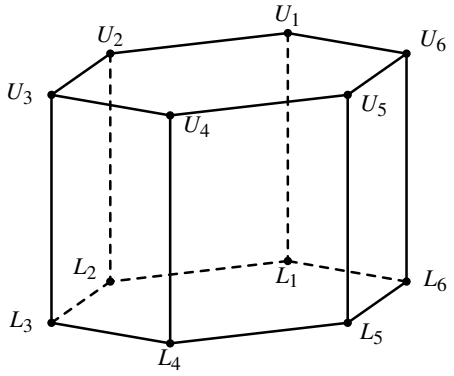
12. Ten cards lie in a row in positions $1, 2, \ldots, 10$. For each i with $1 \leq i \leq 9$, the move s_i exchanges the two cards currently in positions i and $i + 1$. A dealer chooses some of the nine moves (possibly none of them, possibly all nine) and performs each chosen move exactly once, in whatever order she likes. Let N be the number of different arrangements of the ten cards that can result. Find the remainder when N is divided by 1000.

13. Triangle ABC has $AB = 4$, $AC = 5$ and $BC = 6$, and O is the centre of its circumcircle. A point P

inside the triangle satisfies $\angle PAB = \angle PCA$ and $\angle PAC = \angle PBA$. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



14. A hexagonal prism has 12 vertices: $U_1, U_2, \ldots, U_6$ around the top hexagon and $L_1, L_2, \ldots, L_6$ around the bottom one, with U_i directly above L_i. Its six side faces are the rectangles $U_iU_{i+1}L_{i+1}L_i$ for $i = 1, \ldots, 6$ (where $U_7 = U_1$ and $L_7 = L_1$). Each of the 12 vertices is painted red, green or blue. Let N be the number of paintings in which each of the six side faces has all three colours among its four vertices (the two hexagonal faces are unrestricted). Find the remainder when N is divided by 1000.



15. Twelve points are equally spaced around a circle. They are joined in six pairs by six chords, each point being an endpoint of exactly one chord. In how many of these ways is the number of pairs of chords that cross each other (meet at a point inside the circle) exactly two?

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Answer Key — AIME Practice Contest 8

1. 59

2. 840

3. 598

4. 785

5. 674

6. 288

7. 253

8. 83

9. 653

10. 729

11. 312

12. 181

13. 233

14. 592

15. 990

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Solutions — AIME Practice Contest 8

1. Answer: 59**. The idea.** A positive integer has exactly k digits in base b exactly when $b^{k-1} \le n \le b^k - 1$. So "odd length" picks out a union of intervals for each base, and we just intersect the two unions.

Work it out. Base 2, lengths $1, 3, 5, 7, 9$: $n \in [1, 1] \cup [4, 7] \cup [16, 31] \cup [64, 127] \cup [256, 511]$. (Length 11 starts at $1024 > 1000$.)

Base 3, lengths $1, 3, 5, 7$: $n \in [1, 2] \cup [9, 26] \cup [81, 242] \cup [729, 2186]$.

Now intersect interval by interval. $[1, 1]$ meets $[1, 2]$ in $\{1\}$: 1 number. $[4, 7]$ lies in the gap $[3, 8]$ of the base-3 list: 0. $[16, 31] \cap [9, 26] = [16, 26]$: 11 numbers. $[64, 127] \cap [81, 242] = [81, 127]$: 47 numbers. $[256, 511]$ lies inside the gap $[243, 728]$: 0. Nothing from $[729, 1000]$ has odd binary length, since those numbers have 10 binary digits.

Total: $1 + 11 + 47 = 59$.

Check. Spot values: $26 = 11010_2$ (5 digits) $= 222_3$ (3 digits), counted; $27 = 1000_3$ has 4 ternary digits, excluded; $127 = 1111111_2$ (7 digits) $= 11201_3$ (5 digits), counted; 128 has 8 binary digits, excluded. The boundaries behave as claimed.

2. Answer: 840**. The idea.** First find the possible sets of five different amounts, then hand them out. Subtracting $0, 1, 2, 3, 4$ from the amounts in increasing order turns "five different positive numbers with sum 20" into "five positive numbers, repetition allowed, with sum 10", which is a very short list.

Work it out. Sort the amounts as $x_1 < x_2 < x_3 < x_4 < x_5$ and put $y_i = x_i - (i - 1)$. Since the x_i are different integers, $x_{i+1} \ge x_i + 1$, so $y_1 \le y_2 \le \cdots \le y_5$, with $y_1 = x_1 \ge 1$ and

$y_1 + \cdots + y_5 = 20 - (0 + 1 + 2 + 3 + 4) = 10.$

Conversely any such y's give different x's. So we need the ways to write 10 as a sum of five positive parts in non-decreasing order: the parts are 1's plus a little extra. They are

6+1+1+1+1, 5+2+1+1+1, 4+3+1+1+1, 4+2+2+1+1, 3+3+2+1+1, 3+2+2+2+1, 2+2+2+2+2

(7 in all: distribute the 5 units above the five 1's, which is the same as writing 5 as a sum of at most five positive parts, and 5 has exactly 7 such sums). So there are 7 possible sets of amounts, for example $\{1, 2, 3, 4, 10\}$ from the first.

Each set consists of five different numbers, and they can be given to the five children in $5! = 120$ ways. The total is $7 \cdot 120 = 840$.

