MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

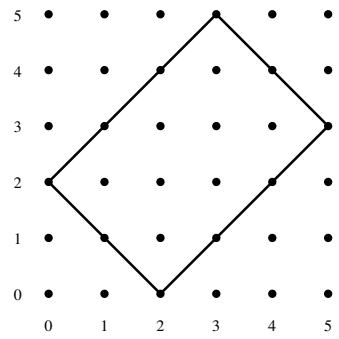
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1. Four different numbers are chosen at random from $\{1, 2, 3, \ldots, 20\}$, with all $\binom{20}{4}$ choices equally likely. The probability that the four numbers can be split into two pairs with equal sums is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

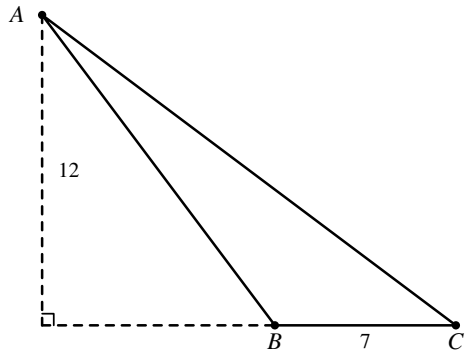
2. How many nonempty subsets S of $\{1, 2, 3, \ldots, 18\}$ have the property that the largest element of S minus the smallest element of S equals the number of elements of S?

3. Real numbers x, y, z satisfy $x^2 - yz = 3, \quad y^2 - zx = 5, \quad z^2 - xy = 7$. Then $x^2 + y^2 + z^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Consider the 36 points (x, y) in the plane whose coordinates x and y both belong to $\{0, 1, 2, 3, 4, 5\}$. How many rectangles, squares included, have all four of their vertices among these 36 points? The sides of a rectangle need not be parallel to the coordinate axes.

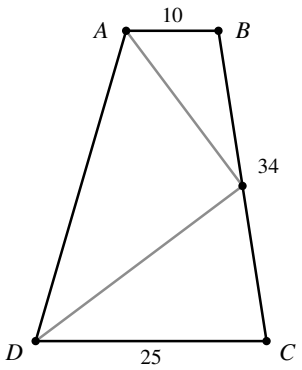


5. In triangle ABC the angle at B is exactly $90°$ larger than the angle at C. The side BC has length 7, and the altitude from A to line BC has length 12. The circumradius R of triangle ABC satisfies $R^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



6. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, with $AB = 10$, $CD = 25$ and $BC = 34$. The bisector of $\angle DAB$ and the bisector of $\angle ADC$ meet at a point that lies on side $\overline{BC}$. Find the area of the trapezoid.

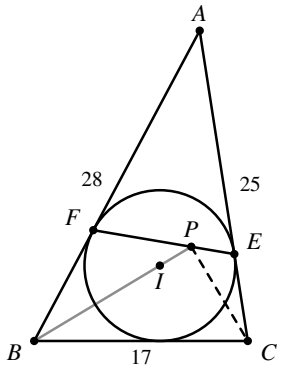
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7. Let $N = 2^5 \cdot 3^4 = 2592$. How many sets S of positive divisors of N have the property that no element of S divides a different element of S? The empty set counts, and so does every set with a single element.

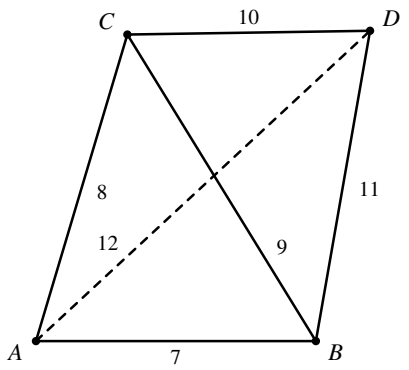
8. Let $\omega = \cos 30^\circ + i \sin 30^\circ$, so that $1, \omega, \omega^2, \ldots, \omega^{11}$ are the twelve complex numbers z with $z^{12} = 1$. How many of the 2^{12} subsets of $\{1, \omega, \omega^2, \ldots, \omega^{11}\}$ (the empty set included, with sum 0) have a sum that is a real number?

9. Triangle ABC has $AB = 28$, $BC = 17$ and $CA = 25$, and I is its incenter. The incircle touches side $\overline{CA}$ at E and side $\overline{AB}$ at F. Line BI meets line EF at the point P. The area of triangle BPC is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



10. Tetrahedron $ABCD$ has edge lengths
$AB = 7,\ CD = 10,\ AC = 8,\ AD = 12,\ BC = 9,\ BD = 11.$
Let d be the distance between the lines AB and CD (the length of the shortest segment joining a point of line AB to a point of line CD). Then $d^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

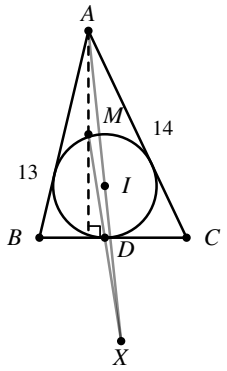
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11. A sequence of integers $a_0, a_1, a_2, \ldots, a_{24}$ has $a_0 = 0$ and
$|a_{k+1}| = |a_k + 1|$ for $k = 0, 1, \ldots, 23$.
(So each term after the first is either a_k+1 or $-(a_k+1)$.) Let N be the number of such sequences $(a_0, a_1, \ldots, a_{24})$ for which the sum of all the terms before the last one equals the last one:
$a_0 + a_1 + a_2 + \cdots + a_{23} = a_{24}$.
Find the remainder when N is divided by 1000.

12. How many quadratic polynomials ax^2+bx+c with integer coefficients and $1 \le a \le 10$ have two different real roots, both strictly between 0 and 1?

