MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

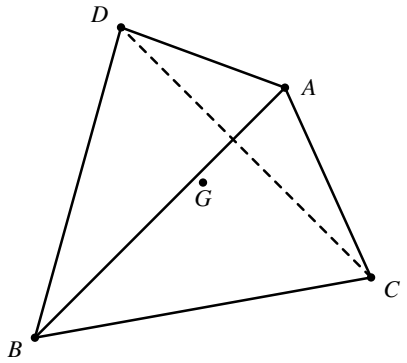
DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. The numbers $1, 2, 3, \ldots, 9$ are placed in the nine cells of a 3×3 grid, one number in each cell. Let N be the number of placements in which the three numbers in each row add up to 15. Find the remainder when N is divided by 1000.

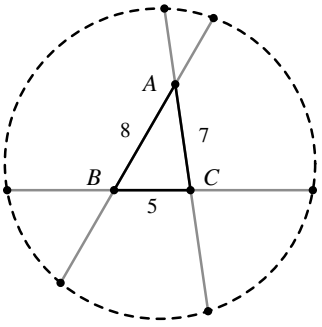
2. Five fair six-sided dice are rolled. The probability that the largest number showing appears on exactly two of the five dice is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. Let $ABCD$ be a regular tetrahedron and let G be its centroid (the point equidistant from all four vertices). A point P is chosen uniformly at random from the interior of the tetrahedron. The probability that P is closer to G than it is to each of the four vertices A, B, C, D can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



4. A finite sequence of positive integers is called a geometric climb if it has at least three terms, it is strictly increasing, and each term after the first equals the previous term multiplied by the same fixed ratio. (The ratio need not be an integer; for example $4, 6, 9$ is a geometric climb with ratio $\frac{3}{2}$.) Let N be the number of geometric climbs whose last term is 1,000,000. Find the remainder when N is divided by 1000.

5. Triangle ABC has $BC = 5$, $CA = 7$ and $AB = 8$. Each side of the triangle is extended past both of its endpoints, each time by the length of the side opposite that endpoint. For example, side $\overline{AB}$ is extended past A by $BC = 5$ and past B by $CA = 7$. This produces six new points, two on each of the lines AB, BC, CA, and these six points lie on one circle. Find the square of the radius of that circle.



6. Each of the nine entries of a 3×3 matrix is chosen independently to be 1 or -1, each with probability $\frac{1}{2}$. The probability that the determinant of the matrix is positive (strictly greater than 0) can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

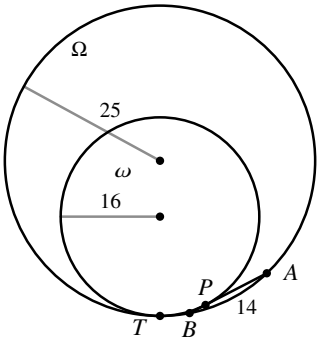
7. Consider the 105^2 ordered pairs (b, c) of integers with $0 \le b \le 104$ and $0 \le c \le 104$. Call a pair good if there is at least one integer n for which $n^2 + bn + c$ is divisible by 105. Let N be the number of good pairs. Find the remainder when N is divided by 1000.

8. Consider the ten-digit positive integers in which every digit is 1, 2 or 3 (for example 3121132213). Let N be the number of these integers that are divisible by 11. Find the remainder when N is divided by 1000.

9. For integers a and b, consider the polynomial $x^4 + ax^3 + bx^2 + ax + 1$. How many ordered pairs (a, b) of integers with $|a| \le 20$ and $|b| \le 20$ make this polynomial have four distinct real roots?

10. How many polynomials $P(x)$ with integer coefficients and degree at most 3 (constant polynomials, including the zero polynomial, count) have all four of the values $P(0)$, $P(1)$, $P(2)$, $P(3)$ in the set $\{0, 1, 2, \ldots, 9\}$?

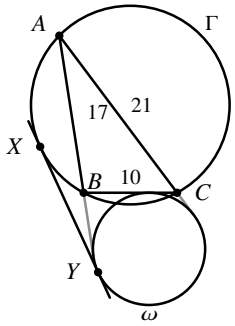
11. Circle Ω has radius 25, and circle ω of radius 16 lies inside Ω and is tangent to it at the point T. A chord $\overline{AB}$ of Ω has length 14 and is tangent to ω at a point P of the chord. Then $TA \cdot TB = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



12. In an election between candidates A and B, each candidate receives exactly 10 of the 20 votes. The votes are counted one at a time, so an order of counting is a sequence of 20 letters consisting of ten A's and ten B's; all $\binom{20}{10}$ such sequences are possible. Let N be the number of these sequences for which, at every stage of the count, neither candidate is ahead of the other by more than 3 votes. Find the remainder when N is divided by 1000.

13. Triangle ABC has $AB = 17$, $BC = 10$ and $CA = 21$. Let Γ be its circumcircle, and let ω be its excircle opposite A: the circle outside the triangle that is tangent to side $\overline{BC}$ and to the extensions of sides $\overline{AB}$ and $\overline{AC}$ beyond B and C. The circles Γ and ω cross each other, and they have two common external tangent lines (lines tangent to both circles with both circles on the same side). One of these lines touches Γ at X and ω at Y. Find XY^2.

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14. A board with 3 rows and 5 columns has a lamp on each of its 15 cells, and all lamps start off. Pressing a cell switches the lamp on that cell and the lamps on every cell sharing a side with it (on becomes off and off becomes on); cells may be pressed any number of times, in any order. Let N be the number of the 2^{15} on/off patterns of the board that can be reached from the all-off board by some sequence of presses. Find the remainder when N is divided by 1000.