Check. The seven sets of amounts are $\{1, 2, 3, 4, 10\}$, $\{1, 2, 3, 5, 9\}$, $\{1, 2, 3, 6, 8\}$, $\{1, 2, 4, 5, 8\}$, $\{1, 2, 4, 6, 7\}$, $\{1, 3, 4, 5, 7\}$ and $\{2, 3, 4, 5, 6\}$, each of sum 20, and no other set of five different positive integers adds to 20 (a computer search agrees). The answer is **840**.

3. Answer: 598**. The idea.** The largest divisor of N smaller than N is N/p, where p is the smallest prime factor of N. So each step multiplies the term by $1 + \frac{1}{p}$. Starting from $2026 = 2 \cdot 1013$ with 1013 prime, the factor 1013 just rides along, and the rest of the number cycles through a pattern that gains a factor 3 every three steps.

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Work it out. If p is the smallest prime factor of N, then N/p is a divisor, and any divisor $d < N$ has $N/d \geq p$ (as $N/d > 1$ has a prime factor, which is at least p), so $d \leq N/p$. Hence $a_{n+1} = a_n \left(1 + \frac{1}{p}\right)$.

The number 1013 is prime (it is not divisible by any prime up to 31, and $32^2 > 1013$). Write $a_n = 1013\, k_n$ with $k_1 = 2$. As long as $k_n > 1$ uses only the primes 2 and 3, the smallest prime factor of a_n is the smallest prime factor of k_n. Now

$$2 \cdot 3^j \xrightarrow{\times\frac{3}{2}} 3^{j+1} \xrightarrow{\times\frac{4}{3}} 4 \cdot 3^j \xrightarrow{\times\frac{3}{2}} 2 \cdot 3^{j+1},$$

so k runs through $2, 3, 4, 6, 9, 12, 18, 27, 36, \ldots$ and always $k_{3j+1} = 2 \cdot 3^j$. Since $100 = 3 \cdot 33 + 1$, $a_{100} = 1013 \cdot 2 \cdot 3^{33}$.

Modulo 1000: $3^{10} = 59049 \equiv 49$, $3^{20} \equiv 49^2 = 2401 \equiv 401$, $3^{30} \equiv 401 \cdot 49 = 19649 \equiv 649$, and $3^{33} \equiv 649 \cdot 27 = 17523 \equiv 523$. Also $1013 \cdot 2 \equiv 26$. So

$$a_{100} \equiv 26 \cdot 523 = 13598 \equiv 598 \pmod{1000}.$$

Check. The first terms are 2026, $2026 + 1013 = 3039$, $3039 + 1013 = 4052$, $4052 + 2026 = 6078$, $6078 + 3039 = 9117 = 9 \cdot 1013$, matching $k = 2, 3, 4, 6, 9$. Running the rule directly for 99 steps on a computer also gives last three digits 598. The answer is **598**.

4. Answer: 785**. The idea.** A tangential quadrilateral has equal sums of opposite sides, which pins down DA. The quadrilateral is then a kite, symmetric about AC, and being cyclic forces right angles at B and D. Both centers lie on the symmetry line AC, so OI is a difference of two distances along AC.

Work it out. Tangent lengths from each vertex are equal, so $AB + CD = BC + DA$, giving $5 + 11 = 11 + DA$ and $DA = 5$.

Triangles ABC and ADC are congruent (SSS: $AB = AD$, $CB = CD$, AC shared), so $\angle B = \angle D$. In a cyclic quadrilateral $\angle B + \angle D = 180°$, so $\angle B = \angle D = 90°$. Hence AC is a diameter, $AC^2 = 5^2 + 11^2 = 146$, and O is the midpoint of AC: $AO = \frac{\sqrt{146}}{2}$.

The kite is symmetric in line AC, so its incircle (unique) is too, and I lies on AC. The incenter also lies on the bisector of $\angle B$ (it is equidistant from BA and BC). In triangle ABC the bisector from B meets AC at the point dividing it in the ratio $AB : BC = 5 : 11$, so $AI = \frac{5}{16}\sqrt{146}$.

Therefore $OI = AO - AI = \left(\frac{8}{16} - \frac{5}{16}\right)\sqrt{146} = \frac{3\sqrt{146}}{16}$ and

$$OI^2 = \frac{9 \cdot 146}{256} = \frac{657}{128}.$$

Since $657 = 9 \cdot 73$ is odd, the fraction is in lowest terms, and $m + n = 657 + 128 = 785$.

Check. Fuss's relation for a bicentric quadrilateral says $\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r^2}$, i.e. $\frac{2(R^2+d^2)}{(R^2-d^2)^2} = \frac{1}{r^2}$. Here the area is $2 \cdot \frac{1}{2} \cdot 5 \cdot 11 = 55$ and the semiperimeter is 16, so $r = \frac{55}{16}$. With $R^2 = \frac{146}{4} = \frac{4672}{128}$ and $d^2 = \frac{657}{128}$: $R^2 - d^2 = \frac{4015}{128}$, $R^2 + d^2 = \frac{5329}{128} = \frac{73^2}{128}$, and $\frac{2 \cdot 73^2 \cdot 128}{4015^2} = \frac{2 \cdot 73^2 \cdot 128}{(5 \cdot 11 \cdot 73)^2} = \frac{256}{3025} = \frac{1}{r^2}$. It checks.