13. Triangle ABC has $AB = 13$ and $AC = 14$, and I is its incenter. The incircle touches side $\overline{BC}$ at D. Let M be the midpoint of the altitude from A (the segment from A to the foot of the perpendicular from A to line BC). Line DM meets line AI at the point X. Suppose that D is exactly the midpoint of segment $\overline{MX}$. Find AX^2.



14. The numbers $1, 2, \ldots, 9$ are placed in the cells of a 3×3 grid, one number in each cell. The placement is called *orderly* if the three numbers in each row, read from left to right, are either increasing or decreasing, and the three numbers in each column, read from top to bottom, are either increasing or decreasing. (Different rows and columns may go in different directions.) How many orderly placements are there?

15. How many functions f from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ to itself have the following property: for every x in the set, the numbers x and $f(f(f(x)))$ have opposite parity (one is even and the other is odd)? Find the remainder when this number is divided by 1000.

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Answer Key — AIME Practice Contest 6

1. 358

2. 816

3. 169

4. 313

5. 629

6. 588

7. 462

8. 336

9. 259

10. 284

11. 785

12. 28

13. 364

14. 736

15. 448

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Solutions — AIME Practice Contest 6

1. Answer: 358**. The idea.** Write the chosen numbers as $a < b < c < d$. Of the three ways to pair them up, only $\{a, d\}$ with $\{b, c\}$ can balance, and $a + d = b + c$ says $b - a = d - c$: the set is two pairs with the same gap, one pair entirely below the other. Count by the gap.

Work it out. Since $a < c$ and $b < d$ we have $a + b < c + d$, and since $a < b$ and $c < d$ we have $a + c < b + d$. So the only possible split is $a + d = b + c$, that is, $b - a = d - c$. Call this common gap $g \geq 1$.

Fix g. The set is determined by a and c, with $b = a+g$ and $d = c+g$; the conditions are $c > b$, that is $c \geq a + g + 1$, and $d \leq 20$, that is $c \leq 20 - g$. For a given a there are $(20 - g) - (a + g + 1) + 1 = 20 - 2g - a$ choices of c, which is positive exactly for $a \leq 19 - 2g$. So the number of sets with gap g is

$$(19 - 2g) + (18 - 2g) + \cdots + 1 = \frac{(19 - 2g)(20 - 2g)}{2}.$$

For $g = 1, 2, \ldots, 9$ this gives $153, 120, 91, 66, 45, 28, 15, 6, 1$, and different gaps give different sets, so there are

$$153 + 120 + 91 + 66 + 45 + 28 + 15 + 6 + 1 = 525$$

good sets out of $\binom{20}{4} = 4845$. The probability is $\frac{525}{4845} = \frac{35}{323}$ (divide by 15; $35 = 5 \cdot 7$ and $323 = 17 \cdot 19$). Hence $m + n = 35 + 323 = 358$.

Check. The same count for $\{1, 2, 3, 4, 5\}$ gives, for $g = 1$, $1 + 2 = 3$ sets and nothing for $g \geq 2$. Directly: $\{1, 2, 3, 4\}$, $\{1, 2, 4, 5\}$ and $\{2, 3, 4, 5\}$ balance ($1+4 = 2+3$, $1+5 = 2+4$, $2+5 = 3+4$), while $\{1, 2, 3, 5\}$ and $\{1, 3, 4, 5\}$ do not: 3 sets, as predicted. The answer is **358**.

2. Answer: 816**. The idea.** A set whose smallest element is a and largest is b lives inside the block $a, a + 1, \ldots, b$, which has only $b - a + 1$ numbers. Having exactly $b - a$ elements means the set is that whole block with exactly one number left out, and the missing number is described by three numbers $a < c < b$.

Work it out. Let S have smallest element a, largest element b and k elements, with $k = b - a$. A one-element set has $b - a = 0 \neq 1$, so $k \geq 2$ and $a < b$. All elements of S lie among the $b - a + 1$ integers from a to b, and S has $b - a$ of them, so exactly one integer c from a to b is missing. It is neither a nor b, since those are in S; so $a < c < b$ and $S = \{a, a + 1, \ldots, b\} \setminus \{c\}$.

Conversely, for any $1 \leq a < c < b \leq 18$ the set $\{a, \ldots, b\} \setminus \{c\}$ has smallest element a, largest element b and $b - a$ elements, so it qualifies. Different triples give different sets (the set determines a, b and the missing c). So the sets correspond exactly to 3-element subsets $\{a, c, b\}$ of $\{1, \ldots, 18\}$, and there are $\binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{6} = 816$.

Check. For $\{1, 2, 3, 4\}$ the formula gives $\binom{4}{3} = 4$, and indeed the qualifying sets are $\{1, 3\}$, $\{2, 4\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$; for instance $\{1, 4\}$ fails ($4 - 1 = 3 \neq 2$) and $\{1, 2, 3\}$ fails ($3 - 1 = 2 \neq 3$).

3. Answer: 169**. The idea.** Subtract the equations in pairs: $(x^2 - yz) - (y^2 - zx)$ factors as $(x - y)(x + y + z)$. Because the right-hand sides $3, 5, 7$ go up in equal steps, the two differences are equal, so x, y, z themselves form an arithmetic progression. Writing $x = y - d$ and $z = y + d$ then makes the system collapse.

Work it out. Let $s = x + y + z$. Subtracting, $(x - y)s = 3 - 5 = -2$ and $(y - z)s = 5 - 7 = -2$. So $s \neq 0$ and $x - y = y - z = -\frac{2}{s}$. Put $d = y - x = z - y = \frac{2}{s}$, so $x = y - d$, $z = y + d$ and $s = 3y$, giving $3yd = 2$. The middle equation becomes $y^2 - (y + d)(y - d) = d^2 = 5$. (The other two are then automatic: $x^2 - yz = d^2 - 3yd = 5 - 2 = 3$ and $z^2 - xy = d^2 + 3yd = 7$.) Hence $d^2 = 5$ and $y^2 = \frac{4}{9d^2} = \frac{4}{45}$. Finally $x^2 + y^2 + z^2 = (y - d)^2 + y^2 + (y + d)^2 = 3y^2 + 2d^2 = \frac{4}{15} + 10 = \frac{154}{15}$, the same for both solutions $d = \pm\sqrt{5}$. So $m + n = 154 + 15 = 169$.

