MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

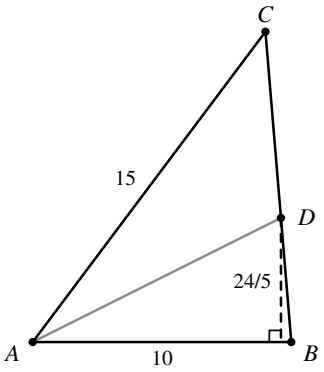
1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. For a positive integer n, let $P(n) = 1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$, the product of k^k for $k = 1, 2, \ldots, n$. Find the least positive integer n for which $P(n)$ is divisible by 10^{1000}.

2. In triangle ABC, $AB = 10$, $AC = 15$, and $\angle BAC$ is acute. The bisector of $\angle BAC$ meets side $\overline{BC}$ at D, and the distance from D to line AB is $\frac{24}{5}$. Find BC^2.

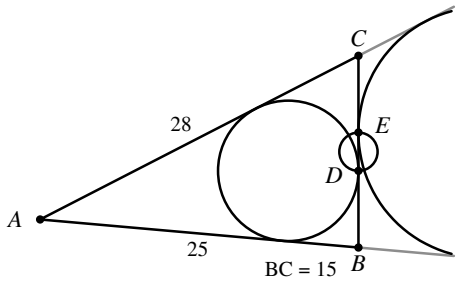


3. How many ordered pairs (x, y) of real numbers satisfy $x^2+y^2 = 2025$ and have the property that $|x|+2|y|$ is an integer?

4. How many pairs of integers (a, b) with $1 \le a < b \le 500$ have the property that the product ab is a perfect square?

5. For each integer a with $1 \le a \le 999$ and $\gcd(a, 1000) = 1$, let a' be the unique integer with $1 \le a' \le 999$ and $aa' \equiv 1 \pmod{1000}$. Call a a *leader* if a is less than or equal to each of the three numbers $1000-a$, a' and $1000 - a'$. How many leaders are there?

6. Triangle ABC has $BC = 15$, $CA = 28$ and $AB = 25$. The circle inscribed in the triangle touches side $\overline{BC}$ at D, and the excircle opposite A (the circle tangent to segment $\overline{BC}$ and to the extensions of sides $\overline{AB}$ and $\overline{AC}$ beyond B and C) touches $\overline{BC}$ at E. Let ω be the circle with diameter $\overline{DE}$. Find the square of the length of a tangent segment from A to ω.



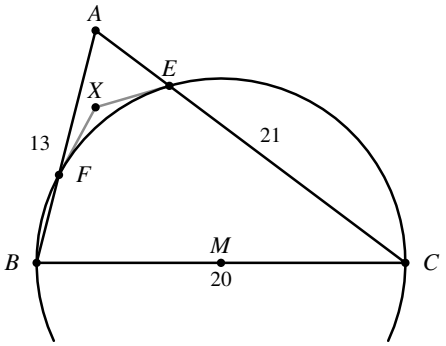
7. A regular heptagon $A_0A_1A_2A_3A_4A_5A_6$ is inscribed in a circle of radius 1. Compute $\left(1 + A_0A_1^2\right)\left(1 + A_0A_2^2\right)\left(1 + A_0A_3^2\right)\left(1$

8. How many subsets of $\{1, 2, 3, \ldots, 15\}$ (the empty set included) contain no two consecutive integers and have an even sum of elements?

9. Triangle ABC has $BC = 11$, $CA = 6$ and $AB = 12$. A straight line cuts the triangle into two pieces that have equal areas and also equal perimeters. It turns out that exactly one such line exists; it meets the boundary of the triangle at two points P and Q. Then $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. For a positive real number x, let
$f(x) = \lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor + \lfloor x^4 \rfloor$,
where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. How many integers n with $1 \le n \le 1000$ are equal to $f(x)$ for at least one positive real number x?

11. Triangle ABC has $AB = 13$, $BC = 20$ and $CA = 21$, and M is the midpoint of $\overline{BC}$. The circle with diameter $\overline{BC}$ meets side $\overline{CA}$ again at E and side $\overline{AB}$ again at F. The tangent lines to this circle at E and at F meet at a point X. Then $XM = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



12. For a positive integer n, let $\varphi(n)$ be the number of integers k with $1 \le k \le n$ and $\gcd(k, n) = 1$, and let $\tau(n)$ be the number of positive divisors of n. Find the sum of all positive integers n for which $\varphi(n) = 2\tau(n)$.

13. A permutation σ of $\{1, 2, 3, \ldots, 100\}$ is chosen uniformly at random. Split σ into its cycles (a fixed point counts as a cycle of length 1), and let X be the number of cycles that contain at least one multiple of 20. The expected value of X^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. The numbers $0, 1, 2, \ldots, 14$ are arranged around a circle. A subset S of these numbers (possibly empty) is called *calm* if no two elements of S differ by 3 or by 5 modulo 15; that is, there are no $x, y \in S$ with $y - x \equiv \pm 3$ or $y - x \equiv \pm 5 \pmod{15}$. How many calm subsets are there?

15. Twelve beads are fixed at the positions $1, 2, \ldots, 12$ around a circle (so position 12 is next to position 1; rotations and reflections count as different). Each bead is painted red, white or blue so that each colour is used on exactly 4 beads and no two neighbouring beads have the same colour. How many such paintings are there?