5. Answer: 674**. The idea.** Group n by the value $m = \lfloor \sqrt{n} \rfloor$: the block for m is $n = m^2 + j$ with $0 \leq j \leq 2m$. There $f(n) = 2m + \left\lfloor \frac{2j}{m} \right\rfloor$, and the extra floor runs through 0, 1, 2, 3, 4 in a pattern that depends on whether m is even or odd.

Work it out. $\dfrac{2(m^2+j)}{m} = 2m + \dfrac{2j}{m}$, so the block sum is $2m(2m+1) + T(m)$ with $T(m) = \sum_{j=0}^{2m} \left\lfloor \frac{2j}{m} \right\rfloor$.

If $m = 2k$ is even, $\lfloor 2j/m \rfloor = \lfloor j/k \rfloor$ for $j = 0, \ldots, 4k$: each of $0, 1, 2, 3$ occurs k times and 4 occurs once, so $T = 6k + 4 = 3m + 4$.

If $m = 2k+1$ is odd, the value q occurs for $qm \le 2j < (q+1)m$: $q = 0$ for $j = 0..k$ ($k+1$ times), $q = 1$ for $j = k+1..2k$ (k times), $q = 2$ for $j = 2k+1..3k+1$ ($k+1$ times), $q = 3$ for $j = 3k+2..4k+1$ (k times), and $q = 4$ once. So $T = k + 2(k+1) + 3k + 4 = 6k + 6 = 3m + 3$.

So the block sum is $4m^2 + 5m + 4$, minus 1 when m is odd. The blocks $m = 1, \ldots, 44$ cover $n = 1, \ldots, 2024$, and $n = 2025 = 45^2$ contributes $f(2025) = \frac{4050}{45} = 90$.

$\sum_{m=1}^{44}(4m^2 + 5m + 4) = 4 \cdot \frac{44 \cdot 45 \cdot 89}{6} + 5 \cdot \frac{44 \cdot 45}{2} + 4 \cdot 44 = 117480 + 4950 + 176 = 122606$.

Subtract 22 for the odd m and add 90: the total is $122606 - 22 + 90 = 122674$. The remainder mod 1000 is 674.

Check. Block $m = 1$ ($n = 1, 2, 3$): $f = 2, 4, 6$, sum $12 = 4+5+4-1$. Block $m = 2$ ($n = 4, \ldots, 8$): $f = 4, 5, 6, 7, 8$, sum $30 = 16 + 10 + 4$. Both match the formula.

6. Answer: 288**. The idea.** If the chord's endpoints are (s, s^2) and (r, r^2) with $d = r - s > 0$, then the height of the region above an integer x is $(r - x)(x - s)$, an integer depending only on the distances to the ends. So both the lattice count and the area depend only on d; find d from the count, then compute the area.

Work it out. The line through the two points is $y = (r+s)x - rs$ (its slope is $\frac{r^2 - s^2}{r - s} = r + s$). The gap between line and parabola is $(r+s)x - rs - x^2 = (r-x)(x-s)$, positive exactly for $s < x < r$.

Lattice points inside need integer x with $s < x < r$. For such x, both x^2 and the line value are integers, so the integers strictly between them number $(r - x)(x - s) - 1$. Writing $k = x - s$, the total is

$\sum_{k=1}^{d-1} (k(d-k) - 1) = \frac{(d-1)d(d+1)}{6} - (d-1) = \frac{(d-1)(d-2)(d+3)}{6}$.

This increases with d for $d \ge 2$: $d = 10, 11, 12, 13$ give $156, 210, 275, 352$. So $d = 12$ is the only possibility.

The area is $\int_s^r (r - x)(x - s)\, dx = \frac{d^3}{6} = \frac{1728}{6} = 288$.

Check. For $d = 3$, say the chord from $(0, 0)$ to $(3, 9)$ on $y = 3x$: column $x = 1$ has $1 < y < 3$, one point; column $x = 2$ has $4 < y < 6$, one point. Total 2, and the formula gives $\frac{2 \cdot 1 \cdot 6}{6} = 2$. For the area, the $d = 3$ region has area $\int_0^3 (3x - x^2)\, dx = \frac{27}{2} - 9 = \frac{9}{2} = \frac{27}{6}$, matching $\frac{d^3}{6}$.

7. Answer: 253**. The idea.** Multiply top and bottom by $1 - z$: for $z \ne 1$, $\dfrac{1}{1 + z + z^2} = \dfrac{1 - z}{1 - z^3}$. Because 3 does not divide 250, cubing just shuffles the 250th roots of unity, so we can rename $w = z^3$ and write z as a power of w. The fraction then becomes a finite geometric sum in w, and sums of powers of roots of unity are easy.

Work it out. First, no denominator vanishes: $1 + z + z^2 = 0$ would make z a primitive cube root of unity, which is not a 250th root of unity since $3 \nmid 250$. The term $z = 1$ contributes $\frac{1}{3}$.