Check. Take $d = \sqrt{5}$, $y = \frac{2}{3\sqrt{5}}$: then $x = y - \sqrt{5} = -\frac{13}{3\sqrt{5}}$ and $z = \frac{17}{3\sqrt{5}}$. Now $x^2 - yz = \frac{169-34}{45} = 3$, $y^2 - zx = \frac{4+221}{45} = 5$, $z^2 - xy = \frac{289+26}{45} = 7$, and $x^2 + y^2 + z^2 = \frac{169+4+289}{45} = \frac{462}{45} = \frac{154}{15}$.

4. Answer: 313**. The idea.** A tilted rectangle with lattice-point vertices has its sides along two perpendicular lattice directions (a, b) and $(-b, a)$. Record how many primitive steps each side takes; then the rectangle sits snugly inside an axis-parallel bounding box whose size depends only on those numbers, and counting positions of that box counts the rectangles.

Work it out. *Sides parallel to the axes.* Choose two of the six vertical lines $x = 0, \ldots, 5$ and two of the six horizontal ones: $\binom{6}{2}^2 = 225$ rectangles.

Tilted rectangles. One pair of sides has positive slope. Its side vector is an integer vector, so it equals $k(a, b)$, where (a, b) has $a, b \geq 1$ and $\gcd(a, b) = 1$ and $k \geq 1$. The other two sides are perpendicular integer vectors, so, taken in the right direction, they equal $m(-b, a)$ with $m \geq 1$ (every integer vector along $(-b, a)$ is a multiple of it, because $(-b, a)$ is also primitive). Each tilted rectangle gives exactly one triple $((a, b), k, m)$. If the lowest vertex is P, the vertices are P, $P + k(a, b)$, $P + m(-b, a)$ and $P + k(a, b) + m(-b, a)$, so the bounding box has width $W = ka + mb$ and height $H = kb + ma$. The box must fit in the 5×5 square, and then it can be placed in $(6 - W)(6 - H)$ ways. Since $W, H \geq a + b$, only $a + b \leq 5$ is possible.

$(a, b) = (1, 1)$: $W = H = k + m = s$ with $2 \leq s \leq 5$, and $s - 1$ choices of (k, m) for each s: $1 \cdot 16 + 2 \cdot 9 + 3 \cdot 4 + 4 \cdot 1 = 50$.

$(a, b) = (1, 2)$: $(W, H) = (k + 2m, 2k + m)$. The pairs $(k, m) = (1, 1), (1, 2), (2, 1)$ give boxes 3×3, 5×4, 4×5, so $9 + 2 + 2 = 13$ rectangles. By the reflection $x \leftrightarrow y$, $(2, 1)$ also gives 13.

$(1, 3)$ and $(3, 1)$: only $k = m = 1$, a 4×4 box: 4 each. $(1, 4)$, $(4, 1)$, $(2, 3)$, $(3, 2)$: only $k = m = 1$, a 5×5 box: 1 each.

Tilted total: $50 + 26 + 8 + 4 = 88$. Altogether $225 + 88 = 313$.

Check. Run the same method on the smaller array $\{0, 1, 2, 3, 4\}^2$, where a box fits in $(5 - W)(5 - H)$ ways: there are $\binom{5}{2}^2 = 100$ axis-parallel rectangles, and the tilted ones number $1 \cdot 9 + 2 \cdot 4 + 3 \cdot 1 = 20$ for $(1, 1)$, 4 each for $(1, 2)$ and $(2, 1)$ (a 3×3 box), and 1 each for $(1, 3)$ and $(3, 1)$, so 30. The total 130 agrees with a direct computer search, as does 313 for the 6×6 array.

5. Answer: 629**. The idea.** Since $\angle B$ is obtuse, the foot D of the altitude from A lies on line BC beyond B. The exterior angle $\angle ABD = 180^\circ - B$ and the condition $B - C = 90^\circ$ give $\angle BAD = B - 90^\circ = C$. So the right triangles ABD and CAD are similar, exactly as for the altitude of a right triangle, but with D outside the side: $AD^2 = BD \cdot CD$.

Work it out. In right triangle ABD, $\angle BAD = 90^\circ - \angle ABD = 90^\circ - (180^\circ - B) = B - 90^\circ = C$. So triangles DBA and DAC (right angles at D, equal angles at A and C) are similar, and $\frac{BD}{AD} = \frac{AD}{CD}$. With $BD = x$ and $CD = x + 7$: $x(x + 7) = 144$, so $x = 9$, $CD = 16$. Then $AB = \sqrt{81 + 144} = 15$ and $AC = \sqrt{256 + 144} = 20$. In triangle ADC, $\sin C = \frac{AD}{AC} = \frac{12}{20} = \frac{3}{5}$, and the law of sines gives $2R = \frac{AB}{\sin C} = \frac{15}{3/5} = 25$. So $R = \frac{25}{2}$, $R^2 = \frac{625}{4}$, and $m + n = 629$.

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Check. $R = \frac{abc}{4K}$ with $a = 7$, $b = 20$, $c = 15$ and area $K = \frac{1}{2} \cdot 7 \cdot 12 = 42$: $R = \frac{2100}{168} = \frac{25}{2}$. And the angles: $\tan C = \frac{12}{16} = \frac{3}{4}$, while $\angle ABD$ has $\tan = \frac{12}{9} = \frac{4}{3}$, so $B = 180^\circ - \arctan\frac{4}{3} = 90^\circ + \arctan\frac{3}{4} = 90^\circ + C$.

