MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

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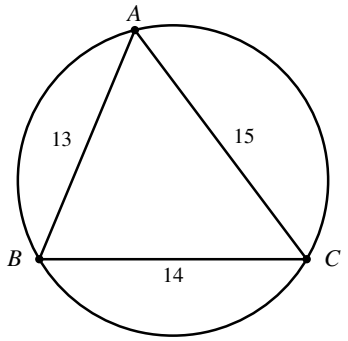
1. How many ordered pairs (a, b) of positive integers satisfy $a + b = 2025$ and $\gcd(a, b) = |a - b|$?

2. For every 3-element subset $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 10\}$, where $a < b < c$, Lina writes down the number $c - a$ (its largest element minus its smallest). She does this once for each such subset. Find the sum of all the numbers Lina writes down.

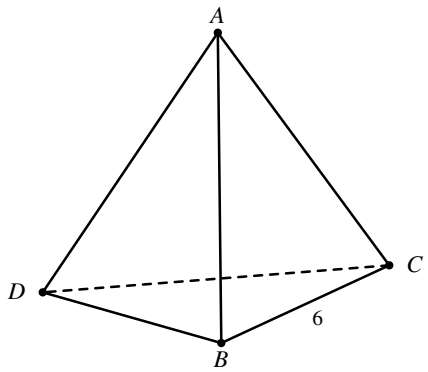
3. Aiko writes down the 901 numbers $\sin 1°, \sin 2°, \sin 3°, \ldots, \sin 901°$ in this order (angles in degrees). Call an integer n with $1 \le n \le 900$ a downhill step if $\sin n° > \sin (n + 1)°$, that is, if the list goes down from its nth entry to its $(n + 1)$st. How many downhill steps are there?

4. The polynomial $P(x) = x^3 - 14x^2 + 63x - 91$ has three positive real roots, and they are the side lengths of a triangle. The circumradius of this triangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. Point P lies on the arc BC of the circumcircle of $\triangle ABC$ that does not contain A, and it satisfies $PB + PC = PA$. Find PA^2.



6. Regular tetrahedron $ABCD$ has edge length 6. Consider all spheres that pass through both A and B and are tangent to the line CD. Among these spheres, the smallest has radius r. Then $r^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



7. For each integer n with $1 \le n \le 100$, let r_n be the remainder when 100 is divided by n (so $0 \le r_n < n$; for example $r_7 = 2$ and $r_{100} = 0$). Find the remainder when $r_1 + r_2 + r_3 + \cdots + r_{100}$ is divided by 1000.

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8. When the polynomial x^{100} is divided by the cubic $(x-1)^2(x-2)$, the remainder is a polynomial ax^2+bx+c with integer coefficients. Find the remainder when c is divided by 1000.

9. Circle ω has diameter $\overline{AB}$ with $AB = 20$. Point C lies on ω with $AC = 12$ (so $BC = 16$). The tangent line to ω at C meets line AB at T. Let Γ be the circle with center T that passes through C, and let line AC meet Γ again at a point $E \neq C$. Then $CE = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

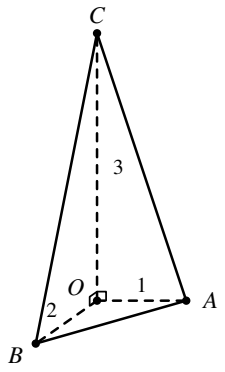
10. Twenty-four identical balls are placed into 8 boxes standing in a row, so that the numbers of balls in any two neighbouring boxes differ by at most 1. Boxes may be empty, and every ball goes into some box. Two placements are different if some box holds a different number of balls. How many placements are there?

11. Consider the polynomials $P(x) = a_0+a_1x+a_2x^2+\cdots+a_{10}x^{10}$ in which every coefficient $a_0, a_1, \ldots, a_{10}$ is -1, 0, or 1, not all of them 0. How many of these polynomials are divisible by $x^4 + x^2 + 1$ (that is, equal $x^4 + x^2 + 1$ times a polynomial)?

12. Every permutation σ of $\{1, 2, \ldots, 9\}$ breaks up into cycles: the cycle containing i is $i, \sigma(i), \sigma(\sigma(i)), \ldots$, followed until it returns to i (a number with $\sigma(i) = i$ forms a cycle of length 1). Let $w(\sigma)$ be the product of the lengths of all the cycles of σ. For example, if σ swaps 1 and 2, sends $3 \to 4 \to 5 \to 3$, and fixes 6, 7, 8 and 9, then $w(\sigma) = 2 \cdot 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 6$. Let N be the sum of $w(\sigma)$ over all 9! permutations σ. Find the remainder when N is divided by 1000.

13. How many ordered pairs (x, y) of integers with $1 \le x \le 110$ and $1 \le y \le 110$ satisfy $x^y \equiv y^x \pmod{11}$? Find the remainder when this number is divided by 1000.