For the other 249 roots, $z^3 \ne 1$ (again because $3 \nmid 250$), so $\frac{1}{1+z+z^2} = \frac{1-z}{1-z^3}$. Since $\gcd(3, 250) = 1$, the map $z \mapsto w = z^3$ permutes the 250th roots of unity and fixes 1, so as z runs over the roots other than 1, so does w. The inverse is $z = w^{167}$, because $3 \cdot 167 = 501 \equiv 1 \pmod{250}$. Therefore

$$\sum_{z\neq 1}\frac{1-z}{1-z^3}=\sum_{w\neq 1}\frac{1-w^{167}}{1-w}=\sum_{w\neq 1}\left(1+w+w^2+\cdots+w^{166}\right).$$

For $1\le j\le 166$ we have $\sum_{w\neq 1}w^j=\left(\sum_{\text{all }w}w^j\right)-1=0-1=-1$, since $250\nmid j$. So the sum is $249\cdot 1+166\cdot(-1)=83$, and the whole sum is

$$83+\tfrac{1}{3}=\frac{250}{3}.$$

Hence $m+n=250+3=253$.

Check. The same method for the 4th roots of unity $1, i, -1, -i$ predicts $\frac{1}{3}+(3-2)=\frac{4}{3}$ (here $3^{-1}\equiv 3 \pmod 4$, giving $1+w+w^2$ and $3+2\cdot(-1)=1$). Directly: $\frac{1}{3}+\frac{1}{1+i-1}+\frac{1}{1-1+1}+\frac{1}{1-i-1}=\frac{1}{3}+\frac{1}{i}+1+\frac{1}{-i}=\frac{4}{3}$. A numerical evaluation of the 250-term sum also gives $83.333\ldots$. The answer is **253**.

8. Answer: 83**. The idea.** Measure the three areas as fractions of $[ABC]$: $\alpha=[PBC]/[ABC]$, $\beta=[PCA]/[ABC]$, $\gamma=[PAB]/[ABC]$, so $\alpha+\beta+\gamma=1$. Each of these is an affine function of P (it is the distance from P to a side, rescaled), so the map $P\mapsto(\alpha,\beta,\gamma)$ is an affine bijection from the triangle onto the triangle $\alpha+\beta+\gamma=1$, $\alpha,\beta,\gamma\ge 0$. Affine maps multiply all areas by the same factor, so (α,β,γ) is uniform on that triangle and the shape of ABC does not matter. The condition is symmetric in the three areas, so we may restrict to the sixth of the triangle where $\alpha\ge\beta\ge\gamma$ and find what fraction of that piece is good.

Work it out. The piece where $\alpha\ge\beta\ge\gamma$ is the triangle with vertices $G=(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ (all equal), $V=(1,0,0)$ and $W=(\frac{1}{2},\frac{1}{2},0)$. Its sides are GV (where $\beta=\gamma$), GW (where $\alpha=\beta$) and VW (where $\gamma=0$). Inside it we need $\beta\le 2\gamma$ and $\alpha\le 2\beta$; both are half-planes, so we cut twice.

First cut, $\beta\le 2\gamma$. The line $\beta=2\gamma$ passes through V (where $\beta=\gamma=0$) and meets GW, whose points are $(u,u,1-2u)$, where $u=2(1-2u)$, i.e. at $Y=(\frac{2}{5},\frac{2}{5},\frac{1}{5})$. The point G satisfies the inequality, so the good part is triangle GVY. Along GW the coordinate γ drops from $\frac{1}{3}$ to 0, and at Y it is $\frac{1}{5}$, so $GY/GW=(\frac{1}{3}-\frac{1}{5})/\frac{1}{3}=\frac{2}{5}$ and $[GVY]=\frac{2}{5}[GVW]$.

Second cut, $\alpha\le 2\beta$. On GV, whose points are $(1-2t,t,t)$, equality holds when $1-2t=2t$: at $X=(\frac{1}{2},\frac{1}{4},\frac{1}{4})$, so $VX/VG=\frac{3}{4}$ (the coordinates β,γ grow linearly from V, reaching $\frac{1}{4}$ out of $\frac{1}{3}$). On VY, whose points are $V+t(Y-V)=(1-\frac{3t}{5},\frac{2t}{5},\frac{t}{5})$, equality holds when $1-\frac{3t}{5}=\frac{4t}{5}$, i.e. $t=\frac{5}{7}$: at $Z=(\frac{4}{7},\frac{2}{7},\frac{1}{7})$, so $VZ/VY=\frac{5}{7}$. The vertex V violates $\alpha\le 2\beta$, so the second cut removes the corner triangle VXZ, whose area is $\frac{3}{4}\cdot\frac{5}{7}=\frac{15}{28}$ of $[GVY]$.

The good region in the piece is the quadrilateral $GXZY$, with area $\left(1-\frac{15}{28}\right)\cdot\frac{2}{5}[GVW]=\frac{13}{28}\cdot\frac{2}{5}[GVW]=\frac{13}{70}[GVW]$. By symmetry the same fraction holds in all six pieces, so the probability is $\frac{13}{70}$ and $m+n=\boxed{83}$.

Check. Redo the area with coordinates (β,γ) (a projection, which scales all areas equally): $G=(\frac{1}{3},\frac{1}{3})$, $V=(0,0)$, $W=(\frac{1}{2},0)$, $X=(\frac{1}{4},\frac{1}{4})$, $Z=(\frac{2}{7},\frac{1}{7})$, $Y=(\frac{2}{5},\frac{1}{5})$. The shoelace formula gives $[GVW]=\frac{1}{12}$ and $[GXZY]=\frac{1}{2}\left(\frac{1}{15}-\frac{1}{28}\right)=\frac{13}{840}$, and $\frac{13}{840}\div\frac{1}{12}=\frac{13}{70}$. A sanity point: the centroid ($\alpha=\beta=\gamma$) is good, and a point very near a vertex (one area near 1) is not, as expected.