15. Four balls with radii 1, 2, 3 and 4 are placed in space so that every two of them are externally tangent. The six points at which two of the balls touch all lie on one sphere. The square of the radius of this sphere is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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Answer Key — AIME Practice Contest 5

1. 592

2. 557

3. 229

4. 125

5. 103

6. 19

7. 520

8. 953

9. 184

10. 850

11. 809

12. 616

13. 219

14. 096

15. 791

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Solutions — AIME Practice Contest 5

1. Answer: 592**. The idea.** First decide which three numbers go into each row, and only then put them in order. There are very few ways to split $1, \ldots, 9$ into three triples of sum 15, and following the largest numbers finds all of them.

Work it out. The triple containing 9 needs two more numbers adding to 6, so it is $\{1, 5, 9\}$ or $\{2, 4, 9\}$.

If it is $\{1, 5, 9\}$, the numbers left are $2, 3, 4, 6, 7, 8$. The triple containing 8 needs two of $2, 3, 4, 6, 7$ adding to 7, and only $3 + 4$ works. The last triple is $\{2, 6, 7\}$, whose sum is indeed 15.

If it is $\{2, 4, 9\}$, the numbers left are $1, 3, 5, 6, 7, 8$. The triple containing 8 needs two of $1, 3, 5, 6, 7$ adding to 7: only $1 + 6$. The last triple is $\{3, 5, 7\}$, again of sum 15.

So there are exactly 2 ways to split the numbers into rows. For each split, the three triples can be assigned to the three rows in $3! = 6$ ways, and the numbers inside each row can be ordered in $3! = 6$ ways, which gives $6 \cdot 6^3 = 1296$ placements. Hence

$N = 2 \cdot 1296 = 2592,$

and the remainder is 592.

Check. The two splits are the rows and the columns of the magic square with rows $2, 7, 6$ / $9, 5, 1$ / $4, 3, 8$: its rows are $\{2, 6, 7\}$, $\{1, 5, 9\}$, $\{3, 4, 8\}$ and its columns are $\{2, 4, 9\}$, $\{3, 5, 7\}$, $\{1, 6, 8\}$. A computer count over all $9!$ placements also gives 2592. The answer is **592**.

2. Answer: 557**. The idea.** Split the outcomes according to the value k of the largest number. Once k is fixed, "appears exactly twice" means: two chosen dice show k, and the other three show something smaller than k.

Work it out. Fix $k \in \{1, 2, \ldots, 6\}$. The two dice showing k can be chosen in $\binom{5}{2} = 10$ ways, and each of the other three dice must show one of the $k - 1$ values below k, in $(k-1)^3$ ways. Every outcome with the required property is counted exactly once, for its own largest value k. So the number of good outcomes is

$10\left(0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3\right) = 10 \cdot 225 = 2250,$

using $1^3 + \cdots + 5^3 = (1 + \cdots + 5)^2 = 225$. There are $6^5 = 7776$ equally likely outcomes, so the probability is

$$\frac{2250}{7776} = \frac{125}{432}$$

(divide by 18; $125 = 5^3$ and $432 = 2^4 \cdot 3^3$ share no factor). Hence $m + n = 125 + 432 = 557$.

Check. The same method counts every possibility for how often the top value appears: once $5 \cdot (1 + 16 + 81 + 256 + 625) = 4895$, twice 2250, three times $10 \cdot 55 = 550$, four times $5 \cdot 15 = 75$, five times 6. These add to $4895 + 2250 + 550 + 75 + 6 = 7776 = 6^5$, as they must. The answer is **557**.

3. Answer: 229**. The idea.** "Closer to A than to G" is one side of the plane that perpendicularly bisects $\overline{AG}$. Because G lies on the altitude from A, that plane is parallel to the opposite face, so it slices off a small tetrahedron similar to $ABCD$ at the corner A.

Work it out. Let h be the height of the tetrahedron. The centroid lies on the altitude from A, one quarter of the way up from face BCD, so $AG = \frac{3}{4}h$. The set of points equidistant from A and G is the plane through the midpoint of $\overline{AG}$ perpendicular to AG. Since AG is along the altitude, this plane is parallel to face BCD, at distance $\frac{1}{2} \cdot \frac{3}{4}h = \frac{3}{8}h$ from A.

The points of the tetrahedron closer to A than to G therefore form the small tetrahedron cut off at A by this plane, similar to $ABCD$ with ratio $\frac{3}{8}$, so with volume $\left(\frac{3}{8}\right)^3 = \frac{27}{512}$ of the whole. The same holds at B, C and D.

These four corner pieces do not overlap: along edge AB the corner at A reaches only $\frac{3}{8}$ of the way to B and the corner at B only $\frac{3}{8}$ of the way to A, and $\frac{3}{8} + \frac{3}{8} < 1$. (More precisely, the corner at A is where the barycentric weight of A exceeds $\frac{5}{8}$, and two weights cannot both exceed $\frac{1}{2}$.) A point is closer to G than to every vertex exactly when it lies in none of the four corners, so the probability is

$1 - 4 \cdot \frac{27}{512} = 1 - \frac{108}{512} = \frac{404}{512} = \frac{101}{128}$.

So $m + n = 101 + 128 = 229$.