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Answer Key — AIME Practice Contest 7

1. 90

2. 145

3. 264

4. 729

5. 102

6. 646

7. 841

8. 805

9. 439

10. 962

11. 71

12. 463

13. 211

14. 391

15. 804

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Solutions — AIME Practice Contest 7

1. Answer: 90**. The idea.** $10^{1000} \mid P(n)$ means both 2^{1000} and 5^{1000} divide $P(n)$. The factor k^k contributes $k \cdot v_5(k)$ fives, where $v_5(k)$ is the exponent of 5 in k; only multiples of 5 contribute, and they contribute a lot (the whole exponent k each time). Twos are far more plentiful, so the fives decide everything.

Work it out. Count fives layer by layer. Every multiple of 5, say $k = 5j$, contributes at least k fives; every multiple of 25, $k = 25j$, contributes k more; multiples of 125 do not occur below 125. So for $n < 125$, $v_5(P(n)) = 5(1 + 2 + \cdots + \lfloor n/5 \rfloor) + 25(1 + 2 + \cdots + \lfloor n/25 \rfloor)$. For $n = 85$: $5 \cdot \frac{17 \cdot 18}{2} + 25 \cdot 6 = 765 + 150 = 915 < 1000$. The value only changes when n passes a multiple of 5, so $n = 86, \ldots, 89$ still give 915. For $n = 90$: $5 \cdot \frac{18 \cdot 19}{2} + 150 = 855 + 150 = 1005 \geq 1000$.

Twos at $n = 90$: already the even numbers give at least $2(1 + 2 + \cdots + 45) = 2070 \geq 1000$ twos. So $10^{1000} \mid P(90)$ but $10^{1000} \nmid P(89)$, and the answer is $\boxed{90}$.

Check. Directly: the multiples of 5 up to 90 sum to $5+10+\cdots+90 = 855$, and $25+50+75 = 150$ is the extra layer; $855 + 150 = 1005$. Removing the 90^{90} factor drops 90 fives, leaving 915, short of 1000.

2. Answer: 145**. The idea.** A point on the bisector of an angle is equally far from both sides of the angle. So D is $\frac{24}{5}$ from line AC as well, and cutting the triangle along $\overline{AD}$ gives its area. The area gives $\sin A$, and then the law of cosines gives BC.

Work it out. Let $h = \frac{24}{5}$ be the distance from D to AB; it is also the distance from D to AC. The segment $\overline{AD}$ splits triangle ABC into triangles ABD and ACD with heights h over the bases AB and AC, so

$[ABC] = \frac{1}{2} \cdot 10 \cdot h + \frac{1}{2} \cdot 15 \cdot h = \frac{25}{2} \cdot \frac{24}{5} = 60.$

Also $[ABC] = \frac{1}{2} \cdot AB \cdot AC \cdot \sin A = 75 \sin A$, so $\sin A = \frac{4}{5}$. Since $\angle A$ is acute, $\cos A = \frac{3}{5}$. By the law of cosines,

$BC^2 = 10^2 + 15^2 - 2 \cdot 10 \cdot 15 \cdot \frac{3}{5} = 100 + 225 - 180 = 145.$

Check. Put $A = (0, 0)$, $B = (10, 0)$ and $C = 15(\frac{3}{5}, \frac{4}{5}) = (9, 12)$; then $BC^2 = 1^2 + 12^2 = 145$. The bisector divides BC in the ratio $BD : DC = AB : AC = 2 : 3$, so $D = B + \frac{2}{5}(C - B) = (\frac{48}{5}, \frac{24}{5})$, whose distance from line AB (the x-axis) is $\frac{24}{5}$, as given. Had $\angle A$ been obtuse, $\cos A = -\frac{3}{5}$ would give 505 instead. The answer is **145**.

3. Answer: 264**. The idea.** Walk around the circle and watch the single quantity $g = |x| + 2|y|$. By symmetry it is enough to understand one quarter of the circle. On that quarter g rises, peaks and falls, so each integer value is hit once or twice, and the count is a matter of reading off where the peak and the two ends are. Care is needed at the ends, which lie on the axes and are shared by neighbouring quarters.

Work it out. On the quarter-circle in the first quadrant write $x = 45\cos\theta$, $y = 45\sin\theta$ with $0° \leq \theta \leq 90°$. Then $g = 45(\cos\theta + 2\sin\theta) = 45\sqrt{5}\cos(\theta - \varphi)$ where $\tan\varphi = 2$. So g increases from $g = 45$ at $\theta = 0°$ (the point $(45, 0)$) up to its maximum $45\sqrt{5} = \sqrt{10125}$ at $\theta = \varphi$, then decreases to $g = 90$ at $\theta = 90°$ (the point $(0, 45)$). Since $100^2 = 10000 < 10125 < 10201 = 101^2$, the maximum lies strictly between 100 and 101.

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Count the points of this closed quarter-circle where g is an integer k:

$k = 45$: only the end $\theta = 0°$ (1 point). $46 \le k \le 89$: exactly one point, on the rising part, since the falling part never drops below 90 (44 points). $k = 90$: one point on the rising part and the end $\theta = 90°$ (2 points). $91 \le k \le 100$: one point on each side of the peak (20 points). That is $1 + 44 + 2 + 20 = 67$ points, including both ends.