9. Answer: 653**. The idea.** We never need the center of ω. The tangency at M turns A and B into points whose powers with respect to ω we know (AM^2 and BM^2), and those powers locate P and Q on the legs. Then Menelaus' theorem locates X on line AB.

Work it out. $AB=\sqrt{20^2+15^2}=25$, so $AM=BM=\frac{25}{2}$. The power of A with respect to ω is AM^2 (tangent) and also $AP\cdot AC$ (secant through P and C): $AP=\frac{AM^2}{AC}=\frac{625/4}{20}=\frac{125}{16}$, $\qquad PC=$

$20-\frac{125}{16}=\frac{195}{16}$. Likewise $BQ\cdot BC=BM^2$ gives $BQ=\frac{625/4}{15}=\frac{125}{12}$, $QC=15-\frac{125}{12}=\frac{55}{12}$. Both lie inside the legs, as the problem says. Now line PQ crosses the sides CA and CB of triangle ABC and meets line AB at X, so by Menelaus $\frac{AP}{PC}\cdot\frac{CQ}{QB}\cdot\frac{BX}{XA}=1$, $\frac{125}{195}\cdot\frac{55}{125}\cdot\frac{BX}{XA}=1$, $\frac{BX}{XA}=\frac{195}{55}=\frac{39}{11}$. Since $XB>XA$, X lies on the extension of AB beyond A, and $XB-XA=25$. So $XA\left(\frac{39}{11}-1\right)=25$, giving $XA=\frac{275}{28}$, and $XM=XA+AM=\frac{275}{28}+\frac{350}{28}=\frac{625}{28}$. The answer is $625+28=\boxed{653}$.

Check. Put $C=(0,0)$, $A=(20,0)$, $B=(0,15)$, so $M=(10,\frac{15}{2})$, $P=(\frac{195}{16},0)$, $Q=(0,\frac{55}{12})$. Line PQ is $\frac{16x}{195}+\frac{12y}{55}=1$ and line AB is $3x+4y=60$; they meet at $X=(\frac{195}{7},-\frac{165}{28})$. Then $XM^2=(\frac{125}{7})^2+(\frac{375}{28})^2=\frac{500^2+375^2}{28^2}=\frac{625^2}{28^2}$. A second check: XM is the tangent from X to ω, so XM^2 must equal $XP\cdot XQ=\frac{1875}{112}\cdot\frac{625}{21}$, and indeed both equal $\frac{390625}{784}$. (Also $\angle PCQ=90^\circ$, so PQ is a diameter: $PQ^2=PC^2+QC^2=\frac{390625}{2304}$, radius $\frac{625}{96}$.)

10. Answer: 729**. The idea.** With $b=n-a$ we have $ab=an-a^2\equiv -a^2 \pmod n$, so the condition is simply $n\mid a^2$. Which a have $n\mid a^2$ is decided prime by prime, and the answer turns out to depend on the largest square dividing n. Summing that over n is easiest after swapping the order of summation.

Work it out. Write $n=\prod p^{e_p}$. Then $n\mid a^2$ exactly when $2v_p(a)\ge e_p$ for every p, that is, when a is a multiple of $r(n)=\prod p^{\lceil e_p/2\rceil}$. The multiples of $r(n)$ in $1,\dots,n-1$ number $\frac{n}{r(n)}-1$. Now $\frac{n}{r(n)}=\prod p^{e_p-\lceil e_p/2\rceil}=\prod p^{\lfloor e_p/2\rfloor}$, which is $q(n)$, the largest integer whose square divides n. So $f(n)=q(n)-1$. To add up $q(n)$, use $m=\sum_{d\mid m}\varphi(d)$ with $m=q(n)$. Also $d\mid q(n)$ exactly when $d^2\mid n$ (compare exponents: $v_p(d)\le\lfloor e_p/2\rfloor$ iff $2v_p(d)\le e_p$). Hence $\sum_{n\le 500}q(n)=\sum_{n\le 500}\ \sum_{d^2\mid n}\varphi(d)=\sum_{d=1}^{22}\varphi(d)\left\lfloor\frac{500}{d^2}\right\rfloor$. The terms for $d=1,2,\dots,22$ are $500,\ 125,\ 2\cdot 55,\ 2\cdot 31,\ 4\cdot 20,\ 2\cdot 13,\ 6\cdot 10,\ 4\cdot 7,\ 6\cdot 6,\ 4\cdot 5,\ 10\cdot 4,\ 4\cdot 3,\ 12\cdot 2,\ 6\cdot 2,\ 8\cdot 2,\ 8,\ 16,\ 6,\ 18,\ 8,\ 12,\ 10$, that is $500+125+110+62+80+26+60+28+36+20+40+12+24+12+16+8+16+6+18+8+12+10=1229$. Subtracting 1 for each of the 500 values of n, $f(1)+\cdots+f(500)=1229-500=\boxed{729}$.

Check. Small cases by hand: for $n=9$ the pairs are $(3,6)$ and $(6,3)$ (18 is divisible by 9), and $q(9)-1=2$. For $n=8$ only $(4,4)$ works, and $q(8)-1=1$. Up to 10, only $n=4,8,9$ contribute, giving $1+1+2=4$; the formula gives $\varphi(1)\cdot 10+\varphi(2)\cdot 2+\varphi(3)\cdot 1-10=10+2+2-10=4$.