14. In tetrahedron $OABC$ the edges OA, OB and OC are pairwise perpendicular, with $OA = 1$, $OB = 2$ and $OC = 3$. A point P is chosen uniformly at random from the interior of the tetrahedron, and d_1, d_2, d_3, d_4 are the distances from P to the planes of the faces OBC, OCA, OAB and ABC. The probability that there is a quadrilateral whose four side lengths are d_1, d_2, d_3 and d_4 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. (Four positive lengths are the side lengths of some quadrilateral exactly when the largest of them is less than the sum of the other three.)



15. Ten students of different heights stand in a row, in positions $1, 2, \ldots, 10$ from left to right. A coach makes two walks along the row. On the first walk she goes from left to right: for $i = 1, 2, \ldots, 9$, in this

order, she compares the students now in positions i and $i+1$ and swaps them if the one on the left is taller. On the second walk she goes from right to left: for $i = 9, 8, \ldots, 1$, in this order, she again compares the students now in positions i and $i + 1$ and swaps them if the one on the left is taller. Let N be the number of the 10! starting orders for which, after both walks, the students stand in order of height with the shortest on the left. Find the remainder when N is divided by 1000.

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Answer Key — AIME Practice Contest 4

1. 28

2. 660

3. 451

4. 17

5. 260

6. 89

7. 701

8. 176

9. 391

10. 277

11. 764

12. 553

13. 550

14. 451

15. 520

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Solutions — AIME Practice Contest 4

1. Answer: 28**. The idea.** Pull out the greatest common divisor. What is left is a pair of coprime numbers whose difference is 1, so the pair is just two consecutive multiples of its gcd.

Work it out. Let $d = \gcd(a, b)$ and write $a = dx$, $b = dy$ with $\gcd(x, y) = 1$. The condition $\gcd(a, b) = |a - b|$ becomes $d = d\,|x - y|$, i.e. $|x - y| = 1$. (Note $a \neq b$ automatically, since 2025 is odd.) So $\{x, y\} = \{k, k + 1\}$ for some $k \geq 1$, and consecutive integers are always coprime, so every such choice works. The sum condition reads $d(2k + 1) = 2025$.

Hence $2k+1$ must be a divisor of $2025 = 3^4 \cdot 5^2$ that is at least 3. Every divisor of 2025 is odd, and 2025 has $5 \cdot 3 = 15$ divisors, so there are 14 choices of $2k + 1$, and each fixes $d = 2025/(2k + 1)$. Each gives one unordered pair $\{kd, (k + 1)d\}$ and two ordered pairs, for a total of $\boxed{28}$.

Check. The divisor $2k + 1 = 3$ gives $d = 675$ and the pair $(675, 1350)$: indeed $\gcd = 675 = 1350 - 675$. The divisor 2025 itself gives $d = 1$, $k = 1012$, and the pair $(1012, 1013)$, with gcd 1 and difference 1. The trivial divisor 1 would need $k = 0$, i.e. $a = 0$, which is not allowed.

2. Answer: 660**. The idea.** Describe each subset by the four gaps it leaves in $1, 2, \ldots, 10$: before a, between a and b, between b and c, and after c. The gaps can be shuffled freely, so by symmetry every gap has the same total, and the spread $c - a$ is just two gaps plus 2.

Work it out. For $a < b < c$ put $g_0 = a - 1$, $g_1 = b - a - 1$, $g_2 = c - b - 1$, $g_3 = 10 - c$. These are nonnegative integers with $g_0 + g_1 + g_2 + g_3 = 10 - 3 = 7$, and conversely every such 4-tuple comes from exactly one subset (read off $a = g_0 + 1$, $b = a + g_1 + 1$, $c = b + g_2 + 1$). So the subsets correspond exactly to the $\binom{10}{3} = 120$ solutions of $g_0 + g_1 + g_2 + g_3 = 7$.

Swapping two coordinates, say $g_0 \leftrightarrow g_1$, maps this set of solutions onto itself, so the total of g_0 over all 120 subsets equals the total of g_1, and likewise for g_2 and g_3. The four totals add up to $120 \cdot 7 = 840$, so each one is 210.

Finally $c - a = g_1 + g_2 + 2$, so the requested sum is $210 + 210 + 2 \cdot 120 = \boxed{660}$.

Check. Count directly by the spread $r = c - a$, which runs from 2 to 9. There are $10 - r$ choices of a, and then b is any of the $r - 1$ numbers strictly between a and $c = a + r$, so $(r - 1)(10 - r)$ subsets have spread r. These counts are 8, 14, 18, 20, 20, 18, 14, 8 (total 120, as it should be), and $\sum r(r - 1)(10 - r) = 16 + 42 + 72 + 100 + 120 + 126 + 112 + 72 = 660$. Equivalently, the average spread is $\frac{660}{120} = \frac{11}{2}$.