The four quarters are mirror images of each other (changing the signs of x or y does not change g), so each has $67 - 2 = 65$ points off the axes. The axis points $(\pm 45, 0)$ (where $g = 45$) and $(0, \pm 45)$ (where $g = 90$) are each shared by two quarters and must be counted once. The total is $4 \cdot 65 + 4 = \boxed{264}$.

Check. For $k = 90$ in the first quadrant, solve $x + 2y = 90$, $x^2 + y^2 = 2025$: substituting gives $y^2 - 72y + 1215 = 0$, so $y = 45$ (the axis point) or $y = 27$, $x = 36$; indeed $36^2 + 27^2 = 2025$ and $36 + 54 = 90$. For $k = 100$ the same substitution gives $y^2 - 80y + 1595 = 0$ with discriminant $20 > 0$, two points; for $k = 101$ it gives $y^2 - \frac{404}{5}y + \frac{8176}{5} = 0$, whose discriminant $\frac{404^2}{25} - \frac{4 \cdot 8176}{5} < 0$, no points. A common slip is to count 4 points for every value from 45 to 100, or to forget the second point at $k = 90$.

4. Answer: 729**. The idea.** Every positive integer can be written uniquely as dx^2 with d squarefree (the *squarefree part* d is the product of the primes appearing to an odd power). A product ab is a perfect square exactly when a and b have the *same* squarefree part. So group the numbers 1 to 500 by squarefree part and count pairs inside each group.

Work it out. Why the criterion holds: if $a = dx^2$ and $b = ey^2$ with d, e squarefree, then ab is a square iff de is a square iff every prime appears in d and e together an even number of times, i.e. $d = e$.

For a squarefree d, the numbers with squarefree part d are $d \cdot 1^2, d \cdot 2^2, \ldots, d \cdot m^2$ with $m = \lfloor \sqrt{500/d} \rfloor$, giving $\binom{m}{2}$ pairs. Only $d \le 125$ can give $m \ge 2$. Tabulate:

$d = 1$: $m = 22$, 231 pairs. $d = 2$: $m = 15$, 105. $d = 3$: $m = 12$, 66. $d = 5$: $m = 10$, 45. $d = 6$: $m = 9$, 36. $d = 7$: $m = 8$, 28. $d = 10$: $m = 7$, 21. $d = 11, 13$: $m = 6$, 15 each, 30. $d = 14, 15, 17, 19$: $m = 5$, 10 each, 40.

For $m = 4$ we need $21 \le d \le 31$, for $m = 3$ we need $32 \le d \le 55$, and for $m = 2$ we need $56 \le d \le 125$. Let $Q(N)$ count squarefree numbers up to N; by inclusion–exclusion over $4, 9, 25, 49, 121$ (adding back 36 and 100), $Q(125) = 125 - 31 - 13 - 5 - 2 - 1 + 3 + 1 = 77$, $Q(55) = 55 - 13 - 6 - 2 - 1 + 1 = 34$, $Q(31) = 31 - 7 - 3 - 1 = 20$, $Q(20) = 20 - 5 - 2 = 13$. So there are $20 - 13 = 7$ squarefree d with $m = 4$ (6 pairs each, 42), $34 - 20 = 14$ with $m = 3$ (3 each, 42), and $77 - 34 = 43$ with $m = 2$ (1 each, 43).

Total: $231 + 105 + 66 + 45 + 36 + 28 + 21 + 30 + 40 + 42 + 42 + 43 = \boxed{729}$.

Check. Count the other way: a pair is (dx^2, dy^2) with $x < y$, so the total is $\sum_{y \ge 2} (y-1)\, Q(\lfloor 500/y^2 \rfloor)$. The terms for $y = 2, 3, \ldots, 22$ are $77, 68, 60, 52, 45, 42, 42, 40, 36, 30, 33, 24, 26, 28, 15, 16, 17, 18, 19, 20, 21$, which again add to 729. For a small case, with $b \le 10$ the pairs are $(1, 4), (1, 9), (4, 9), (2, 8)$: 4 of them, matching $\binom{3}{2} + \binom{2}{2} = 4$.

5. Answer: 102**. The idea.** The four numbers a, $1000 - a$, a', $1000 - a'$ are the images of a under four maps on the units mod 1000: the identity, negation, inversion, and "negate the inverse". These maps compose among themselves (each undoes itself, and any two give the third), so they split the units into groups (orbits) $\{a, -a, a', -a'\}$. A leader is exactly the smallest member of its group, and each group has exactly one smallest member. So we are counting groups.

Work it out. There are $\varphi(1000) = 400$ units. Look at when members of a group coincide.

$a \equiv -a$ would need $2a \equiv 0$, impossible for odd a. $a \equiv -a'$ means $a^2 \equiv -1 \pmod{1000}$, impossible since odd squares are $\equiv 1 \pmod 8$.