11. Answer: 312**. The idea.** Sort people and chairs into residue classes mod 3. The rule says that nobody ends up in a chair of their own class. Before choosing anything else, decide how many people go from each class to each other class. Those flows are forced up to a single parameter, and after them everything factors into choosing who goes where and then seating each class.

Work it out. The classes are $R_0=\{3,6,9\}$ (size 3), $R_1=\{1,4,7,10\}$ and $R_2=\{2,5,8,11\}$ (size 4 each); the same sets describe the chairs. Let x_{ij} be the number of people from class i who sit in chairs of class j ($i\ne j$). Everyone leaves their class, and every chair is filled, so $x_{01}+x_{02}=3$, $x_{10}+x_{12}=4$, $x_{20}+x_{21}=4$, $x_{10}+x_{20}=3$, $x_{01}+x_{21}=4$, $x_{02}+x_{12}=4$. Put $t=x_{01}$. Then $x_{02}=3-t$, $x_{12}=1+t$, $x_{10}=3-t$, $x_{20}=t$, $x_{21}=4-t$, and the remaining equation checks. So $t\in\{0,1,2,3\}$.

For a given t: choose which t of the 3 people in R_0 go to R_1 ($\binom{3}{t}$ ways; the rest go to R_2); which $3-t$ of the 4 people in R_1 go to R_0 ($\binom{4}{3-t}$); which t of the 4 in R_2 go to R_0 ($\binom{4}{t}$). Now each chair class receives a set of exactly as many people as it has chairs, and they can be seated in 3!, 4!, 4! ways. So $N=3!\,4!\,4!\sum_{t=0}^{3}\binom{3}{t}\binom{4}{3-t}\binom{4}{t}=3456\,(1\cdot 4\cdot 1+3\cdot 6\cdot 4+3\cdot 4\cdot 6+1\cdot 1\cdot 4)=3456\cdot 152=525312$.

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The remainder is $\boxed{312}$.

Check. The same method for nine people (three classes of 3) forces $x_{01} = x_{12} = x_{20} = t$ and $x_{02} = x_{21} = x_{10} = 3 - t$, giving $(3!)^3 \sum_t \binom{3}{t}^3 = 216 \cdot 56 = 12096$. That agrees with a direct count, and it also shows why the uneven case needs care: with sizes 3, 4, 4 the flows are no longer symmetric. A tiny case: people 1, 2, 3 must form a 3-cycle (x_{ij} all 1 or all 0 around), and the formula gives $1!\,1!\,1!\,(1+1) = 2$, the two 3-cycles.

12. Answer: 181**. The idea.** Two moves s_i and s_j with $|i-j| \geq 2$ touch different positions, so they can be done in either order with the same effect. So the result depends only on which moves are used and, for each pair of used neighbours s_i, s_{i+1}, which of the two comes first. We show conversely that the final arrangement reveals all of this. Then counting arrangements means counting these choices.

Work it out. *Cards move in one direction.* Suppose the card at position p is first moved right by s_p. At $p+1$ the only moves touching it are s_p (already used) and s_{p+1}: if s_{p+1} is still to come, the card moves right again, otherwise it never moves again. By induction every card that moves travels in one direction only, and a card is moved exactly when s_{p-1} or s_p is used; the first of these to happen sets its direction.

The arrangement determines the choices. Move s_i is used exactly when the set of cards in positions $1, \ldots, i$ changes: each use of s_i sends one card rightward across the gap between i and $i+1$, and such a card never comes back. If s_{p-1} and s_p are both used, the card starting at p moves right iff s_p came before s_{p-1}, and we can read its direction from its final position. So different choices give different arrangements.

Counting. The used moves form runs of consecutive indices. For a run of k moves there are $k-1$ neighbouring pairs, and any choice of their relative orders can be realised (a path has no cycles, so we can always order the moves to match). That gives 2^{k-1} options per run. So $N = a_9$, where $a_m = \sum_S \prod_{\text{runs of } S} 2^{\text{length}-1}$ over all subsets S of m moves in a row. Condition on the last move: if it is unused we get a_{m-1}; if it ends a run of length k, preceded by an unused move (or by nothing), we get $2^{k-1} a_{m-k-1}$ (with $a_{-1} = a_0 = 1$). With $b_m = a_m - a_{m-1} = \sum_{k \geq 1} 2^{k-1} a_{m-k-1}$ we have $b_m = a_{m-2} + 2b_{m-1}$, which simplifies to $a_m = 3a_{m-1} - a_{m-2}$. Starting from $a_0 = 1$, $a_1 = 2$: $5, 13, 34, 89, 233, 610, 1597, 4181$. So $N = a_9 = 4181$ and the remainder is $\boxed{181}$.

Check. Three cards: nothing (1 arrangement), s_1 or s_2 alone (2), and both in either order, which gives the two different 3-cycles 231 and 312. That is $5 = a_2$. Four cards by the run formula: $1 + 3 + (2 + 2 + 1) + 4 = 13 = a_3$. The values are the odd-indexed Fibonacci numbers F_{2m+1}, consistent with $a_m = 3a_{m-1} - a_{m-2}$.