6. Answer: 588**. The idea.** The bisectors meeting on BC forces the fourth side: $AD = AB + CD$. Extend the bisector from D until it hits line AB; the parallel sides make an isosceles triangle and a pair of congruent triangles, and the lengths add up. After that the trapezoid is known and its height comes from one triangle.

Work it out. Let the bisectors meet at P on BC, and let line DP meet line AB at Q. Because $AB \parallel CD$, $\angle AQD = \angle QDC = \angle ADQ$, so triangle AQD is isosceles with $AQ = AD$. The bisector AP of its apex angle is also the median, so P is the midpoint of DQ. Now triangles PBQ and PCD have $PQ = PD$, vertical angles at P, and $\angle PQB = \angle PDC$ (alternate angles), so they are congruent and $BQ = CD = 25$, with Q beyond B. Hence $AD = AQ = AB + BQ = 10 + 25 = 35$.

The trapezoid now has bases 10 and 25 and legs 35 and 34. Sliding leg BC parallel to itself until B lands on A produces a triangle with sides 35, 34 and $25 - 10 = 15$, whose height to the side 15 is the trapezoid's height h. By Heron's formula ($s = 42$) its area is $\sqrt{42 \cdot 7 \cdot 8 \cdot 27} = \sqrt{63504} = 252$, so $h = \frac{2 \cdot 252}{15} = \frac{168}{5}$. The area of $ABCD$ is $\frac{10+25}{2} \cdot \frac{168}{5} = 588$.

Check. With $D = (0, 0)$, $C = (25, 0)$, $A = \left(\frac{49}{5}, \frac{168}{5}\right)$, $B = \left(\frac{99}{5}, \frac{168}{5}\right)$ one checks $AD = 35$ and $BC = 34$. The midpoint of BC is $P = \left(\frac{112}{5}, \frac{84}{5}\right)$, with $PA = 21$, $PD = 28$ and $21^2 + 28^2 = 35^2$, so $\angle APD = 90^\circ$ as it must be (the two bisected angles add to 180°). Also the area is $2[APD] = PA \cdot PD = 588$, since $[ABP] + [PCD] = [ABP] + [PBQ] = [APQ] = [APD]$.

7. Answer: 462**. The idea.** Such a set S (an *antichain*) is determined by the set T of all divisors of N that divide some element of S: S is exactly the set of elements of T that divide nothing else in T. These sets T, drawn on the 6×5 grid of exponents, are "staircases", and staircases are counted by lattice paths.

Work it out. Write each divisor as $2^a 3^b$ with $0 \le a \le 5$, $0 \le b \le 4$. Call a set T of divisors *closed* if every divisor of a member of T is also in T. The map $S \mapsto T = \{d : d \mid s \text{ for some } s \in S\}$ is a bijection from antichains to closed sets; its inverse sends T to its maximal elements, and the maximal elements of a closed set always form an antichain that generates it.

For a closed set T and each a, the exponents b with $2^a 3^b \in T$ form an initial run $0, 1, \ldots, c_a - 1$ with $0 \le c_a \le 5$, and closure forces $c_0 \ge c_1 \ge \cdots \ge c_5$. Conversely every such sequence gives a closed set. So we count non-increasing sequences of 6 terms from $\{0, 1, 2, 3, 4, 5\}$. Such a sequence is fixed by how many times each of the 6 values occurs, and those multiplicities are any 6 non-negative integers adding to 6: $\binom{6+6-1}{6} = \binom{11}{6} = 462$.

Check. For $N = 6 = 2 \cdot 3$ the same count gives $\binom{4}{2} = 6$, and the antichains are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{6\}$, $\{2, 3\}$: six of them. Equivalently, the boundary of the staircase is a path of 6 right-steps and 5 up-steps, and there are $\binom{11}{5} = 462$ of those.

8. Answer: 336**. The idea.** The sum is real exactly when its imaginary part is 0. The imaginary parts of the twelve roots are 0, $\pm\frac{1}{2}$, ± 1 and $\pm\frac{\sqrt{3}}{2}$, and since $\sqrt{3}$ is irrational, the rational part and the $\sqrt{3}$ part of the total must vanish separately. That splits the count into small independent pieces.

Work it out. The imaginary part of ω^k is $\sin(30k^\circ)$:

$k = 0, 6$: 0; $k = 1, 5$: $\frac{1}{2}$; $k = 7, 11$: $-\frac{1}{2}$; $k = 3$: 1; $k = 9$: -1; $k = 2, 4$: $\frac{\sqrt{3}}{2}$; $k = 8, 10$: $-\frac{\sqrt{3}}{2}$.

For a subset S let n_1, n_2 be the numbers of chosen elements among $\{\omega, \omega^5\}$ and $\{\omega^7, \omega^{11}\}$, let $n_3, n_4 \in \{0, 1\}$ record whether ω^3 and ω^9 are chosen, and let n_5, n_6 count the chosen elements among $\{\omega^2, \omega^4\}$ and $\{\omega^8, \omega^{10}\}$. The imaginary part of the sum is

$\frac{1}{2}(n_1 - n_2) + (n_3 - n_4) + \frac{\sqrt{3}}{2}(n_5 - n_6)$.

If $n_5 \neq n_6$ this equals a rational number plus a nonzero rational multiple of $\sqrt{3}$, which is irrational, so it is not 0. Hence the sum is real exactly when

$n_5 = n_6$ and $n_1 - n_2 = 2(n_4 - n_3)$.

The $\sqrt{3}$ part. Choosing j elements from each pair can be done in $\binom{2}{j}^2$ ways, so there are 1+4+1 = 6 choices.