Check. In coordinates $A = (1, 1, 1)$, $B = (1, -1, -1)$, $C = (-1, 1, -1)$, $D = (-1, -1, 1)$, G is the origin and $|P - A|^2 < |P|^2$ becomes $x + y + z > \frac{3}{2}$. Face BCD is $x + y + z = -1$ and A is at $x + y + z = 3$, so the plane is $\frac{3-3/2}{3-(-1)} = \frac{3}{8}$ of the way down from A, as claimed. The answer is **229**.

4. Answer: 125**. The idea.** Write the ratio in lowest terms as $\frac{p}{q}$. A geometric climb with $j + 1$ terms then has the shape $kq^j, kq^{j-1}p, \ldots, kp^j$ for a positive integer k, so the last term must be a multiple of p^j. For each allowed p, the numerator q can be anything below p and coprime to it, and the totals collapse by the identity $\sum_{d|n} \varphi(d) = n$.

Work it out. Take a climb $a_0 < a_1 < \cdots < a_j$ with ratio $\frac{p}{q}$, where $p > q \geq 1$ and $\gcd(p, q) = 1$, and $j \geq 2$. Since $a_j = a_0 \frac{p^j}{q^j}$ is an integer and $\gcd(p^j, q^j) = 1$, q^j divides a_0; write $a_0 = kq^j$. Then $a_i = kq^{j-i}p^i$ for every i, all integers, and the last term is kp^j. Conversely any $k \geq 1$ and coprime $p > q \geq 1$ give a climb. The climb determines p, q and k, so we are counting triples (j, p, q) with $p^j \mid 10^6$ (then $k = 10^6/p^j$ is forced), $1 \leq q < p$ and $\gcd(p, q) = 1$.

For a fixed $p \geq 2$ the number of q is $\varphi(p)$. And $p^j \mid 2^6 5^6$ exactly when $p \mid 10^{\lfloor 6/j \rfloor}$. So the climbs with $j + 1$ terms number

$\sum_{p|10^m,\ p\geq 2} \varphi(p) = 10^m - 1$, where $m = \lfloor 6/j \rfloor$,

using $\sum_{d|n} \varphi(d) = n$ and removing $p = 1$ (for which $\varphi(1) = 1$).

Now go through j: $j = 2$ gives $m = 3$ and 999 climbs; $j = 3$ gives $m = 2$ and 99; $j = 4, 5, 6$ give $m = 1$ and 9 each; for $j \geq 7$, $m = 0$ and there are none (a ratio above 1 would need $p^7 \mid 10^6$). So

$N = 999 + 99 + 9 + 9 + 9 = 1125$,

and the remainder is 125.

Check. The three-term count can be seen directly: a climb $a, b, 10^6$ needs $b^2 = 10^6 a$, i.e. $1000 \mid b$, so $b = 1000t$ and $a = t^2$ for $t = 1, \ldots, 999$: that is 999 climbs, as found. At the other extreme, a seven-term climb needs $p^6 \mid 10^6$, so $p \in \{2, 5, 10\}$, with $q = 1$ for $p = 2$, $q \in \{1, 2, 3, 4\}$ for $p = 5$ and $q \in \{1, 3, 7, 9\}$ for $p = 10$: $1 + 4 + 4 = 9$ climbs, as found. (For example $p = 2$ gives $15625, 31250, \ldots, 1000000$.) The answer is **125**.

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5. Answer: 103**. The idea.** Measure every new point from the place where the incircle touches its line. The tangent lengths from the vertices make that distance the same, the semiperimeter s, for all six points, and a point on a tangent line at distance s from the touch point is at distance $\sqrt{r^2+s^2}$ from the incenter.

Work it out. Write $a = BC = 5$, $b = CA = 7$, $c = AB = 8$ and $s = \frac{a+b+c}{2} = 10$. Let I be the incenter, r the inradius, and let the incircle touch $\overline{AB}$ at F. The two tangent segments from A are equal, and the standard computation gives $AF = s - a$ and $BF = s - b$.

The new point A_1 beyond A on line AB has $AA_1 = a$. Since A lies between F and A_1, $FA_1 = FA + AA_1 = (s-a) + a = s$. Likewise the new point B_1 beyond B has $FB_1 = (s-b) + b = s$. The same happens at every vertex: the point beyond vertex V on a side is extended by the side opposite V, and the tangent length from V is s minus that same side, so every one of the six points is at distance exactly s from the touch point on its own line.

Since $IF \perp AB$ and $IF = r$, the Pythagorean theorem gives $IA_1^2 = r^2 + s^2$, and the same for all six points. So they lie on the circle centred at I with radius squared $r^2 + s^2$.

By Heron, $[ABC] = \sqrt{10 \cdot 5 \cdot 3 \cdot 2} = \sqrt{300} = 10\sqrt{3}$, so $r = \frac{[ABC]}{s} = \sqrt{3}$ and

$r^2 + s^2 = 3 + 100 = 103$.

Check. Put $A = (0,0)$, $B = (8,0)$; then $C = (\frac{11}{2}, \frac{5\sqrt{3}}{2})$, the incircle touches AB at $F = (s-a, 0) = (5,0)$ and $I = (5, \sqrt{3})$. The new points on line AB are $(-5,0)$ and $(15,0)$, both 10 from F, and $10^2 + (\sqrt{3})^2 = 103$ for each. The answer is **103**.

6. Answer: 19**. The idea.** Multiplying a row or a column by -1 only flips the sign of the determinant, so we may normalise the matrix to have first row and first column all equal to 1 and study the 16 normalised matrices. A separate symmetry (swapping two rows) shows positive and negative determinants are equally likely.