$a \equiv a'$ means $a^2 \equiv 1 \pmod{1000}$. Modulo 8 all four odd residues $1, 3, 5, 7$ work; modulo 125, $a^2 - 1 = (a-1)(a+1)$ and 5 cannot divide both factors, so $a \equiv \pm 1$. By the Chinese Remainder Theorem there are $4 \cdot 2 = 8$ such a (namely $1, 249, 251, 499, 501, 749, 751, 999$). For these the group is $\{a, 1000 - a\}$, of size 2; the 8 numbers form 4 groups.

Every other unit lies in a group of size exactly 4 (the four members are pairwise distinct by the above), so the remaining 392 units form 98 groups. Total groups, hence leaders: $4 + 98 = \boxed{102}$.

Check. Burnside's lemma gives the same count: the four maps fix 400, 0, 8 and 0 units, and $\frac{400+0+8+0}{4} = 102$. Small examples: $a = 1$ is a leader ($1 \le 999, 1, 999$); $a = 3$ has $a' = 667$, and $3 \le 997, 667, 333$, so it leads $\{3, 997, 667, 333\}$.

6. Answer: 646. **The idea.** The two touch points are reflections of each other in the midpoint M of BC (the classical fact $BD = CE$). So ω is centred at M, and the squared tangent length is the power of A: $AM^2 - (\text{radius})^2$. The median formula then collapses to a clean expression in the semiperimeter.

Work it out. Let $a = 15$, $b = 28$, $c = 25$, $s = 34$. Tangent lengths from B to the incircle are $s - b$, so $BD = s - b = 6$. For the excircle opposite A, the tangent length from A is s (the two tangents from A have total length $AB + BE + EC + CA = 2s$ split equally), so $BE = s - c = 9$ and $CE = s - b = 6$. Hence $BD = CE = 6$, $DE = a - 2(s - b) = b - c = 3$, and ω is centred at M with radius $\frac{b-c}{2} = \frac{3}{2}$.

The power of A with respect to ω is $AM^2 - \frac{(b-c)^2}{4} = \frac{2b^2+2c^2-a^2}{4} - \frac{b^2-2bc+c^2}{4} = \frac{(b+c)^2-a^2}{4} = \frac{(b+c-a)(b+c+a)}{4} = s(s-a)$. With numbers: $34 \cdot 19 = \boxed{646}$.

Check. Coordinates: $B = (0, 0)$, $C = (15, 0)$, $A = \left(\frac{11}{5}, y\right)$ with $y^2 = 625 - \frac{121}{25} = \frac{15504}{25}$. Then $M = (7.5, 0)$, $AM^2 = (7.5 - 2.2)^2 + \frac{15504}{25} = 28.09 + 620.16 = 648.25$, and $648.25 - 2.25 = 646$.

7. Answer: 841. **The idea.** Put the heptagon at the 7th roots of unity. A factor $1 + |1 - \omega|^2$ equals $3 - 2\cos\theta$, and $3 - 2\cos\theta$ factors as $\frac{(x - e^{i\theta})(x - e^{-i\theta})}{x}$ for the number x with $x + \frac{1}{x} = 3$. Then the product over all seventh roots of unity is a value of $x^7 - 1$.

Work it out. Let $\omega = e^{2\pi i/7}$ and $A_k = \omega^k$. Then $A_0 A_k^2 = |1 - \omega^k|^2 = 2 - 2\cos\theta_k$ with $\theta_k = \frac{2\pi k}{7}$, so each factor is $3 - 2\cos\theta_k$.

Let $x = \frac{3+\sqrt{5}}{2}$, so $x + \frac{1}{x} = 3$. For any θ, $(x - e^{i\theta})(x - e^{-i\theta}) = x^2 - 2x\cos\theta + 1 = x\,(3 - 2\cos\theta)$. Including $k = 0$ (whose factor is $3 - 2 = 1$) changes nothing, so the product is $\prod_{k=0}^{6} \frac{(x - \omega^k)(x - \omega^{-k})}{x} = \frac{(x^7-1)^2}{x^7} = x^7 + x^{-7} - 2$, using $\prod_{k=0}^{6}(x - \omega^k) = x^7 - 1$ (the ω^{-k} run over the same roots).

Let $t_n = x^n + x^{-n}$. Then $t_0 = 2$, $t_1 = 3$ and $t_{n+1} = 3t_n - t_{n-1}$ (multiply t_n by $x + \frac{1}{x}$). So t_n runs $2, 3, 7, 18, 47, 123, 322, 843$, giving $t_7 = 843$ and the product $843 - 2 = \boxed{841}$.

Check. Try the same method on a square inscribed in the unit circle: the chords from one vertex have squares $2, 4, 2$, so the product is $3 \cdot 5 \cdot 3 = 45$, and the formula gives $t_4 - 2 = 47 - 2 = 45$. Also the factors for k and $7 - k$ are equal, so the product is a square: numerically $(3 - 2\cos\frac{2\pi}{7})(3 - 2\cos\frac{4\pi}{7})(3 - 2\cos\frac{6\pi}{7}) \approx 1.753 \cdot 3.445 \cdot 4.802 \approx 29$, and $29^2 = 841$.

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8. Answer: 805**. The idea.** Let E_n and O_n count the subsets of $\{1,\dots,n\}$ with no two consecutive elements and even or odd sum. Their total is a Fibonacci number, and their difference $d_n = E_n - O_n$ obeys a recursion just as simple, because adding the element n flips the parity of the sum exactly when n is odd.