3. Answer: 451**. The idea.** Turn the difference of two sines into a product. One factor is a fixed positive number; the other is a cosine whose sign is easy to read off.

Work it out. By the sum-to-product formula, $\sin(n + 1)^\circ - \sin n^\circ = 2\cos\left(n + \frac{1}{2}\right)^\circ \sin\frac{1}{2}^\circ$. Since $\sin\frac{1}{2}^\circ > 0$, we have $\sin n^\circ > \sin(n + 1)^\circ$ exactly when $\cos(n + \frac{1}{2})^\circ < 0$ (the cosine is never 0: it vanishes only at odd multiples of 90°, which are integers, while $n + \frac{1}{2}$ is not). The cosine is negative exactly when $n + \frac{1}{2}$ lies strictly between 90 and 270 modulo 360, that is, when $n \bmod 360 \in \{90, 91, \ldots, 269\}$: 180 residues.

In $1 \leq n \leq 900$: the blocks 90..269 and 450..629 give 180 each, and the third block 810..989 is cut off at 900, giving 810..900, that is, 91 values. Total $180 + 180 + 91 = \boxed{451}$.

Check. Picture the sine curve: it goes downhill from its peak at 90° to its trough at 270°, which is half of every cycle. Two and a half cycles would suggest about 450; the extra 1 comes from

the last stretch $810°..901°$ lying entirely on a downhill part (it starts exactly at the peak $810° = 2 \cdot 360° + 90°$). Also $n = 90$ counts ($\sin 90° = 1 > \sin 91°$) and $n = 89$ does not.

4. Answer: 17**. The idea.** We never need the roots themselves. The circumradius is $R = \frac{abc}{4K}$; the product abc is read off by Vieta, and Heron's formula for the area K needs $(s-a)(s-b)(s-c)$, which is just the value of P at the semiperimeter s.

Work it out. Let the sides be a, b, c. By Vieta, $a + b + c = 14$ and $abc = 91$, so $s = 7$. Because P is monic with roots a, b, c, we have $P(x) = (x-a)(x-b)(x-c)$ for every x, and in particular $(s-a)(s-b)(s-c) = P(7) = 343 - 686 + 441 - 91 = 7$. Heron's formula gives $K^2 = s(s-a)(s-b)(s-c) = 7 \cdot 7 = 49$, so $K = 7$. Then $R = \frac{abc}{4K} = \frac{91}{28} = \frac{13}{4}$, and $m + n = 13 + 4 = \boxed{17}$.

Check. The hypotheses really hold: $P(x) = (x-3)(x-5)(x-6) - 1$, so $P(3) = P(5) = P(6) = -1$, $P(4) = 1$ and $P(7) = 7$. Hence there are roots in $(3,4)$, $(4,5)$ and $(6,7)$, all positive, and the largest is less than $7 = s$, which is exactly the triangle inequality. Numerically the sides are about 3.198, 4.555 and 6.247; Heron gives area 7.000, and as a bonus the inradius is $K/s = 1$.

5. Answer: 260**. The idea.** $ABPC$ is a cyclic quadrilateral, so Ptolemy's theorem gives a second linear relation among PA, PB, PC. The two relations together force $PB = PC$, so P is the midpoint of the arc.

Work it out. In cyclic quadrilateral $ABPC$ (vertices in this order around the circle), the diagonals are AP and BC, and Ptolemy's theorem says $PA \cdot BC = PB \cdot AC + PC \cdot AB$, i.e. $14\,PA = 15\,PB + 13\,PC$. Substituting $PA = PB + PC$: $14PB + 14PC = 15PB + 13PC$, so $PB = PC$. Thus P is the midpoint of arc BC, and $PA = 2PB$.

Now compute PB. The circumradius is $R = \frac{abc}{4K} = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$, and $\cos A = \frac{13^2 + 15^2 - 14^2}{2 \cdot 13 \cdot 15} = \frac{198}{390} = \frac{33}{65}$. The inscribed angle $\angle PAB$ is $\frac{A}{2}$, so the chord $PB = 2R \sin\frac{A}{2}$, and $PB^2 = 4R^2 \cdot \frac{1 - \cos A}{2} = 4 \cdot \frac{4225}{64} \cdot \frac{16}{65} = 65$. Hence $PA^2 = 4 \cdot PB^2 = \boxed{260}$.

Check. Coordinates: $B = (0,0)$, $C = (14,0)$, $A = (5,12)$. The circumcenter is $(7, \frac{33}{8})$, so the arc midpoint below BC is $P = (7, \frac{33}{8} - \frac{65}{8}) = (7, -4)$. Then $PB^2 = 49 + 16 = 65$ and $PA^2 = 4 + 256 = 260$, and indeed $PA = 2\sqrt{65} = PB + PC$.