Work it out. Given any matrix M, negate the columns whose top entry is -1, and then negate the rows (other than the first) whose first entry is now -1. The result is the matrix N whose rows are $(1,1,1)$, $(1,a,b)$ and $(1,c,d)$, with $a,b,c,d \in \{1,-1\}$, and $\det M = \pm \det N$. Each of the 16 matrices N arises from exactly $2^9/16 = 32$ matrices M, so $P(\det M \neq 0)$ is the fraction of the 16 matrices N with $\det N \neq 0$.

Subtract the first row from the other two:

$\det N = (a-1)(d-1) - (b-1)(c-1)$.

Each factor $a-1, b-1, \ldots$ is 0 or -2. Writing $a' = \frac{1-a}{2} \in \{0,1\}$ and so on, $\det N = 4(a'd' - b'c')$, which is 4 times the determinant of a 2×2 matrix of 0s and 1s. Such a determinant is nonzero exactly when $a'd' \neq b'c'$: one product is 1 and the other 0, i.e. $a' = d' = 1$ with $b'c' = 0$ (3 ways) or $b' = c' = 1$ with $a'd' = 0$ (3 ways). That is 6 of 16, so $P(\det M \neq 0) = \frac{6}{16} = \frac{3}{8}$.

Swapping the second and third rows is a bijection on all 512 matrices that negates the determinant, so positive and negative determinants are equally common. Hence $P(\det M > 0) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$, and $m + n = 3 + 16 = 19$.

Check. In counts: $512 \cdot \frac{3}{8} = 192$ matrices are invertible, 96 with positive determinant; every nonzero determinant equals ± 4. For example the matrix with rows $(1,1,1)$, $(1,-1,1)$, $(1,1,-1)$ has determinant $(-2)(-2) - 0 = 4 > 0$. The answer is **019**.

7. Answer: 520**. The idea.** Split $105 = 3 \cdot 5 \cdot 7$ with the Chinese Remainder Theorem, and count prime by prime. Modulo a prime p, a monic quadratic has a root exactly when it factors

as $(x-r)(x-s)$, and different unordered pairs $\{r,s\}$ give different polynomials. So instead of counting coefficient pairs, count root pairs.

Work it out. *One prime.* Fix a prime p and consider x^2+bx+c with b,c taken mod p. If it has a root r mod p, divide by $x-r$: $x^2+bx+c \equiv (x-r)(x-s)$ with $s \equiv -b-r$. Conversely every product $(x-r)(x-s)$ is such a polynomial, with $b \equiv -(r+s)$ and $c \equiv rs$. Two products are the same polynomial only when they have the same roots: if $(x-r)(x-s) \equiv (x-r')(x-s')$, then plugging in $x=r$ gives $(r-r')(r-s') \equiv 0$, so r is r' or s' (mod a prime, a product is 0 only if a factor is), and then the other roots agree since their sums agree. So the polynomials with a root mod p correspond to unordered pairs $\{r,s\}$ with repetition allowed: there are $\binom{p}{2}+p = \frac{p(p+1)}{2}$ of them. That gives 6 for $p=3$, 15 for $p=5$ and 28 for $p=7$.

Gluing. By the Chinese Remainder Theorem, a pair (b,c) mod 105 is the same as a choice of (b,c) mod 3, mod 5 and mod 7 independently. And $105 \mid n^2+bn+c$ for some n exactly when there are roots n_3, n_5, n_7 modulo 3, 5, 7 separately: given those, CRT produces one n that matches all three. So

$N = 6 \cdot 15 \cdot 28 = 2520,$

and the remainder when N is divided by 1000 is 520.

Check. For $p=3$ list them: the root pairs $\{0,0\},\{1,1\},\{2,2\},\{0,1\},\{0,2\},\{1,2\}$ give x^2, x^2+x+1, x^2+2x+1, x^2+2x, x^2+x, $x^2+2 \pmod 3$: six different polynomials, and the other three, x^2+1, x^2+x+2, x^2+2x+2, have no root mod 3. A direct computer check over all 105^2 pairs also gives 2520. The answer is **520**.

8. Answer: 953**. The idea.** A number is divisible by 11 exactly when its alternating digit sum is. With digits only 1, 2, 3, that alternating sum is too small to be a nonzero multiple of 11, so it must be exactly 0: the problem becomes counting pairs of 5-digit strings with equal digit sums.

Work it out. Since $10 \equiv -1 \pmod{11}$, the number is congruent to $A-B$, where A is the sum of the five digits in the odd places (counting from the right) and B the sum of the five digits in the even places. Each of A, B lies between 5 and 15, so $-10 \le A-B \le 10$, and the only multiple of 11 in that range is 0. Hence we need $A=B$, and the two groups of five digits can be chosen independently. Let $N(s)$ be the number of strings of five digits from $\{1,2,3\}$ with sum s; these are the coefficients of $(x+x^2+x^3)^5$, that is, of $(1+x+x^2)^5$ shifted. Building up the rows of this trinomial triangle (1, 1, 1; then 1, 2, 3, 2, 1; then 1, 3, 6, 7, 6, 3, 1; then 1, 4, 10, 16, 19, 16, 10, 4, 1), the fifth row is $1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1$ for $s = 5, 6, \ldots, 15$. The count is $\sum_s N(s)^2 = 2(1+25+225+900+2025) + 51^2 = 2 \cdot 3176 + 2601 = 8953$. The remainder is 953.