Work it out. *The total.* A valid subset of $\{1,\dots,n\}$ either omits n (then it is a valid subset of $\{1,\dots,n-1\}$) or contains n (then it omits $n-1$, and the rest is a valid subset of $\{1,\dots,n-2\}$). So the totals $t_n = E_n + O_n$ satisfy $t_n = t_{n-1} + t_{n-2}$ with $t_0 = 1$ (the empty set) and $t_1 = 2$. These are the Fibonacci numbers $1, 2, 3, 5, 8, \dots$, and $t_{15} = 1597$.

The difference. Give each subset the sign $+1$ if its sum is even and -1 if odd, so d_n is the total sign. The same split applies; in the second case adding n multiplies the sign by $(-1)^n$. Hence

$$d_n = d_{n-1} + (-1)^n d_{n-2}, \qquad d_0 = 1, \quad d_1 = 0$$

(for $n = 1$: the empty set counts $+1$ and $\{1\}$ counts -1). Running the recursion:

$$d_2, d_3, \dots, d_{15} = 1, 1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, 13.$$

So $E_{15} - O_{15} = 13$ and $E_{15} + O_{15} = 1597$, giving

$$E_{15} = \frac{1597 + 13}{2} = 805.$$

Check. For $n = 4$ the valid subsets are $\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{2,4\}$, with sums $0, 1, 2, 3, 4, 4, 5, 6$: five even and three odd, so $d_4 = 2$ and $t_4 = 8$, matching the recursions. A direct computer count for $n = 15$ also gives 805. The answer is **805**.

9. Answer: 439**. The idea.** The line cuts off a small triangle at one vertex V. The two conditions give the sum and the product of the two cut lengths from V, so they are the roots of a quadratic, and we can test each vertex. The roots themselves are irrational, but PQ^2 only needs their sum and product, and it collapses to a clean formula.

Work it out. Let the line cut off triangle VPQ at vertex V, where the sides at V have lengths x and y and the opposite side has length z. Put $p = VP$ and $q = VQ$. Equal perimeters: $p + q + PQ = (x - p) + (y - q) + z + PQ$, so $p + q = \frac{x+y+z}{2} = s = \frac{29}{2}$. Equal areas: $\frac{1}{2}pq \sin V = \frac{1}{2} \cdot \frac{1}{2}xy \sin V$, so $pq = \frac{xy}{2}$. So p and q are the roots of $t^2 - \frac{29}{2}t + \frac{xy}{2} = 0$, and we need $p \le x$, $q \le y$ (in some order).

At B ($x = 12$, $y = 11$): $pq = 66$, but $pq \le \left(\frac{29}{4}\right)^2 < 53$, so there are no real roots. At C ($x = 6$, $y = 11$): $pq = 33$, the roots are $\frac{29 \pm \sqrt{313}}{4} \approx 11.67, 2.83$, and 11.67 is longer than both sides at C. At A ($x = 12$, $y = 6$): $pq = 36$, the roots are $\frac{29 \pm \sqrt{265}}{4} \approx 11.32, 3.18$, and $AP \approx 11.32 \le 12$ on AB with $AQ \approx 3.18 \le 6$ on AC works (the other order does not). A line through a vertex would need that vertex's median and $x = y$, which fails. So the line is unique and cuts off a triangle at A, with $z = BC = 11$.

By the law of cosines, $2xy \cos A = x^2 + y^2 - z^2$, so

$$PQ^2 = p^2 + q^2 - 2pq \cos A = (p+q)^2 - pq(2 + 2\cos A) = s^2 - \frac{xy}{2} \cdot \frac{2xy + x^2 + y^2 - z^2}{xy} = s^2 - \frac{(x+y)^2 - z^2}{2}.$$

Since $(x + y)^2 - z^2 = (2s)(2s - 2z)$, this is $PQ^2 = s^2 - 2s(s - z) = s(2z - s) = \frac{29}{2} \cdot \frac{15}{2} = \frac{435}{4}$. So $m + n = 435 + 4 = 439$.

Check. Directly: $\cos A = \frac{144+36-121}{144} = \frac{59}{144}$, $p + q = \frac{29}{2}$, $pq = 36$, so $p^2 + q^2 = \frac{841}{4} - 72 = \frac{553}{4}$ and $PQ^2 = \frac{553}{4} - 72 \cdot \frac{59}{144} = \frac{553}{4} - \frac{59}{2} = \frac{435}{4}$.

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10. Answer: 962**. The idea.** As x increases from 0, f never decreases, and it only jumps at points where one of x, x^2, x^3, x^4 becomes an integer; the size of the jump is the number of these four powers that are integers there. A jump of size s skips $s - 1$ values. So the whole problem is to find where two or more of the four powers are integers at once.

Work it out. Just above 0, $f = 0$. Each floor $\lfloor x^j \rfloor$ is constant between consecutive points where x^j is an integer and goes up by exactly 1 at each such point. So every integer value from 0 upward is attained, except the values jumped over.

When do two powers hit integers together? If x is an integer, all four do. Suppose x is not an integer.

• If x^i and x^j are both integers and $\gcd(i, j) = 1$, choose integers u, v with $ui + vj = 1$; then $x = (x^i)^u (x^j)^v$ is rational. But a rational number that is not an integer, written in lowest terms $\frac{a}{b}$ with $b > 1$, has no integer powers at all. So this cannot happen, and among the exponents $1, 2, 3, 4$ the only pair with a common factor is $2, 4$.