6. Answer: 89**. The idea.** The center of any sphere through A and B lies in the perpendicular bisector plane of AB, and in a regular tetrahedron that plane contains the whole opposite edge CD. So the problem collapses to a plane: find a point whose distance to a line equals its distance to A.

Work it out. Let M and N be the midpoints of AB and CD. Since $CA = CB$ and $DA = DB$, the plane Π perpendicular to AB through M contains C and D. The segment MN is perpendicular to both AB and CD, and $MN^2 = MC^2 - NC^2 = 27 - 9 = 18$, so $MN = 3\sqrt{2}$ (here $MC = 3\sqrt{3}$ is the height of an equilateral face).

Put coordinates in Π: N at the origin, line CD along the first axis, $M = (0, 3\sqrt{2})$. A center $O = (t, y)$ in Π has distance $|y|$ to line CD (the tangency point is the foot $(t, 0)$), and $OA^2 = OM^2 + MA^2 = t^2 + (y - 3\sqrt{2})^2 + 9$ because $MA = 3$ is perpendicular to Π. Tangency with radius OA means $OA^2 = y^2$: $t^2 + y^2 - 6\sqrt{2}\,y + 18 + 9 = y^2 \quad\Longrightarrow\quad y = \frac{t^2 + 27}{6\sqrt{2}}$. This is positive, and the radius $r = y$ is smallest when $t = 0$: $r = \frac{27}{6\sqrt{2}} = \frac{9\sqrt{2}}{4}$, so $r^2 = \frac{81}{8}$ and $m + n = \boxed{89}$.

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Check. The smallest sphere touches CD at its midpoint N and has its center on segment NM at distance $\frac{9\sqrt{2}}{4} \approx 3.18$ from N, while $MN \approx 4.24$. Its distance to M is $3\sqrt{2} - \frac{9\sqrt{2}}{4} = \frac{3\sqrt{2}}{4}$, and $\frac{18}{16} + 9 = \frac{81}{8}$, matching r^2. Note it is bigger than the sphere with diameter AB (radius 3), which cannot reach line CD at distance $3\sqrt{2}$.

7. Answer: 701. **The idea.** Write each remainder as $r_n = 100 - n\lfloor 100/n \rfloor$. The sum of $n\lfloor 100/n \rfloor$ counts every pair (d, e) with $de \le 100$, weighted by d, and such a sum over a hyperbola is quickest when split at $\sqrt{100} = 10$.

Work it out. Summing, $\sum r_n = 100 \cdot 100 - S$, where $S = \sum_{n=1}^{100} n\lfloor 100/n \rfloor = \sum_{de \le 100} d$ (for each d there are $\lfloor 100/d \rfloor$ choices of e). Split the pairs (d, e) with $de \le 100$ into those with $d \le 10$ and those with $e \le 10$; the pairs with both $d \le 10$ and $e \le 10$ are counted twice. So $S = \sum_{d=1}^{10} d \left\lfloor \frac{100}{d} \right\rfloor + \sum_{e=1}^{10} T\left(\left\lfloor \frac{100}{e} \right\rfloor\right) - 10 \cdot T(10)$, where $T(k) = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$.

First sum: $100 + 100 + 99 + 100 + 100 + 96 + 98 + 96 + 99 + 100 = 988$. For the second, $\lfloor 100/e \rfloor = 100, 50, 33, 25, 20, 16, 14, 12, 11, 10$, so the terms are $5050 + 1275 + 561 + 325 + 210 + 136 + 105 + 78 + 66 + 55 = 7861$. The overlap is $10 \cdot 55 = 550$. Hence $S = 988 + 7861 - 550 = 8299$, and $\sum r_n = 10000 - 8299 = 1701$. The remainder is $\boxed{701}$.

Check. Spot values: $r_{51}, \ldots, r_{100}$ are $49, 48, \ldots, 0$, adding to 1225, so the other fifty remainders must add to $1701 - 1225 = 476$. For $34 \le n \le 50$ we get $r_n = 100 - 2n$, i.e. $32, 30, \ldots, 0$, adding to 272, leaving 204 for $n \le 33$. Those are $0, 0, 1, 0, 0, 4, 2, 4, 1, 0, 1, 4, 9, 2, 10, 4, 15, 10, 5, 0, 16, 12, 8, 4, 0, 22, 19, 16,$ which indeed add to 204.

8. Answer: 176. **The idea.** Write $x^{100} = (x-1)^2(x-2)Q(x) + R(x)$ with $\deg R \le 2$. We cannot find Q, but we do not need it: the divisor vanishes at $x = 2$, and it vanishes *together with its derivative* at the double root $x = 1$. So R must match x^{100} in value at 2, and in both value and slope at 1. Three conditions pin down the three coefficients of R.