13. Answer: 233**. The idea.** The two angle conditions say that triangles ABP and CAP are similar, which gives PB and PC as fixed multiples of PA and shows that $\angle BPC = 2\angle A$. That is also the angle $\angle BOC$, so B, P, O, C lie on one circle, and Ptolemy's theorem on that circle gives OP.

Work it out. Let $a = BC = 6$, $b = CA = 5$, $c = AB = 4$, and $x = PA$. Then $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{25+16-36}{40} = \frac{1}{8}$, so $A < 90^\circ$ and $\cos 2A = 2 \cdot \frac{1}{64} - 1 = -\frac{31}{32}$. The triangle is acute ($36 < 16 + 25$), so O is inside it.

A similarity. In triangle ABP the angles at A and B are $\angle PAB$ and $\angle PBA$. In triangle CAP the angles at C and A are $\angle PCA = \angle PAB$ and $\angle PAC = \angle PBA$. So triangle ABP is similar to triangle CAP, with $A \to C$, $B \to A$, $P \to P$. The ratio is $\frac{AB}{CA} = \frac{c}{b}$, so $\frac{PB}{PA} = \frac{PA}{PC} = \frac{c}{b} = \frac{4}{5}$, $\quad PB = \frac{4}{5}x, \quad PC = \frac{5}{4}x$.

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The angle at P. In triangle ABP, $\angle APB = 180^\circ - \angle PAB - \angle PBA = 180^\circ - \angle PAB - \angle PAC = 180^\circ - A$. In triangle APC, $\angle APC = 180^\circ - \angle PAC - \angle PCA = 180^\circ - \angle PAC - \angle PAB = 180^\circ - A$. The three angles around P add to 360°, so $\angle BPC = 2A$.

Finding PA. The law of cosines in triangle BPC: $36 = x^2\left(\frac{16}{25} + \frac{25}{16} + 2\cdot\frac{4}{5}\cdot\frac{5}{4}\cdot\frac{31}{32}\right) = x^2\left(\frac{16}{25} + \frac{25}{16} + \frac{31}{16}\right) = x^2\left(\frac{16}{25} + \frac{7}{2}\right) = \frac{207}{50}x^2$, so $x^2 = \frac{1800}{207} = \frac{200}{23}$.

A circle through O. The central angle $\angle BOC$ is $2A = \angle BPC$, and O and P are on the same side of BC. So B, P, O, C lie on one circle. On that circle $OB = OC = R$ (the circumradius), so O is the midpoint of its arc BC on that side. Since $PB < PC$, the point P lies on the arc between B and O, and the order around the circle is B, P, O, C. Ptolemy's theorem for $BPOC$ (diagonals BO and PC): $BO \cdot PC = BP \cdot OC + PO \cdot BC$, so $OP = \frac{R\,(PC-PB)}{a}$.

The numbers. With $s = \frac{15}{2}$, Heron gives $[ABC]^2 = \frac{15}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2} = \frac{1575}{16}$, so $[ABC] = \frac{15\sqrt{7}}{4}$ and $R = \frac{abc}{4[ABC]} = \frac{120}{15\sqrt{7}} = \frac{8}{\sqrt{7}}$, $R^2 = \frac{64}{7}$. Also $PC - PB = \left(\frac{5}{4} - \frac{4}{5}\right)x = \frac{9}{20}x$, so $(PC-PB)^2 = \frac{81}{400}\cdot\frac{200}{23} = \frac{81}{46}$. Therefore $OP^2 = \frac{64}{7}\cdot\frac{81}{46}\cdot\frac{1}{36} = \frac{144}{322} = \frac{72}{161}$, and $m + n = 72 + 161 = \boxed{233}$.

Check. Coordinates: $B = (0,0)$, $C = (6,0)$, $A = \left(\frac{9}{4}, \frac{5\sqrt{7}}{4}\right)$, and $O = \left(3, \frac{1}{\sqrt{7}}\right)$ (indeed $9 + \frac{1}{7} = \frac{64}{7}$). From $PB^2 = \frac{16}{25}x^2 = \frac{128}{23}$ and $PC^2 = \frac{25}{16}x^2 = \frac{625}{46}$ we get $P = \left(\frac{429}{184}, \frac{25\sqrt{7}}{184}\right)$, and then $OP^2 = \left(\frac{123}{184}\right)^2 + \left(\frac{9}{184\sqrt{7}}\right)^2 = \frac{7\cdot 15129+81}{7\cdot 33856} = \frac{105984}{236992} = \frac{72}{161}$. A second check with the circumcircle: the power of P is $R^2 - OP^2 = \frac{64}{7} - \frac{72}{161} = \frac{1400}{161} = \frac{200}{23} = PA^2$, which says P is the midpoint of the chord of the circumcircle through A and P, a known property of this point.

14. Answer: 592**. The idea.** Call the vertical pair (U_i, L_i) column i. A column is either *split* (two different colours, missing exactly one colour) or *solid* (one colour twice). Give every column a *pair*: for a split column, the two colours it uses; for a solid column of colour c, the two colours other than c. The face between two columns rules out exactly the combinations that fail to show all three colours, and in terms of pairs the rule becomes very simple.

Work it out. *The rule.* A face shows all three colours iff its two columns together use all three. Two split columns: the second must contain the colour missing from the first, which happens exactly when their pairs are different. A solid column of colour c next to a split column: the split column must use both other colours, i.e. the two pairs are equal. Two solid columns together show at most two colours, so they are never adjacent. In short, going around the ring of six columns: *split–split neighbours have different pairs, solid–split neighbours have equal pairs, and no two solid columns are adjacent.* A split column also has an orientation (which of its two colours is on top), 2 ways; a solid column is determined by its pair.