• For x^2 and x^4: if $x^2 = m$ is an integer then $x^4 = m^2$ is too. Here m is not a perfect square, and $x = \sqrt{m}$ makes exactly x^2 and x^4 integers ($x^3 = m\sqrt{m}$ is irrational).

So the jumps larger than 1 are: jumps of 4 at the integers $x = k$, each skipping 3 values ($f(k) - 3, f(k) - 2, f(k) - 1$), and jumps of 2 at $x = \sqrt{m}$ with m not a square, each skipping the single value $f(\sqrt{m}) - 1$.

Counting skips up to 1000*.* At the integers, $f(k) = k + k^2 + k^3 + k^4$ is $4, 30, 120, 340, 780$ for $k = 1, \ldots, 5$ and 1554 for $k = 6$. So $k = 1, \ldots, 5$ each skip 3 values in $[1, 1000]$: 15 values.

At $x = \sqrt{m}$, $f(\sqrt{m}) = \lfloor \sqrt{m} \rfloor + m + \lfloor m\sqrt{m} \rfloor + m^2$, which increases with m. For $m = 28$: $5 + 28 + 148 + 784 = 965$, since $28\sqrt{28} = \sqrt{21952}$ and $148^2 = 21904 < 21952 < 149^2$. For $m = 29$: $5 + 29 + 156 + 841 = 1031$, since $156^2 = 24336 < 29^3 = 24389 < 157^2$. So the skipped value $f(\sqrt{m}) - 1$ is at most 1000 exactly for $m \le 28$. The non-squares m from 2 to 28 number $27 - 4 = 23$ (removing $4, 9, 16, 25$).

All these skipped values are different (they sit at different jumps of an increasing function), so $15 + 23 = 38$ of the numbers $1, \ldots, 1000$ are missed, and

$$1000 - 38 = 962$$

are attained.

Check. Near $x = 1$: just below 1 all four floors are 0, and at $x = 1$ they all become 1, so f jumps from 0 to 4 and the values $1, 2, 3$ are never taken. Next, at $x = \sqrt{2}$ we get $f = 1 + 2 + 2 + 4 = 9$, while just below $\sqrt{2}$ the value is $1 + 1 + 2 + 3 = 7$: the value 8 is skipped, as predicted. A computer evaluation of f at every jump point also finds exactly 962 attained values in $[1, 1000]$. The answer is **962**.

11. Answer: 71**. The idea.** E and F are the feet of the altitudes from B and C (angles in a semicircle), so they lie on the circle with diameter AH, where H is the orthocentre. The tangents at E and F meet at the centre of that circle, the midpoint N of $\overline{AH}$. Then N and M are both on the nine-point circle and are opposite ends of a diameter of it.

Work it out. Since $\angle BEC = \angle BFC = 90°$, E and F are the feet of the altitudes from B and C. The triangle is acute ($21^2 = 441 < 13^2 + 20^2 = 569$), so these feet lie on the sides. Also $\angle AEH = \angle AFH = 90°$, so E and F lie on the circle Γ with diameter AH, centre N.

Show NE is tangent to the circle ω on BC. Since $NE = NA$, $\angle NEA = \angle NAE = \angle HAC = 90° - C$. Since $ME = MC$, $\angle MEC = \angle C$. As A, E, C are collinear, $\angle NEM = 180° - (90° - C) - C = 90°$. So $NE \perp ME$, and NE is the tangent to ω at E. The same holds at F, so $X = N$.

The nine-point circle passes through M, E and N. Because $\angle NEM = 90°$, the chord NM subtends a right angle there, so NM is a diameter of the nine-point circle. Its radius is $\frac{R}{2}$, so $XM = R$, the circumradius of ABC.

With $s = 27$, Heron gives area $K = \sqrt{27 \cdot 14 \cdot 7 \cdot 6} = \sqrt{15876} = 126$, and $R = \frac{abc}{4K} = \frac{13 \cdot 20 \cdot 21}{504} = \frac{5460}{504} = \frac{65}{6}$. So $m + n = 65 + 6 = 71$.

Check. Coordinates: $B = (0, 0)$, $C = (20, 0)$, $A = \left(\frac{16}{5}, \frac{63}{5}\right)$ (then $AB^2 = \frac{256+3969}{25} = 169$ and $AC^2 = \frac{7056+3969}{25} = 441$). The altitude from A is $x = \frac{16}{5}$. $\overrightarrow{AC} \propto (4, -3)$, so the altitude from B is $4x = 3y$, and $H = \left(\frac{16}{5}, \frac{64}{15}\right)$. Then $N = \left(\frac{16}{5}, \frac{253}{30}\right)$ and, with $M = (10, 0)$, $NM^2 = \left(\frac{204}{30}\right)^2 + \left(\frac{253}{30}\right)^2 = \frac{41616+64009}{900} = \frac{105625}{900} = \left(\frac{65}{6}\right)^2$.