Work it out. Differentiating, $100x^{99} = \left[(x-1)^2(x-2)Q(x)\right]' + R'(x)$, and every term of the bracket's derivative still contains a factor $x - 1$. So $R(1) = 1$ and $R'(1) = 100$. Write R around $x = 1$ as $R(x) = 1 + 100(x-1) + k(x-1)^2$; this has the right value and slope at 1 for every k. The condition at 2 gives $R(2) = 1 + 100 + k = 2^{100}$, so $k = 2^{100} - 101$. Then $c = R(0) = 1 - 100 + k = 2^{100} - 200$. Now find $2^{100} \bmod 1000$. Modulo 8 it is 0. Modulo 125, Euler's theorem with $\varphi(125) = 100$ gives $2^{100} \equiv 1$. The number in 0..999 that is 0 mod 8 and 1 mod 125 is 376 ($376 = 3 \cdot 125 + 1 = 47 \cdot 8$). Hence $c \equiv 376 - 200 = 176 \pmod{1000}$.

Check. A small version: x^4 divided by $(x-1)^2(x-2)$. The same method gives $k = 16 - 5 = 11$ and $R(x) = 1 + 4(x-1) + 11(x-1)^2 = 11x^2 - 18x + 8$, so $c = 2^4 - 8 = 8$. Direct division: $x^4 = (x^3 - 4x^2 + 5x - 2)(x+4) + (11x^2 - 18x + 8)$, which matches. The answer is $\boxed{176}$.

9. Answer: 391. **The idea.** Γ is not an arbitrary circle: it is the Apollonius circle of A and B through C, the set of points X with $XA : XB = CA : CB = 3 : 4$. Once we know where Γ crosses line AB, the power of the point A with respect to Γ finishes the problem.

Work it out. Since TC is tangent to ω, the tangent–chord angle gives $\angle TCA = \angle CBA$, so triangles TCA and TBC are similar (they share the angle at T). Hence $TA/TC = TC/TB = CA/BC = 3/4$. Since $TA < TB$, T lies beyond A; writing $TA = 9u$, $TC = 12u$, $TB = 16u$ and using $TB - TA = 7u = 20$ gives $u = 20/7$, so $TA = 180/7$, $TC = 240/7$, $TB = 320/7$.

So Γ has radius $240/7$ and its center is $180/7$ from A. The point A is inside Γ, and its power is

$r^2 - TA^2 = \left(\frac{240}{7}\right)^2 - \left(\frac{180}{7}\right)^2 = \frac{60^2(16-9)}{49} = \frac{3600}{7}$. The chord of Γ through A along line AC is split by A into pieces $AC = 12$ and AE, with $AC \cdot AE = 3600/7$. So $AE = 300/7$, and because A is inside Γ, E is on the far side of A from C: $CE = 12 + \frac{300}{7} = \frac{384}{7}$. The answer is $384 + 7 = 391$.

Check. The Apollonius picture gives the same power. Γ meets line AB at the feet P, Q of the internal and external bisectors from C, which divide AB in the ratio $3:4$: $AP = \frac{3}{7} \cdot 20 = \frac{60}{7}$ and $AQ = 3 \cdot 20 = 60$ (since $QB - QA = 20$ with $QA : QB = 3:4$). Then $AP \cdot AQ = \frac{3600}{7}$, matching. Coordinates ($A = (-10, 0)$, $C = (-\frac{14}{5}, \frac{48}{5})$, $T = (-\frac{250}{7}, 0)$) also give $CE = 384/7$.

10. Answer: 277**. The idea.** Record the first box and the seven steps between neighbours. The total is then a fixed multiple of the first box plus a weighted sum of the steps, with weights $7, 6, \ldots, 1$. Measuring how far each step falls short of $+1$ turns the count into coefficients of $\prod_{w=1}^{7}(1 + x^w + x^{2w})$.

Work it out. Let the counts be $a_1, \ldots, a_8$ and $e_i = a_{i+1} - a_i \in \{-1, 0, 1\}$. Each step e_i is included in $a_{i+1}, \ldots, a_8$, that is, in $w = 8 - i$ of the counts, so $a_1 + \cdots + a_8 = 8a_1 + \sum_{i=1}^{7}(8-i)\, e_i = 24$. First, no step sequence can ever produce a negative box: if $a_j \le -1$ then $a_i \le -1 + |i - j|$ for all i, and the total would be at most $-8 + \sum_i |i - j| \le -8 + 28 = 20 < 24$. So we may count all $(a_1, e_1, \ldots, e_7)$ with $a_1 \ge 0$ satisfying the equation. Let a step of weight w cost $w(1 - e) \in \{0, w, 2w\}$. The weights add to 28, so the equation says the costs add to $D = 28 - (24 - 8a_1) = 4 + 8a_1$. The number of ways is the coefficient $c(D)$ of x^D in $G(x) = \prod_{w=1}^{7}(1 + x^w + x^{2w})$, which has degree 56 and is symmetric: $c(D) = c(56 - D)$. So $a_1 \in \{0, 1, \ldots, 6\}$ and the answer is $c(4) + c(12) + c(20) + c(28) + c(36) + c(44) + c(52) = 2c(4) + 2c(12) + 2c(20) + c(28)$.