The rational part. If $n_3 = n_4$ (both chosen or neither: 2 ways), we need $n_1 = n_2$, again 6 ways: 12 in all. If $n_4 = 1, n_3 = 0$ we need $n_1 - n_2 = 2$, so both of ω, ω^5 and neither of ω^7, ω^{11}: 1 way. Symmetrically $n_3 = 1, n_4 = 0$ gives 1 way. Total 14.

The real roots. 1 and $\omega^6 = -1$ are free: 4 ways.

So the number of subsets is $6 \cdot 14 \cdot 4 = 336$.

Check. The subsets closed under conjugation (each ω^k chosen together with ω^{12-k}) certainly have real sums; there are $2^2 \cdot 2^5 = 128$ of them, fewer than 336 as they should be. An example that is not closed under conjugation: $\{\omega, \omega^5, \omega^9\}$ has imaginary part $\frac{1}{2} + \frac{1}{2} - 1 = 0$. A direct computer check of all 4096 subsets also gives 336. The answer is **336**.

9. Answer: 259**. The idea.** The point where the bisector from B crosses the touch chord EF is exactly the foot of the perpendicular from C to that bisector. This is a classical angle chase. Once $\angle BPC = 90°$, triangle BPC is a right triangle with hypotenuse BC and an acute angle $\frac{B}{2}$.

Work it out. Write A, B, C for the angles. Here $s = 35$, $AE = AF = s - a = 18$, and $\cos B = \frac{17^2+28^2-25^2}{2\cdot 17\cdot 28} = \frac{448}{952} = \frac{8}{17}$, so $\sin B = \frac{15}{17}$. In this triangle P lies on segment EF, beyond I as seen from B.

Triangle AEF is isosceles, so $\angle AFE = 90° - \frac{A}{2}$ and $\angle BFP = 90° + \frac{A}{2}$. In triangle BFP, $\angle FBP = \frac{B}{2}$, so

$\angle BPF = 180° - \frac{B}{2} - 90° - \frac{A}{2} = \frac{C}{2}$.

Since E and F are on opposite sides of P, and I and B are on the same side, $\angle EPI = 180° - \frac{C}{2}$. Also $\angle ECI = \frac{C}{2}$. These angles sum to 180°, so P, E, C, I are concyclic. Since $\angle IEC = 90°$ (radius to the tangency point), IC is a diameter of this circle, and therefore $\angle IPC = 90°$ too. So $CP \perp BP$.

In right triangle BPC with hypotenuse $BC = 17$ and $\angle PBC = \frac{B}{2}$, we get $BP = 17\cos\frac{B}{2}$ and $CP = 17\sin\frac{B}{2}$. Hence

$[BPC] = \frac{1}{2} \cdot 17^2 \sin\frac{B}{2}\cos\frac{B}{2} = \frac{17^2 \sin B}{4} = \frac{289\cdot 15}{68} = \frac{255}{4}$.

So $m + n = 255 + 4 = 259$.

Check. Coordinates: $B = (0, 0)$, $C = (17, 0)$, $A = \left(\frac{224}{17}, \frac{420}{17}\right)$ give $I = (10, 6)$, $E = \left(\frac{1354}{85}, \frac{588}{85}\right)$, $F = \left(\frac{80}{17}, \frac{150}{17}\right)$. Line BI is $y = \frac{3}{5}x$, and it meets EF at $P = \left(\frac{25}{2}, \frac{15}{2}\right)$. Then $\overrightarrow{PB} \cdot \overrightarrow{PC} = \left(-\frac{25}{2}\right)\left(\frac{9}{2}\right) + \left(\frac{15}{2}\right)^2 = 0$, and the area is $\frac{1}{2} \cdot 17 \cdot \frac{15}{2} = \frac{255}{4}$.

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10. Answer: 284**. The idea.** With $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{CD}$ and $\mathbf{w} = \overrightarrow{AC}$, the distance between the two lines is the height of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ over the base spanned by $\mathbf{u}$ and $\mathbf{v}$: $d = \frac{|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})|}{|\mathbf{u}\times\mathbf{v}|}$.

Both pieces come from dot products, and every dot product between edge vectors is determined by the six edge lengths.

Work it out. Put $\mathbf{b} = \overrightarrow{AB}$, $\mathbf{c} = \overrightarrow{AC}$, $\mathbf{d} = \overrightarrow{AD}$. From $|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$:

$\mathbf{b}\cdot\mathbf{c} = \frac{49+64-81}{2} = 16, \quad \mathbf{b}\cdot\mathbf{d} = \frac{49+144-121}{2} = 36, \quad \mathbf{c}\cdot\mathbf{d} = \frac{64+144-100}{2} = 54.$

The numerator: $\mathbf{v} = \mathbf{d} - \mathbf{c}$, so $\mathbf{c}\cdot(\mathbf{b}\times(\mathbf{d}-\mathbf{c})) = \mathbf{c}\cdot(\mathbf{b}\times\mathbf{d})$, the triple product of $\mathbf{b}, \mathbf{c}, \mathbf{d}$. Its square is the Gram determinant

$$\det\begin{pmatrix} 49 & 16 & 36 \\ 16 & 64 & 54 \\ 36 & 54 & 144 \end{pmatrix} = 49\cdot 6300 - 16\cdot 360 + 36\cdot(-1440) = 251100.$$

(This is $36V^2$, so $V = 15\sqrt{31}$.)

The denominator: $\mathbf{u}\cdot\mathbf{v} = \mathbf{b}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{c} = 36 - 16 = 20$, so $|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = 49\cdot 100 - 400 = 4500$.

Therefore $d^2 = \frac{251100}{4500} = \frac{279}{5}$, and $m + n = 279 + 5 = 284$.