Check. $\sum_s N(s)^2$ is also the central coefficient of $(1+x+x^2)^{10}$ (a string of ten digits has digit sum 20 exactly when its first five digits have sum s and its last five have sum $20-s$, and $N(20-s) = N(s)$ by the symmetry $d \mapsto 4-d$), and that coefficient is indeed 8953. The row sums check the table: $1+5+15+\cdots+1 = 243 = 3^5$. A direct computer test of all $3^{10} = 59049$ numbers also finds 8953.

9. Answer: 184**. The idea.** The coefficients read the same forwards and backwards, so 0 is not a root and x is a root exactly when $1/x$ is. Dividing by x^2 and substituting $y = x + \frac{1}{x}$ turns the quartic into a quadratic in y. Each real y with $|y| > 2$ gives two distinct real x; $|y| = 2$ gives the double root $x = \pm 1$; and $|y| < 2$ gives no real x.

Work it out. Dividing by x^2: $x^2 + \frac{1}{x^2} + a\left(x + \frac{1}{x}\right) + b = 0$, i.e. $g(y) = y^2 + ay + (b-2) = 0$. The equation $x + \frac{1}{x} = y$ is $x^2 - yx + 1 = 0$, with discriminant $y^2 - 4$. Different values of y give

different x (since y is determined by x), so we need g to have two distinct real roots, both with $|y| > 2$. Note $g(2) = b + 2a + 2$ and $g(-2) = b - 2a + 2$.

Case 1: one root below -2, one above 2. This happens exactly when $g(2) < 0$ and $g(-2) < 0$, i.e. $b < -2 - 2|a|$ (then the discriminant is automatically positive). With $b \geq -20$ this gives $18 - 2|a|$ values of b for $|a| \leq 8$: in total $18 + 2(16 + 14 + \cdots + 2) = 18 + 144 = 162$.

Case 2: both roots below -2. We need $g(-2) > 0$, the vertex $-\frac{a}{2} < -2$ (so $a > 4$), and a positive discriminant $a^2 - 4(b-2) > 0$. So $2a - 2 < b < \frac{a^2}{4} + 2$ with $b \leq 20$. For $a = 5, 6$: $8 < b < 8.25$ and $10 < b < 11$, nothing. For $a = 7$: $b = 13, 14$. For $a = 8$: $b = 15, 16, 17$. For $a = 9$: $b = 17, \ldots, 20$. For $a = 10$: $b = 19, 20$. For $a \geq 11$: $b > 20$, nothing. That is $2 + 3 + 4 + 2 = 11$.

Case 3: both roots above 2. Replacing x by $-x$ swaps a and $-a$, so this also gives 11.

The total is $162 + 11 + 11 = 184$.

Check. Take $(a, b) = (7, 13)$: $g(y) = y^2 + 7y + 11$ has roots $\frac{-7 \pm \sqrt{5}}{2} \approx -2.38, -4.62$, both below -2, giving four real x. For $(7, 15)$ the discriminant $49 - 52$ is negative, correctly excluded. A computer check of the real roots of all 41^2 quartics gives 184.

10. Answer: 850**. The idea.** A polynomial of degree at most 3 is fixed by its four values, so we are counting the quadruples of digits (v_0, v_1, v_2, v_3) that are the values of a polynomial with *integer* coefficients. Not every quadruple works: $P(2) - P(0)$ must be even, for instance. The clean way to see exactly which ones work is to write P in the basis $1,\ x,\ x(x-1),\ x(x-1)(x-2)$ instead of $1, x, x^2, x^3$.

Work it out. Write $P(x) = a + bx + c\,x(x-1) + d\,x(x-1)(x-2)$. Expanding, the coefficient of x^3 is d, that of x^2 is $c - 3d$, that of x is $b - c + 2d$ and the constant is a. So if a, b, c, d are integers, so are the coefficients, and conversely (solve from the top: d, then c, then b, then a). Hence P has integer coefficients exactly when a, b, c, d are integers. Now $P(0) = a$, $\quad P(1) = a + b$, $\quad P(2) = a + 2b + 2c$, $\quad P(3) = a + 3b + 6c + 6d$, so $a = v_0$, $b = v_1 - v_0$, $2c = v_2 - 2v_1 + v_0$ and $6d = v_3 - 3v_2 + 3v_1 - v_0$. The conditions are therefore:

- c is an integer $\iff v_2 \equiv v_0 \pmod 2$;
- given that, $6d$ is even $\iff v_3 + v_2 + v_1 + v_0$ is even $\iff v_3 \equiv v_1 \pmod 2$;
- $6d$ is a multiple of 3 $\iff v_3 - v_0 \equiv 0 \pmod 3$.

So we need $v_0 \equiv v_3 \pmod 3$, $v_2 \equiv v_0 \pmod 2$ and $v_1 \equiv v_3 \pmod 2$, and every such quadruple gives exactly one polynomial. Choose v_0 and v_3 first. The digits with remainder 0, 1, 2 mod 3 are $\{0, 3, 6, 9\}$, $\{1, 4, 7\}$, $\{2, 5, 8\}$, so there are $4^2 + 3^2 + 3^2 = 34$ choices of the pair. Then v_2 is any of the 5 digits with the parity of v_0, and v_1 any of the 5 digits with the parity of v_3. The count is $34 \cdot 5 \cdot 5 = 850$.