12. Answer: 463**. The idea.** Both φ and τ are multiplicative, so the ratio $g(n) = \varphi(n)/\tau(n)$ is the product of the ratios for the prime powers in n. For a prime power p^e this ratio is $g(p^e) = \dfrac{p^{e-1}(p-1)}{e+1}$. Almost all of these are at least 2, and the few small ones can be combined in only a handful of ways to make exactly 2.

Work it out. *The small values.* For $p = 2$: $g(2) = \frac{1}{2}$, $g(4) = \frac{2}{3}$, $g(8) = 1$, $g(16) = \frac{8}{5}$, $g(32) = \frac{8}{3}$, and $g(2^{e+1})/g(2^e) = \frac{2(e+1)}{e+2} > 1$, so $g(2^e) > 2$ for $e \geq 5$. For $p = 3$: $g(3) = 1$, $g(9) = 2$, $g(27) = \frac{9}{2}$, increasing after that. For $p = 5$: $g(5) = 2$, $g(25) = \frac{20}{3}$. For $p \geq 7$: $g(p) = \frac{p-1}{2} \geq 3$, and higher powers are larger still. So for an odd prime power, g is 1 only for 3, equals 2 only for 9 and 5, and is otherwise more than 2.

Write $n = 2^e m$ with m odd, so we need $g(2^e) \cdot g(m) = 2$, where $g(m)$ is 1 (for $m = 1$ or 3) or at least 2.

• $g(2^e) = 1$ ($e = 0$ or 3): we need $g(m) = 2$, so m is 5, 9 or $3 \cdot 5 = 15$ (a factor 3 contributes 1, but 3 and 9 cannot both occur). This gives $n = 5, 9, 15, 40, 72, 120$.

• $g(2^e) = \frac{1}{2}$ ($e = 1$): we need $g(m) = 4$. No odd prime power has $g = 4$, and a product of two factors equal to 2 must be $g(9)g(5)$; so $m = 45$ and $n = 90$.

• $g(2^e) = \frac{2}{3}$ ($e = 2$): we need $g(m) = 3$, so $m = 7$ or $m = 3 \cdot 7$, and $n = 28$ or 84.

• $e = 4$: we need $g(m) = \frac{5}{4}$, impossible since $g(m) = 1$ or $g(m) \geq 2$. For $e \geq 5$, $g(2^e) > 2$ is already too big.

So the solutions are $5, 9, 15, 28, 40, 72, 84, 90, 120$, and their sum is

$(5 + 9 + 15) + (28 + 84) + (40 + 72 + 120) + 90 = 29 + 112 + 232 + 90 = 463$.

Check. $n = 84 = 2^2 \cdot 3 \cdot 7$: $\varphi(84) = 2 \cdot 2 \cdot 6 = 24$ and $\tau(84) = 3 \cdot 2 \cdot 2 = 12$, so $\varphi = 2\tau$. $n = 90 = 2 \cdot 3^2 \cdot 5$: $\varphi(90) = 1 \cdot 6 \cdot 4 = 24$, $\tau(90) = 2 \cdot 3 \cdot 2 = 12$. A computer search of all $n \leq 200000$ finds exactly these nine numbers. The answer is **463**.

13. Answer: 211**. The idea.** Follow σ from a multiple of 20 until you first land on another multiple of 20. This "first return" map τ is a permutation of the five numbers $20, 40, 60, 80, 100$, and its cycles are exactly the traces of the cycles of σ that meet those numbers. So X is the number of

cycles of τ. The key fact is that τ is *uniformly* random. That gives the whole distribution of X, which linearity alone could not (it gives only $E[X]$).

Work it out. For a marked number m (a multiple of 20), let $\tau(m)$ be the first marked number in the sequence $\sigma(m), \sigma^2(m), \ldots$. This is well defined, since the cycle of m returns to m. Each cycle of σ containing marked numbers lists them in the cyclic order of one cycle of τ, and different cycles of σ give different cycles of τ. So $X =$ (number of cycles of τ).

Uniformity: build σ by starting from any permutation τ of the five marked numbers (written as cycles) and inserting the unmarked numbers $1, 2, \ldots$ one at a time. When k numbers are already placed, a new number can go immediately after any one of them in its cycle (k ways) or start a new cycle by itself (1 way), giving $k+1$ choices. Every permutation of $\{1, \ldots, 100\}$ arises exactly once, and inserting unmarked numbers never changes the first-return map on the marked ones. So each τ comes from exactly $6 \cdot 7 \cdots 100 = \frac{100!}{5!}$ permutations σ, and τ is uniform on the 120 permutations of five elements.

The numbers of permutations of 5 elements with $1, 2, 3, 4, 5$ cycles are $24, 50, 35, 10, 1$ (for example, $24 = 4!$ five-cycles, and $50 = 30 + 20$ from the types $4+1$ and $3+2$). Hence

$E[X^2] = \frac{24\cdot1+50\cdot4+35\cdot9+10\cdot16+1\cdot25}{120} = \frac{724}{120} = \frac{181}{30}$,

and $m + n = 181 + 30 = 211$.