Multiply the factors in one at a time, keeping coefficients up to x^{28}. After the factors for $w = 1, \ldots, 6$ the coefficients of $x^0, x^1, \ldots, x^{28}$ are $1, 1, 2, 2, 4, 5, 7, 8, 11, 12, 16, 17, 21, 22, 26, 27, 31, 31, 34, 33, 36, 35,$ The last factor gives $c(D) = c_6(D) + c_6(D-7) + c_6(D-14)$, so $c(4) = 4$, $c(12) = 21 + 5 = 26$, $c(20) = 36 + 22 + 7 = 65$, and $c(28) = 26 + 35 + 26 = 87$. The total is $2(4 + 26 + 65) + 87 = \boxed{277}$.

Check. $c(4) = 4$ by hand: costs adding to 4 come from weight 4 at cost 4; weights $1, 3$ at cost $1+3$; weight 2 at cost 4; or weight 1 at cost 2 with weight 2 at cost 2. These give the four rows with $a_1 = 0$: $0, 1, 2, 3, 3, 4, 5, 6$; $0, 1, 2, 3, 4, 4, 5, 5$; $0, 1, 2, 3, 4, 5, 4, 5$; $0, 1, 2, 3, 4, 5, 5, 4$. Each sums to 24. By symmetry the rows with $a_1 = 6$ are the complements $6 - a_i$ of these, e.g. $6, 5, 4, 3, 3, 2, 1, 0$. A computer search over all rows confirms 277.

11. Answer: 764**. The idea.** $x^4 + x^2 + 1$ divides $x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1)$. So we may replace every x^6 by 1 without changing divisibility: fold the eleven coefficients into six sums by exponent mod 6, and then test a polynomial of degree at most 5, which is divisible by a quartic only in a very specific way.

Work it out. Put $c_r = a_r + a_{r+6}$ for $r = 0, 1, 2, 3, 4$ and $c_5 = a_5$. Then $P(x) \equiv S(x) = c_0 + c_1 x + \cdots + c_5 x^5 \pmod{x^6 - 1}$, hence also modulo $x^4 + x^2 + 1$, and P is divisible exactly when S is. A multiple of $x^4 + x^2 + 1$ of degree at most 5 is $(u + vx)(x^4 + x^2 + 1) = u + vx + ux^2 + vx^3 + ux^4 + vx^5$. So the condition is $c_0 = c_2 = c_4$ and $c_1 = c_3 = c_5$. Each c_r with $r \le 4$ is a sum of two coefficients in $\{-1, 0, 1\}$; the value 0 arises in 3 ways, ± 1 in 2 ways each and ± 2 in 1 way each. *Even classes:* the common value s of c_0, c_2, c_4 is free, giving $3^3 + 2 \cdot 2^3 + 2 \cdot 1^3 = 45$ choices of $a_0, a_2, a_4, a_6, a_8, a_{10}$. *Odd classes:* $c_5 = a_5$ forces the common value $t \in \{-1, 0, 1\}$, giving $3^2 + 2 \cdot 2^2 = 17$ choices of a_1, a_3, a_5, a_7, a_9. That is $45 \cdot 17 = 765$ polynomials, one of which is the zero polynomial. The answer is 764.

Check. A spot check: x^4+x^2+1 itself has $c_0 = c_2 = c_4 = 1$ and $c_1 = c_3 = c_5 = 0$, as required, and so does $x^{10}+x^8+x^6$ ($c_4 = c_2 = c_0 = 1$), which is $x^6(x^4+x^2+1)$. A computer division test on all 3^{11} coefficient choices also finds 764 nonzero multiples.

12. Answer: 553**. The idea.** Do not try to sum the products over cycle types. A product of lengths is itself a count: a cycle of length k can be cut open at any one of its k members, and read from there it becomes a list. So $w(\sigma)$ is the number of ways to choose a starting member in every cycle of σ, and N counts permutations together with such a choice. Those turn out to be exactly the ways of splitting $1,\dots,9$ into lists, which are easy to count.

Work it out. Given σ and a chosen start s in each cycle, write that cycle as the list $s, \sigma(s), \sigma(\sigma(s)),\dots$, stopping just before s comes back. The result is an unordered collection of nonempty lists that uses each of $1,\dots,9$ exactly once. Conversely, a collection of lists closes up into a permutation (each entry is sent to the next entry of its list, the last entry back to the first) with the first entry of each list marked, and this undoes the first step. So N is the number of ways to split 9 labelled objects into an unordered collection of nonempty ordered lists.