Check. The values $(0, 1, 0, 3)$ pass all three tests, and indeed $a = 0$, $b = 1$, $c = -1$, $d = 1$ give $P(x) = x - x(x-1) + x(x-1)(x-2) = x^3 - 4x^2 + 4x$, with $P(1) = 1$, $P(2) = 0$, $P(3) = 3$. The values $(0, 1, 0, 1)$ fail the mod-3 test, and their interpolating polynomial $\frac{2}{3}x^3 - 3x^2 + \frac{10}{3}x$ indeed has fractional coefficients. A computer that interpolates all 10^4 quadruples exactly finds 850 with integer coefficients.

11. Answer: 809**. The idea.** The homothety centred at T with ratio $\frac{16}{25}$ carries Ω onto ω. It turns the tangent length from A to ω into a fixed fraction of TA, so TA and TB are proportional to AP and BP. The product $AP \cdot PB$ is then a power of a point in Ω.

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Work it out. Let line TA meet ω again at A'. The homothety sends A to A', so $TA' = \frac{16}{25}TA$ and $AA' = \frac{9}{25}TA$. The power of A with respect to ω is $AP^2 = AA' \cdot AT = \frac{9}{25}TA^2$, so $TA = \frac{5}{3}AP$. In the same way $TB = \frac{5}{3}BP$, hence $TA \cdot TB = \frac{25}{9}\,AP \cdot PB$.

Now locate P. Let O and O' be the centres; $OO' = 25 - 16 = 9$. The chord has half-length 7, so it lies at distance $\sqrt{625 - 49} = 24$ from O. Let θ be the angle between $\overrightarrow{OO'}$ and the unit normal of the chord pointing away from O. The distance from O' to the chord is $24 - 9\cos\theta = 16$ (the other sign would need $9\cos\theta = 40$), so $\cos\theta = \frac{8}{9}$. If M is the midpoint of the chord, then P and M are the feet of O' and O on the chord, so $MP = 9\sin\theta = \sqrt{17} < 7$ and P really lies on the chord. Then $OP^2 = 24^2 + 17 = 593$, and the power of P with respect to Ω gives $AP \cdot PB = 625 - 593 = 32$. Therefore $TA \cdot TB = \frac{25}{9} \cdot 32 = \frac{800}{9}$, and $m + n = 809$.

Check. Coordinates: $O = (0, 0)$, $T = (25, 0)$, $O' = (9, 0)$, chord on the line $\frac{8}{9}X + \frac{\sqrt{17}}{9}Y = 24$. Its endpoints on $X^2 + Y^2 = 625$ are 14 apart, the line is 16 from O', and computing $TA \cdot TB$ directly gives $\frac{800}{9} \approx 88.9$. As a rough check, $AP \cdot PB = 32$ with $AP + PB = 14$ gives $AP, PB = 7 \mp \sqrt{17}$, and $TA \cdot TB = \frac{25}{9}(49 - 17)$, the same value.

12. Answer: 616**. The idea.** Follow the lead L = (votes for A so far) − (votes for B so far). It starts and ends at 0, moves by ± 1 at each vote, and must stay between -3 and 3. After an even number of votes L is even, so it is -2, 0 or 2; after an odd number it is ± 1 or ± 3, which is automatically allowed. So it is enough to look at the count two votes at a time, as a walk on the three states $-2, 0, 2$.

Work it out. Two consecutive votes move the lead as follows. From 0: back to 0 in 2 ways (AB or BA), to 2 in 1 way (AA) and to -2 in 1 way (BB). From 2: to 0 in 1 way (BB), to 2 in 2 ways (AB passes through 3, BA through 1, both allowed), and AA is forbidden since it reaches 4. From -2 symmetrically. After m pairs of votes, let x_m be the number of valid beginnings ending with lead 0 and y_m the number ending with lead 2 (the same as with lead -2, by swapping A and B). Then $x_{m+1} = 2x_m + 2y_m, \qquad y_{m+1} = x_m + 2y_m, \qquad x_0 = 1,\ y_0 = 0$. Tabulating (x_m, y_m) for $m = 0, 1, \ldots, 9$: $(1, 0)$, $(2, 1)$, $(6, 4)$, $(20, 14)$, $(68, 48)$, $(232, 164)$, $(792, 560)$, $(2704, 1912)$, $(9232, 6528)$, $(31520$ The last step is $x_{10} = 2(31520 + 22288) = 107616$. So $N = 107616$ and the remainder is 616.

(Eliminating y gives $x_{m+1} = 4x_m - 2x_{m-1}$, so $x_m = \frac{1}{2}\left[(2 + \sqrt{2})^m + (2 - \sqrt{2})^m\right]$; the three states $-2, 0, 2$ have transition eigenvalues $2 - \sqrt{2}$, 2, $2 + \sqrt{2}$, and the one with eigenvalue 2 never touches the middle state.)

Check. With two votes each ($m = 2$) the lead can never exceed 2, and indeed $x_2 = 6 = \binom{4}{2}$. With four votes each ($m = 4$) the only bad orders are $AAAABBBB$ and $BBBBAAAA$ (a lead of 4 needs four equal votes before any of the other kind), and indeed $x_4 = 68 = \binom{8}{4} - 2$. A computer run through all 184756 orders finds exactly 107616 good ones.

13. Answer: 219**. The idea.** For two circles with radii R, ρ and centres d apart, a common external tangent has length $\sqrt{d^2 - (R - \rho)^2}$ between its points of contact. So everything hinges on the distance from the circumcentre O to the excentre I_A. That distance comes from the power of I_A with respect to Γ, which the arc midpoint of BC computes beautifully: $OI_A^2 = R^2 + 2R\rho$. Then $XY^2 = R^2 + 2R\rho - (R - \rho)^2 = \rho(4R - \rho)$.