Counting. Choose the set of s solid columns, no two adjacent on the 6-cycle: $1, 6, 9, 2$ ways for $s = 0, 1, 2, 3$. The $k = 6 - s$ split columns, in cyclic order, are consecutive either directly (then their pairs must differ) or across one solid column (then both equal that solid column's pair, and in particular each other). Merge the pairs of split columns joined across a solid column: we need a proper colouring, with 3 "pair colours", of a cycle of length $j = k - s = 6 - 2s$. That can be done in $P(j) = 2^j + 2(-1)^j$ ways for $j \geq 2$, and $P(0) = 3$ (all pairs equal). The solid columns then have forced pairs, and the split columns have 2^k orientations. So $N = 1\cdot 2^6 P(6) + 6\cdot 2^5 P(4) + 9\cdot 2^4 P(2) + 2\cdot 2^3 P(0) = 64\cdot 66 + 192\cdot 18 + 144\cdot 6 + 16\cdot 3$. That is $4224 + 3456 + 864 + 48 = 8592$, so the remainder is $\boxed{592}$.

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Check. The same method for a triangular prism: a solid column would need its two neighbours to have equal pairs, but those neighbours are adjacent split columns, which need different pairs. So all columns are split, with $P(3) \cdot 2^3 = 6 \cdot 8 = 48$ paintings. Directly: the three pairs must all differ ($3! = 6$ ways) and each column has 2 orientations, and any two different pairs together contain all three colours. The 6-column count also matches a transfer-matrix computation: summing the 6th powers of the eigenvalues $2 \pm \sqrt{6}$, $-1 \pm \sqrt{3}$ (each twice) and 0 (three times) gives $7760 + 2 \cdot 416 = 8592$.

15. Answer: 990**. The idea.** Look at the chords involved in the two crossings. The two crossing pairs either share a chord (one chord crossed by two others) or use four different chords. Every remaining chord crosses nothing, so it must stay inside one region cut out by the special chords, and that forces its endpoints into one arc between consecutive special endpoints. Each arc then carries a non-crossing matching, counted by Catalan numbers $1, 1, 2, 5, \ldots$. Everything reduces to coefficients of powers of $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$.

Work it out. Useful coefficients: $C^2 = 1 + 2x + 5x^2 + 14x^3 + \cdots$, $C^4 = 1 + 4x + 14x^2 + 48x^3 + \cdots$, and $C^6 = C^4 C^2 = 1 + 6x + 27x^2 + 110x^3 + \cdots$ (e.g. $[x^3] = 48 + 14 \cdot 2 + 4 \cdot 5 + 14 = 110$).

Type 1: chord b crossed by a and by c (and a, c do not cross). Starting at an endpoint b_1 of b and going clockwise, the six special endpoints are forced to be $b_1, a_1, c_1, b_2, c_2, a_2$. Any two of the six arcs between them are separated by one of a, b, c: a separates the two arcs next to b_1 from the other four, c separates the two arcs next to b_2 from the rest, and b splits the six into 3 + 3. So a chord crossing none of a, b, c has both ends in one arc. Each arc holds an even number $2k_i$ of points with $\sum k_i = 3$, matched non-crossingly in $\prod \mathrm{Cat}(k_i)$ ways. Counting pairs (matching, chosen endpoint of b): the endpoint is any of 12 points and the arc sizes are free, so $2N_1 = 12\,[x^3]C^6 = 1320$ and $N_1 = 660$.

Type 2: two disjoint crossing pairs $\{a, b\}$ and $\{c, d\}$. The pair $\{c, d\}$ crosses neither a nor b, so its four endpoints lie in one arc between consecutive endpoints of a and b, and vice versa. So each pair's endpoints form a block of four consecutive special points. The eight special points split into block 1, gap g_1, block 2, gap g_2. Inside each block, the three small arcs are separated from each other and from everything else by that block's two chords, so each holds a non-crossing matching of its own. But the two gaps lie in the *same* region (outside all four chords), and a chord joining gap g_1 to gap g_2 crosses nothing. So the $2m$ gap points together, in circular order, carry one non-crossing matching, in $\mathrm{Cat}(m)$ ways, and they can be split between g_1 and g_2 in $2m + 1$ ways. Counting pairs (matching, choice of one of the two blocks), with the chosen block's first endpoint any of 12 points: $2N_2 = 12 \sum_{m=0}^{2} (2m+1)\mathrm{Cat}(m)\,[x^{2-m}]C^6 = 12\,(1 \cdot 1 \cdot 27 + 3 \cdot 1 \cdot 6 + 5 \cdot 2 \cdot 1) = 12 \cdot 55$, so $N_2 = 330$.

The total is $660 + 330 = \boxed{990}$.

Check. With 8 points the same method gives $2N_1 = 8\,[x^1]C^6 = 48$ and $2N_2 = 8 \cdot 1$, so $24 + 4 = 28$ matchings with exactly two crossings. Listing all 105 matchings of 8 points confirms this: the numbers with $0, 1, 2, \ldots, 6$ crossings are $14, 28, 28, 20, 10, 4, 1$. For 12 points, a computer run over all 10395 matchings confirms 990.

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