Check. Another route uses $V = \frac{1}{6}\,AB\cdot CD\cdot d\,\sin\theta$, where θ is the angle between the lines. Here $\cos\theta = \frac{20}{70} = \frac{2}{7}$, so $\sin^2\theta = \frac{45}{49}$ and $d^2 = \frac{36\cdot 6975}{4900\cdot 45/49} = \frac{279}{5}$. The formula $\mathbf{u}\cdot\mathbf{v} = \frac{AD^2+BC^2-AC^2-BD^2}{2} = \frac{144+81-64-121}{2} = 20$ agrees. Explicit coordinates give the same value.

11. Answer: 785**. The idea.** Squaring the rule gives $a_{k+1}^2 = a_k^2 + 2a_k + 1$, which telescopes: the sum of the terms is controlled entirely by the last term, so the condition becomes a quadratic equation for a_{24} with two roots of opposite signs. After that, track only the absolute values $b_k = |a_k|$. They perform a walk that moves up or down by 1 and never goes below 0, and the walk records every sign except the sign of the last term, which the quadratic then pins down.

Work it out. Summing $a_{k+1}^2 - a_k^2 = 2a_k + 1$ for $k = 0, \ldots, 23$ gives $a_{24}^2 = 2(a_0 + \cdots + a_{23}) + 24$. The condition says the bracket equals a_{24}, so $a_{24}^2 - 2a_{24} - 24 = 0$, i.e. $(a_{24} - 6)(a_{24} + 4) = 0$.

Conversely, if $a_{24} = 6$ or $a_{24} = -4$, the identity gives $a_0 + \cdots + a_{23} = \frac{a_{24}^2 - 24}{2} = a_{24}$. So we must count the sequences with $a_{24} = 6$ or $a_{24} = -4$.

Now let $b_k = |a_k| \geq 0$. If $a_k \geq 0$ then $b_{k+1} = a_k + 1 = b_k + 1$. If $a_k < 0$ then $b_{k+1} = |1 - b_k| = b_k - 1$ (as $b_k \geq 1$). So $(b_0, \ldots, b_{24})$ is a walk from 0 with steps ± 1 that never goes below 0. Conversely, every such walk comes from sequences: for $k \leq 23$, if $b_k > 0$ the step out of b_k tells us the sign of a_k (up means $a_k = b_k$, down means $a_k = -b_k$), and if $b_k = 0$ then $a_k = 0$ and the walk is forced up, as it must be. So the walk determines $a_0, \ldots, a_{23}$, and only the sign of a_{24} is left open. For $a_{24} = 6$ the walk ends at 6; for $a_{24} = -4$ it ends at 4; in each case the sign is fixed, so each walk gives exactly one sequence.

Let $W(j)$ be the number of 24-step walks from 0 to j that never go below 0. There are $\binom{24}{(24-j)/2}$ unrestricted walks (choose the down-steps). A walk that touches -1 can be reflected up to its first visit to -1, giving a walk from -2 to j, which has $\frac{24-j}{2} - 1$ down-steps; this is a bijection (reflection principle). So

$W(6) = \binom{24}{9} - \binom{24}{8} = 1307504 - 735471 = 572033,$

$W(4) = \binom{24}{10} - \binom{24}{9} = 1961256 - 1307504 = 653752.$

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Hence $N = 572033 + 653752 = 1225785$, and the remainder is 785.

Check. Try the same question with 8 steps: $a_8^2 - 2a_8 - 8 = 0$ gives $a_8 = 4$ or -2, and the formula gives $W(4) + W(2) = \left(\binom{8}{2} - \binom{8}{1}\right) + \left(\binom{8}{3} - \binom{8}{2}\right) = 20 + 28 = 48$. Listing all 8-step sequences by computer gives 48 as well. For 24 steps, a computer run over all sign choices gives $N = 1225785$.

12. Answer: 28**. The idea.** Describe the polynomial $f(x) = ax^2 + bx + c$ by a and its two values $p = f(0)$ and $q = f(1)$ instead of by a, b, c. Two different roots in $(0, 1)$ turn out to mean exactly that p and q are positive integers with $\sqrt{p} + \sqrt{q} < \sqrt{a}$, and that leaves very few possibilities.

Work it out. Put $p = f(0) = c$ and $q = f(1) = a + b + c$. For fixed a, the pair (p, q) determines $c = p$ and $b = q - p - a$, so we count triples (a, p, q) of integers.

The condition is necessary. Suppose the roots are $r \neq s$ in $(0, 1)$. Then $f(x) = a(x - r)(x - s)$, so $p = ars$ and $q = a(1 - r)(1 - s)$ are positive integers, and by the Cauchy–Schwarz inequality

$$\sqrt{p} + \sqrt{q} = \sqrt{a}\left(\sqrt{r}\sqrt{s} + \sqrt{1 - r}\sqrt{1 - s}\right) \leq \sqrt{a}\,\sqrt{r + (1 - r)}\,\sqrt{s + (1 - s)} = \sqrt{a},$$

with equality only if $(\sqrt{r}, \sqrt{1 - r})$ and $(\sqrt{s}, \sqrt{1 - s})$ are proportional, that is, $r = s$. So $\sqrt{p} + \sqrt{q} < \sqrt{a}$.

The condition is sufficient. Suppose $p, q \geq 1$ are integers with $\sqrt{p} + \sqrt{q} < \sqrt{a}$, that is, $a > p + q + 2\sqrt{pq}$. With $b = q - p - a$ and $c = p$ the discriminant is

$$b^2 - 4ac = (a + p - q)^2 - 4ap = (a - p - q)^2 - 4pq > 0,$$

because $a - p - q > 2\sqrt{pq} > 0$. The vertex $x = -\frac{b}{2a} = \frac{a+p-q}{2a}$ lies strictly between 0 and 1 because $|p - q| < p + q < a$. An upward parabola that is positive at 0 and at 1 and dips below the axis between them has both roots in $(0, 1)$.