Work it out. *Basic data.* $s = 24$, and Heron gives $[ABC] = \sqrt{24 \cdot 14 \cdot 3 \cdot 7} = 84$. The circumradius is $R = \frac{abc}{4[ABC]} = \frac{10 \cdot 21 \cdot 17}{336} = \frac{85}{8}$, and the exradius is $\rho = \frac{[ABC]}{s - a} = \frac{84}{14} = 6$.

The power of I_A. The bisector of angle A passes through the incentre I and the excentre I_A, and meets Γ again at M, the midpoint of arc BC. It is a classical fact that M is the centre of the circle through B, C, I, I_A (angle chasing gives $MB = MI$, and $\angle IBI_A = 90^\circ$ because the internal and external bisectors at B are perpendicular, so M is the midpoint of the hypotenuse II_A). Hence $MI_A = MB = 2R\sin\frac{A}{2}$ (chord MB subtends $\frac{A}{2}$ at A). Also $AI_A = \frac{\rho}{\sin(A/2)}$, since I_A is ρ from line AB. The point I_A lies outside Γ with A and M on one ray from it, so its power is $OI_A^2 - R^2 = I_AM \cdot I_AA = 2R\sin\frac{A}{2} \cdot \frac{\rho}{\sin(A/2)} = 2R\rho$.

The tangent. Drop perpendiculars from O and I_A to the tangent line, at X and Y. Both centres are on the same side, so $OXYI_A$ is a right trapezoid with parallel sides R and ρ, and $XY^2 = OI_A^2 - (R-\rho)^2 = 4R\rho - \rho^2 = \rho(4R-\rho)$. Here $XY^2 = 6\left(\frac{85}{2} - 6\right) = 6 \cdot \frac{73}{2} = 219$. (The external tangents exist because $d^2 - (R-\rho)^2 = \rho(4R-\rho) > 0$, and the circles cross because $d^2 = R^2 + 2R\rho < (R+\rho)^2$.)

Check. Coordinates: $B = (0,0)$, $C = (10,0)$, $A = \left(-\frac{13}{5}, \frac{84}{5}\right)$. Then $O = \left(5, \frac{75}{8}\right)$ (its distance to B is $\sqrt{25 + \frac{5625}{64}} = \frac{85}{8}$) and $I_A = \frac{-10A+21B+17C}{-10+21+17} = (7,-6)$, which is 6 below BC as it should be. So $OI_A^2 = 2^2 + \left(\frac{123}{8}\right)^2 = \frac{15385}{64}$, matching $R^2 + 2R\rho = \frac{7225}{64} + \frac{8160}{64}$, and $XY^2 = \frac{15385}{64} - \left(\frac{37}{8}\right)^2 = \frac{14016}{64} = 219$. Solving directly for the two external tangent lines by machine gives 219 for each.

14. Answer: 096**. The idea.** Think of a pattern as a vector of 15 bits, added modulo 2. A press adds a fixed vector, so the order of presses does not matter and pressing a cell twice does nothing: what matters is only the *set* S of cells pressed an odd number of times, and the result is a linear function $f(S)$ over the integers mod 2. The reachable patterns form the image of f, and its size is 2^{15} divided by the number of *quiet* sets, the sets S with $f(S) = 0$ (each reachable pattern is hit by exactly that many sets). Quiet sets can be found by chasing column by column, and a small matrix identity mod 2 shows that there are more of them than one would expect.

Work it out. Record the pressed cells column by column as vectors $x_1, \dots, x_5 \in \{0,1\}^3$ (top, middle, bottom), and put $x_0 = x_6 = 0$. A lamp in column k is switched by presses in its own column at itself or a vertical neighbour, and by presses beside it in columns $k \pm 1$. So the change in column k is $x_{k-1} + Tx_k + x_{k+1}$ (mod 2), where $T = I + A$ is the 3×3 matrix with rows $(1,1,0)$, $(1,1,1)$, $(0,1,1)$, and A is the matrix with rows $(0,1,0)$, $(1,0,1)$, $(0,1,0)$ (the vertical-neighbour relation inside a column). S is quiet exactly when $x_{k+1} = Tx_k + x_{k-1}$ for $k = 1, \dots, 5$ (signs do not matter mod 2). Once the first column $x_1 = v$ is chosen, these equations force $x_2, \dots, x_6$, and S is quiet exactly when the forced x_6 is 0. Writing $x_k = p_k(T)v$, we get $p_0 = 0$, $p_1 = 1$ and $p_{k+1} = Tp_k + p_{k-1}$: $p_2 = T$, $p_3 = T^2 + 1$, $p_4 = T^3 + 2T = T^3$, $p_5 = T^4 + T^2 + 1$, $p_6 = T^5 + T^3 + T + T^3 = T^5 + T$, all mod 2. Now A^2 has rows $(1,0,1)$, $(0,2,0)$, $(1,0,1)$, and A^3 has rows $(0,2,0)$, $(2,0,2)$, $(0,2,0)$, which is 0 mod 2. So $(T+I)^3 = A^3 \equiv 0$ and $p_6(T) = T^5 + T = T(T^4 + I) = T(T+I)^4 \equiv 0 \pmod 2$. So *every* one of the $2^3 = 8$ choices of the first column gives a quiet set. There are exactly 8 quiet sets, the image of f has $2^{15}/8 = 2^{12} = 4096$ elements, and the remainder is 096.