Count the collections with exactly k lists. Line up all nine objects (9! ways) and cut the line into k nonempty pieces ($\binom{8}{k-1}$ ways, choosing $k-1$ of the 8 gaps). Each collection is produced $k!$ times, once for each order of its lists, so there are $L_k = \frac{9!}{k!}\binom{8}{k-1}$ of them: $L_1 = 362880$, $L_2 = 181440\cdot 8 = 1451520$, $L_3 = 60480\cdot 28 = 1693440$, $L_4 = 15120\cdot 56 = 846720$, $L_5 = 3024\cdot 70 = 211680$, $L_6 = 504\cdot 56 = 28224$, $L_7 = 72\cdot 28 = 2016$, $L_8 = 9\cdot 8 = 72$, $L_9 = 1$. Their sum is $N = 4596553$ (the last three digits alone: $880+520+440+720+680+224+16+72+1 = 3553$). The remainder is 553.

Check. For $\{1,2,3\}$: the identity has $w = 1$, the three swaps have $w = 2\cdot 1 = 2$, and the two 3-cycles have $w = 3$, a total of $1+6+6 = 13$. The list count gives $\frac{3!}{1!}\binom{2}{0}+\frac{3!}{2!}\binom{2}{1}+\frac{3!}{3!}\binom{2}{2} = 6+6+1 = 13$, as it should. A computer pass over all 9! permutations, multiplying their cycle lengths, also gives 4596553. The answer is $\boxed{553}$.

13. Answer: 550**. The idea.** The value of x^y mod 11 depends only on x mod 11 and on y mod 10 (Fermat: units have order dividing 10, and $0^y = 0$ for $y \geq 1$). Because $110 = 11\cdot 10$ and $\gcd(11,10) = 1$, the Chinese Remainder Theorem makes x mod 11 and x mod 10 *independent*. So the two sides of the congruence are built from four independent, uniformly distributed pieces, the left side from (x mod 11, y mod 10) and the right side from (y mod 11, x mod 10), and the count becomes a sum of squares.

Work it out. As x runs over 1..110, the pair $(x \bmod 11, x \bmod 10)$ runs over all 110 pairs in $\mathbb{Z}_{11}\times\mathbb{Z}_{10}$ exactly once, and likewise for y. For a residue v mod 11 let $A(v)$ be the number of pairs $(r,e) \in \mathbb{Z}_{11}\times\mathbb{Z}_{10}$ with $r^e \equiv v$, where e stands for any positive exponent in its class (so $e = 0$ means an exponent that is a multiple of 10). Then x^y mod 11 takes the value v for $A(v)$ of the 110 choices of (x mod 11, y mod 10), and y^x takes the value v for $A(v)$ of the 110 independent choices of (y mod 11, x mod 10). So the number of pairs is $\sum_v A(v)^2$.

$A(0) = 10$: $r = 0$ with any e. For units, take the primitive root $g = 2$ and write $r = g^i$, $v = g^b$ with $i, b \in \mathbb{Z}_{10}$. Then $A(g^b)$ counts $(i,e) \in \mathbb{Z}_{10}^2$ with $ie \equiv b \pmod{10}$. For fixed i, the congruence $ie \equiv b$ has $\gcd(i,10)$ solutions if $\gcd(i,10) \mid b$ and none otherwise. The values of $\gcd(i,10)$ are 10 ($i = 0$), 5 ($i = 5$), 2 ($i = 2,4,6,8$) and 1 ($i = 1,3,7,9$). So $b = 0$: $10+5+4\cdot 2+4\cdot 1 = 27$; b even, $b \neq 0$: $4\cdot 2+4 = 12$ (four such b); $b = 5$: $5+4 = 9$; b odd, $b \neq 5$: 4 (four such b). These add to $10+27+48+9+16 = 110$, as they must. The count is $10^2+27^2+4\cdot 12^2+9^2+4\cdot 4^2 = 100+729+576+81+64 = 1550$. The remainder is 550.

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Check. The 110 diagonal pairs $x = y$ are included, and the count 1550 is well above 110, as it should be since many values coincide (for instance every pair with $x \equiv y \equiv 1 \pmod{11}$ works). A direct computer check of all 110^2 pairs gives 1550. The answer is $\boxed{550}$.

14. Answer: 451**. The idea.** A quadrilateral fails to exist exactly when one distance is at least the sum of the other three. The distances are linear functions of the position of P, so each of these four bad events is the part of the tetrahedron on one side of a plane. That plane cuts off a small corner tetrahedron, whose share of the volume is a product of three edge fractions. The edge fractions come from the fact that the four distances, weighted by the face areas, always add up to the same total.