Counting. Since $\sqrt{p} + \sqrt{q} \geq 2$, we need $a \geq 5$. If $p + q \geq 6$ then $pq \geq p + q - 1 \geq 5$, so $(\sqrt{p} + \sqrt{q})^2 = p + q + 2\sqrt{pq} \geq 6 + 2\sqrt{5} > 10$: too big for every $a \leq 10$. The remaining pairs, with the value of $(\sqrt{p} + \sqrt{q})^2$ and the $a \leq 10$ that exceed it:

$(1, 1)$: 4, so $a = 5, \ldots, 10$: 6 polynomials.

$(1, 2), (2, 1)$: $3 + 2\sqrt{2} \approx 5.83$, so $a = 6, \ldots, 10$: $2 \cdot 5 = 10$.

$(1, 3), (3, 1)$: $4 + 2\sqrt{3} \approx 7.46$, so $a = 8, 9, 10$: $2 \cdot 3 = 6$.

$(2, 2)$: 8, so $a = 9, 10$: 2.

$(1, 4), (4, 1)$: 9, so $a = 10$: 2.

$(2, 3), (3, 2)$: $5 + 2\sqrt{6} \approx 9.90$, so $a = 10$: 2.

The total is $6 + 10 + 6 + 2 + 2 + 2 = 28$.

Check. The smallest case $a = 5$, $p = q = 1$ is $5x^2 - 5x + 1$, with roots $\frac{5 \pm \sqrt{5}}{10} \approx 0.28$ and 0.72. The borderline pair $(1, 4)$ with $a = 9$ gives $b = 4 - 1 - 9 = -6$ and $9x^2 - 6x + 1 = (3x - 1)^2$: a double root, correctly excluded by the strict inequality. A computer search over all integer b, c for $a \leq 10$ also finds 28. The answer is **028**.

13. Answer: 364**. The idea.** The side BC is not given, so a direct coordinate attack means carrying an unknown through messy formulas. The key is a classical fact: the line from the touch point D through the midpoint of the altitude always passes through the excenter I_A opposite A, which lies on line AI. So $X = I_A$ for every triangle with $AB \neq AC$. The same homothety that proves this also gives the ratio $DX : DM$, and the condition turns into a simple equation for BC.

Work it out. Write $a = BC$, $b = CA = 14$, $c = AB = 13$, $s = \frac{a+b+c}{2}$, K for the area, $r_a = \frac{K}{s-a}$ for the radius of the A-excircle and $h = \frac{2K}{a}$ for the altitude from A. Let the A-excircle touch BC at E, and let E' be the point of the excircle diametrically opposite E. The segment EE' is perpendicular to BC, has length $2r_a$ and has midpoint I_A.

Step 1: A, D, E′ are collinear. The homothety centred at A with positive ratio $\frac{r_a}{r}$ takes the incircle to the excircle. It maps the point of the incircle farthest from A in the direction perpendicular to BC, namely D, to the point of the excircle farthest from A in that direction, namely E'. So D lies on segment AE', with A above BC and E' below it.

Step 2: $X = I_A$ and the ratio. Let H be the foot of the altitude. The homothety centred at D that sends A to E' has negative ratio $-\frac{DE'}{DA} = -\frac{2r_a}{h}$ (compare the distances of E' and A from line BC). It sends line AH to the parallel line through E' and fixes line BC, so it sends H to E, and therefore sends the midpoint M of AH to the midpoint I_A of $E'E$. Hence D, M, I_A are collinear, with D between M and I_A and $\frac{DI_A}{DM} = \frac{2r_a}{h}$. Since $AB \neq AC$, the line DM is not line AI, so it meets line AI only at I_A: $X = I_A$.

Step 3: find a. Now $\frac{DX}{DM} = \frac{2r_a}{h} = \frac{2K/(s-a)}{2K/a} = \frac{a}{s-a}$. The point D is the midpoint of MX exactly when this ratio is 1, i.e. $a = s - a$, i.e. $b + c = 3a$. So $a = \frac{27}{3} = 9$ and $s = 18$.

Step 4: AX. The excircle touches line AB at a point T with $AT = s$, and $\angle TAI_A = \frac{A}{2}$, so $AI_A = \frac{s}{\cos(A/2)}$. With $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$,

$AX^2 = \frac{s^2\,bc}{s(s-a)} = \frac{s\,bc}{s-a} = \frac{18\cdot 182}{9} = 364.$

Check. Coordinates: $B = (0,0)$, $C = (9,0)$, $A = (3, 4\sqrt{10})$ (indeed $3^2 + 160 = 13^2$ and $6^2 + 160 = 14^2$). The foot of the altitude is $(3,0)$, so $M = (3, 2\sqrt{10})$. $BD = s - b = 4$ gives $D = (4,0)$. The excircle has $BE = s - c = 5$ and $r_a = \frac{K}{s-a} = \frac{18\sqrt{10}}{9} = 2\sqrt{10}$, so $I_A = (5, -2\sqrt{10})$. The midpoint of M and I_A is $(4,0) = D$, as required, and $AX^2 = 2^2 + (6\sqrt{10})^2 = 364$.

14. Answer: 736. **The idea.** In a monotone triple the smallest and largest entries are at the ends. So 1 and 9 both sit at an end of their row and of their column, which puts them in corners. Rotating the grid lets us fix 1 in the top-left corner (a factor of 4). The corner holding 9 then fixes most of the directions, and each remaining case is a count of orderings compatible with a small set of inequalities.

Work it out. Label the grid

$a \quad b \quad c$
$d \quad e \quad f$
$g \quad h \quad i$

with $a = 1$. Rotations send the corners to one another and preserve orderliness, so the total is 4 times the number of placements with $a = 1$. Now row 1 increases to the right and column 1 increases downward. The 9 is in corner c, g or i.