Check. Take $v = (1,0,0)$. Mod 2, $T^2 = I + A^2$ and $T^3 = I + A + A^2$, and the chase gives the columns $x_1, \dots, x_5 = (1,0,0)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(0,0,1)$. Press those eight cells. The top-left lamp is switched by the presses at itself and at its right-hand neighbour (the cell below it is not pressed): twice, so it ends off. The middle lamp of column 3 is not pressed itself, but all four of its neighbours are: four switches. Checking all 15 lamps the same way, each is switched an even number of times. By machine, forming all 2^{15} press sets gives exactly 4096 distinct patterns.

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(For comparison, the 3×3 board has no quiet set except the empty one, so there every pattern is reachable.)

15. Answer: 791**. The idea.** Coordinates are hopeless here; work with vectors from the centre O of the sphere instead. First show that from each ball's centre the squared distance to O is $\rho^2 + r_i^2$ (where ρ is the radius we want): the sphere meets each ball at right angles. That pins down every dot product between the four centre vectors. Four vectors in space are always linearly dependent, and writing out that dependence gives one equation for ρ.

Work it out. Let the balls have centres P_i and radii r_i ($i = 1, 2, 3, 4$), and let T_{ij} be the point where balls i and j touch. It lies on segment P_iP_j with $P_iT_{ij} = r_i$, $T_{ij}P_j = r_j$ and $P_iP_j = r_i + r_j$. Put $u_i = OP_i^2 - r_i^2$.

Step 1: $u_i = \rho^2$. Stewart's theorem in triangle OP_iP_j with the cevian OT_{ij} gives $OT_{ij}^2 = \frac{r_j\,OP_i^2 + r_i\,OP_j^2}{r_i+r_j} - r_ir_j = \frac{r_ju_i + r_iu_j}{r_i+r_j}$, because the r_i^2 and r_j^2 terms contribute $\frac{r_jr_i^2 + r_ir_j^2}{r_i+r_j} = r_ir_j$. All six of these equal ρ^2, so $r_j(u_i - \rho^2) + r_i(u_j - \rho^2) = 0$, i.e. $w_i + w_j = 0$ for every pair, where $w_i = \frac{u_i - \rho^2}{r_i}$. Then $w_1 = -w_2 = w_3 = -w_1$, so every $w_i = 0$ and $OP_i^2 = \rho^2 + r_i^2$.

Step 2: the dot products. Let $\mathbf{p}_i = \overrightarrow{OP_i}$. Then $|\mathbf{p}_i|^2 = \rho^2 + r_i^2$, and from $|\mathbf{p}_i - \mathbf{p}_j|^2 = (r_i + r_j)^2$, $\mathbf{p}_i \cdot \mathbf{p}_j = \frac{1}{2}\left[(\rho^2 + r_i^2) + (\rho^2 + r_j^2) - (r_i + r_j)^2\right] = \rho^2 - r_ir_j \quad (i \neq j)$.

Step 3: the dependence. Four vectors in space are linearly dependent: $\sum_j \lambda_j \mathbf{p}_j = \mathbf{0}$ with the λ_j not all 0. Put $\Lambda = \sum_j \lambda_j$ and $M = \sum_j r_j\lambda_j$. Dotting with $\mathbf{p}_i$: $\lambda_i(\rho^2 + r_i^2) + \sum_{j\neq i} \lambda_j(\rho^2 - r_ir_j) = \rho^2\Lambda - r_iM + 2r_i^2\lambda_i = 0$. With $k_i = \frac{1}{r_i}$ this says $\lambda_i = \frac{1}{2}\left(k_iM - \rho^2\Lambda k_i^2\right)$. Multiply by r_i and add: $M = \frac{1}{2}\left(4M - \rho^2\Lambda \sum k_i\right)$, so $M = \frac{1}{2}\rho^2\Lambda \sum k_i$. If $\Lambda = 0$, then $M = 0$ and every $\lambda_i = 0$, which is impossible; so $\Lambda \neq 0$. Adding the λ_i themselves: $\Lambda = \frac{1}{2}M \sum k_i - \frac{1}{2}\rho^2\Lambda \sum k_i^2 = \frac{1}{4}\rho^2\Lambda\,(\sum k_i)^2 - \frac{1}{2}\rho^2\Lambda \sum k_i^2$. Dividing by Λ: $\rho^2 = \frac{4}{(\sum k_i)^2 - 2\sum k_i^2}$.

Step 4: the numbers. With $k_i = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$: $\sum k_i = \frac{25}{12}$ and $\sum k_i^2 = \frac{144+36+16+9}{144} = \frac{205}{144}$, so $(\sum k_i)^2 - 2\sum k_i^2 = \frac{625-410}{144} = \frac{215}{144}$ and $\rho^2 = \frac{576}{215}$. Since $576 = 2^6 \cdot 3^2$ and $215 = 5 \cdot 43$, the fraction is in lowest terms, and $m + n = 791$.

Check. Four unit balls have centres at the vertices of a regular tetrahedron of edge 2, touching at the edge midpoints; the sphere through the six midpoints has radius $\frac{2}{2\sqrt{2}}$, so $\rho^2 = \frac{1}{2}$, and the formula gives $\frac{4}{16-8} = \frac{1}{2}$. For the radii $1, 2, 3, 4$, placing the four centres exactly in coordinates, forming the six contact points and solving for the sphere through them gives $\rho^2 = \frac{576}{215} \approx 2.68$, with all six points on it.

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