Check. The same table gives $E[X] = \frac{24+100+105+40+5}{120} = \frac{137}{60} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$, the familiar mean number of cycles of a random permutation of 5 things. Equivalently $E[t^X] = \frac{t(t+1)(t+2)(t+3)(t+4)}{120}$, and $E[X^2] = E[X(X-1)] + E[X]$, where $E[X(X-1)]$ is the second derivative at $t = 1$: $2\sum_{i<j} \frac{1}{(i+1)(j+1)}$ over $0 \le i < j \le 4$, which equals $\left(\frac{137}{60}\right)^2 - \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}\right) = \frac{18769-5269}{3600} = \frac{15}{4}$. Then $\frac{15}{4} + \frac{137}{60} = \frac{225+137}{60} = \frac{181}{30}$.

14. Answer: 391**. The idea.** By the Chinese Remainder Theorem, a residue x mod 15 is the same as the pair (x mod 5, x mod 3). A step of ±3 changes only the mod-5 coordinate (by ∓2), and a step of ±5 changes only the mod-3 coordinate (by ∓1). So the forbidden pairs form a 5×3 grid wrapped into a torus. Calm sets are its independent sets, and a transfer matrix counts them.

Work it out. Replace the mod-5 coordinate i by $3i$ mod 5; since $3 \cdot 2 \equiv 1$, steps of ±2 become steps of ±1. Now we have 5 columns $C_0, \ldots, C_4$ in a cycle, each a triangle of 3 mutually forbidden cells (within a column, any two mod-3 values differ by ±1). Two cells in neighbouring columns are forbidden only if they have the same mod-3 value.

So each column holds at most one element: its state is $\varnothing$ or one of 3 cells, 4 states in all. Neighbouring columns are compatible unless both hold the same cell. The transfer matrix is $T = J - \mathrm{diag}(0,1,1,1)$, where J is the 4×4 all-ones matrix, and the count is $\mathrm{tr}\, T^5$.

Eigenvalues: vectors $(0, u)$ with $u \perp (1,1,1)$ give $T(0,u) = (0,-u)$, so -1 is an eigenvalue twice. On the span of $(1,0,0,0)$ and $(0,1,1,1)$, T acts by $\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$, with characteristic polynomial $\lambda^2 - 3\lambda - 1$ and roots $\lambda_\pm = \frac{3\pm\sqrt{13}}{2}$. Let $L_n = \lambda_+^n + \lambda_-^n$. Then $L_0 = 2$, $L_1 = 3$ and $L_n = 3L_{n-1} + L_{n-2}$, so $L_2 = 11$, $L_3 = 36$, $L_4 = 119$, $L_5 = 393$. Therefore

$\mathrm{tr}\, T^5 = 2(-1)^5 + L_5 = 393 - 2 = 391$.

Check. $\mathrm{tr}\, T = 1$ matches $-1 - 1 + 3$, and $\mathrm{tr}\, T^2$ is the sum of the squared entries of T, $16 - 3 = 13$, matching $1+1+11$. Directly: calm sets of size at most 1 number $1+15 = 16$; size 2: $\binom{15}{2} - 30 = 75$

(there are $15 \cdot 4/2 = 30$ forbidden pairs). A computer check over all 2^{15} subsets confirms the total 391.

15. Answer: 804**. The idea.** Place the red beads first. They are pairwise non-adjacent and cut the ring into four runs of non-red beads, each of length at least 1. Inside a run the colours must alternate white/blue, so each run has just 2 paintings. An even run always has as many whites as blues. An odd run has one extra of one colour, +1 or −1. We need the whites and blues to balance overall.

Work it out. Fix a painting and one of its red beads. Reading clockwise from that red bead, the runs between consecutive reds have lengths $g_1, g_2, g_3, g_4 \geq 1$ with sum $12 - 4 = 8$. Conversely, a position for the chosen red bead (12 ways), a composition $(g_1, \ldots, g_4)$, and a painting of each run determine the painting. So $4N = 12 \cdot W$, where W is the number of (composition, run painting) pairs with 4 whites and 4 blues, and $N = 3W$.

For a composition with e even parts and o odd parts: each even run has 2 balanced paintings. Each odd run has an excess of +1 or −1, and exactly half of the odd runs must have +1, which gives $\binom{o}{o/2}$ choices (o is even since the sum 8 is even). So the composition contributes $2^e \binom{o}{o/2}$.

The partitions of 8 into 4 positive parts, with their number of orderings and contributions:

5+1+1+1: 4 orderings, $o = 4$: $\binom{4}{2} = 6$ each, total 24.

4+2+1+1: 12 orderings, $e = 2, o = 2$: $4 \cdot 2 = 8$ each, total 96.

3+3+1+1: 6 orderings, $o = 4$: 6 each, total 36.

3+2+2+1: 12 orderings, $e = 2, o = 2$: 8 each, total 96.

2+2+2+2: 1 ordering, $e = 4$: 16.

The orderings total 4+12+6+12+1 = 35 = $\binom{7}{3}$, as they should. So $W = 24+96+36+96+16 = 268$ and $N = 3 \cdot 268 = 804$.

Check. Without the balance condition, each composition would give $2^4 = 16$ run paintings, so $3 \cdot 35 \cdot 16 = 1680$ proper colourings would have exactly 4 red beads. A quick check of that: choosing 4 pairwise non-adjacent positions on a 12-cycle can be done in $\frac{12}{12-4}\binom{8}{4} = 105$ ways, and $105 \cdot 16 = 1680$. The balance condition keeps 268 of the 560 pairs, and a computer check of all 3^{12} colourings gives 804.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.