9 at i. Rows $1, 3$ and columns $1, 3$ all increase toward i. If row 2 and column 2 also increase, every row and column increases: this is a standard Young tableau of shape 3×3, and the hook length formula gives $\frac{9!}{5\cdot4\cdot3\cdot4\cdot3\cdot2\cdot3\cdot2\cdot1} = 42$. If row 2 decreases and column 2 increases, we get the chain $b < c < f < e < d < g < h$ covering all seven cells, so there is exactly 1 placement. By the diagonal reflection, row 2 increasing with column 2 decreasing also gives 1. If both decrease, then $f < e < b < c < f$, which is impossible. Subtotal 44.

9 at c. Now column 3 decreases downward: $9 > f > i$, and $1 < d < g$. The map "reflect left to right, then replace each x by $10-x$" keeps 1 at a and 9 at c, preserves orderliness and the direction of row 2, and reverses column 2. So we may take column 2 increasing ($b < e < h$) and double the count.

Row 2 decreasing ($d > e > f$): then $h > e > f > i$, so row 3 must decrease: $g > h > i$. The poset is $g > h > e$, $g > d > e$, and $e > b$, $e > f > i$. So $g = 8$, e has exactly three cells below it and so $e = 5$, $\{d, h\} = \{6, 7\}$ in 2 ways, and b, f, i fill $\{2, 3, 4\}$ with $f > i$ in 3 ways. That gives 6.

Row 2 increasing ($d < e < f$), row 3 increasing ($g < h < i$): the chain $d < g < h < i < f$ together with $b < e < h$ and $d < e$. So $f = 8$ and then $i = 7$ and $h = 6$ are forced, and b, d, e, g fill $\{2, 3, 4, 5\}$ with $d < g$, $d < e$, $b < e$. That gives 5 orders ($dbge$, $dbeg$, $dgbe$, $bdge$, $bdeg$).

Row 2 increasing, row 3 decreasing ($g > h > i$): now each of f, g, h lies above both e and i, and e lies above b and d. So the top three values go to $\{f, g, h\}$ with $g > h$ (3 ways), and the bottom four to $\{b, d, e, i\}$ with e above b, d (2 orders of b, d, e with e last, then 4 places for i: 8 ways). That gives 24.

So with 9 at c there are $2(6 + 5 + 24) = 70$ placements. With 9 at g there are also 70, by the diagonal reflection.

Total with $a = 1$: $44 + 70 + 70 = 184$. Altogether $4 \cdot 184 = 736$.

Check. A computer run over all $9! = 362880$ placements finds 736. It also confirms the case split: $42 + 1 + 1$ with 9 opposite 1, and $29 + 29 + 6 + 6$ with 9 at c.

15. Answer: 448**. The idea.** Give each x a *type*: the parities of x, $f(x)$ and $f(f(x))$, written as a word in E and O. If x has type pqr, then $f(x)$ has type $q\,r\,\bar{p}$, where $\bar{p}$ is the opposite of p, because $f(f(f(x)))$ has the opposite parity to x. So f must send each type class into the *next* type class. The eight types fall into two orbits under this rule, which splits the count into a few clean cases.

Work it out. The rule $pqr \mapsto qr\bar{p}$ has the orbits

EEE → EEO → EOO → OOO → OOE → OEE → EEE and EOE → OEO → EOE.

Conversely, suppose we label each number with a type whose first letter is its own parity, and f sends every number labelled t to a number labelled $\text{next}(t)$. Then the parities of $f(x)$, $f(f(x))$ and $f(f(f(x)))$ are the first letters of the next three labels, namely q, r and $\bar{p}$. So the labels are the true types and the condition holds. Valid functions therefore correspond to pairs (labelling, f respecting it). For a fixed labelling, the number of such f is $\prod_t |\text{class}(\text{next}(t))|^{|\text{class}(t)|}$.

This product is 0 if some nonempty class is followed by an empty one. So within each orbit, the classes are either all empty or all nonempty. The 4 evens go to types EEE, EEO, EOO, EOE, and the 4 odds to OOO, OOE, OEE, OEO.

Case 1: the 6-orbit is empty. All evens are EOE and all odds OEO, and f sends evens to odds and odds to evens freely: $4^4 \cdot 4^4 = 65536$.

Case 2: both orbits are used. Each of the eight classes has exactly one element: $4! \cdot 4! = 576$ labellings, each allowing 1 function. Total 576.

Case 3: only the 6-orbit is used. The even classes EEE, EEO, EOO have sizes 2, 1, 1 in some order, and so do the odd classes: $3 \cdot 3$ size patterns, each with $\frac{4!}{2!} \cdot \frac{4!}{2!} = 144$ labellings. Going around the 6-cycle $x_1 \to x_2 \to \cdots \to x_6 \to x_1$ of class sizes, the product is $\prod x_{i+1}^{x_i}$. Each class of size 2 contributes $2^{(\text{size of the class before it})}$ and everything else contributes 1. So the product is

$2 \cdot 2 = 4$, unless the class before one of the 2s is the other 2, which gives $4 \cdot 2 = 8$. That happens only when the two doubled classes are consecutive across an even/odd boundary (EOO then OOO, or OEE then EEE): 2 patterns give 8 and 7 give 4. Total $144 \cdot (2 \cdot 8 + 7 \cdot 4) = 144 \cdot 44 = 6336$.

Altogether $65536 + 576 + 6336 = 72448$, and the remainder is 448.

Check. Case 1 is the easy family: f always changes parity, so three steps change it an odd number of times. The analogous two-step question ($f(f(x))$ of opposite parity to x) gives a single 4-cycle of types and a count of 20160. For the three-step condition, a computer search over all 8^8 functions confirms 72448.