Work it out. Let S_1, S_2, S_3, S_4 be the areas of faces OBC, OCA, OAB, ABC, and let V be the volume. Cutting the tetrahedron into four pyramids with apex P gives $S_1d_1+S_2d_2+S_3d_3+S_4d_4 = 3V$ for every P inside. Here $V = \frac{1\cdot2\cdot3}{6} = 1$, $S_1 = 3$, $S_2 = \frac{3}{2}$, $S_3 = 1$, and the distance from O to plane ABC (the plane $6x + 3y + 2z = 6$ when O is the origin and A, B, C lie on the axes; its normal $(6, 3, 2)$ has length 7) is $\frac{6}{7}$, so $S_4 = 3V \div \frac{6}{7} = \frac{7}{2}$. The areas are in the ratio $6 : 3 : 2 : 7$.

Two bad events cannot happen together: $d_i \geq d_j + d_k + d_l$ and $d_j \geq d_i + d_k + d_l$ add up to $0 \geq 2(d_k + d_l)$, which is false inside. Let V_i be the vertex opposite face i. The bad event for d_i is bounded by the plane $d_i = d_j + d_k + d_l$, which has V_i on the bad side (there $d_i > 0$ and the other three are 0) and the other three vertices on the good side. So this plane cuts the three edges at V_i, and the bad region is the corner tetrahedron formed by V_i and the three cut points. If those points are the fractions t_j, t_k, t_l of the way along the edges from V_i, the corner has $t_jt_kt_l$ of the whole volume.

On edge V_iV_j only d_i and d_j are nonzero (the edge lies in the other two faces), so $S_id_i+S_jd_j = 3V$ there, and the cut point has $d_i = d_j = \frac{3V}{S_i+S_j}$. Since d_i falls linearly from $\frac{3V}{S_i}$ at V_i to 0 at V_j, the cut point is the fraction $t = 1 - \frac{S_i}{S_i+S_j} = \frac{S_j}{S_i+S_j}$ of the way from V_i. With weights $6, 3, 2, 7$ for the faces opposite A, B, C, O, the four corners take at A: $\frac{3}{9} \cdot \frac{2}{8} \cdot \frac{7}{13} = \frac{7}{156}$; at B: $\frac{6}{9} \cdot \frac{2}{5} \cdot \frac{7}{10} = \frac{14}{75}$; at C: $\frac{6}{8} \cdot \frac{3}{5} \cdot \frac{7}{9} = \frac{7}{20}$; at O: $\frac{6}{13} \cdot \frac{3}{10} \cdot \frac{2}{9} = \frac{2}{65}$. Over the common denominator 3900 these are $175 + 728 + 1365 + 120 = 2388$, so the bad probability is $\frac{2388}{3900} = \frac{199}{325}$ and the good probability is $\frac{126}{325}$. The answer is $126 + 325 = 451$.

Check. Directly at corner C: with $P = (x, y, z)$ the distances are x, y, z and $\frac{6-6x-3y-2z}{7}$, and $z \geq x + y + \frac{6-6x-3y-2z}{7}$ simplifies to $9z \geq x + 4y + 6$. On edge CO this is $z \geq \frac{2}{3}$ (fraction $\frac{7}{9}$ from C); on edge CA, at $(s, 0, 3 - 3s)$, it is $s \leq \frac{3}{4}$; on edge CB, at $(0, 2s, 3 - 3s)$, it is $s \leq \frac{3}{5}$. These are the factors $\frac{7}{9}$, $\frac{6}{8}$, $\frac{3}{5}$ used above. For a regular tetrahedron every factor is $\frac{1}{2}$ and the answer would be $1 - 4 \cdot \frac{1}{8} = \frac{1}{2}$, the classical chance that three random cuts of a stick give four pieces forming a quadrilateral. An exact volume computation of the good region also gives $\frac{126}{325}$. The answer is $\boxed{451}$.

15. Answer: 520**. The idea.** Call the heights $1, \ldots, 10$ and let $r(x)$ be the number of students shorter than x standing to the right of x; the row is sorted exactly when every $r(x)$ is 0. The second walk lowers every positive $r(x)$ by exactly 1, so it finishes the job exactly when the first walk leaves every $r(x) \leq 1$. The first walk has a simple exact description, from which each finished row has a countable number of starting rows. The count then becomes a weighted count of rows with all $r(x) \leq 1$, done with two states.

Work it out. *The second walk.* Going right to left, the coach carries along the shortest student

seen so far. Fix a student x. If $r(x) \geq 1$, the carried student arriving at x is the shortest one to the right of x, so it is shorter than x and passes x, and it is the only student who crosses x from right to left. If $r(x) = 0$, then x itself is picked up and carried left past taller students only. Either way $r(x)$ becomes $\max(r(x) - 1, 0)$. So the row ends sorted exactly when, after the first walk, every student has at most one shorter student to the right.

The first walk. Call x a *leader* if it is taller than everyone to its left. Write the starting row as $m_1 W_1 m_2 W_2 \cdots m_k W_k$, where $m_1 < m_2 < \cdots < m_k = 10$ are the leaders and each block W_j (possibly empty) holds the students between them. Going left to right, the coach carries m_1 past W_1 until it meets the taller m_2, then carries m_2 past W_2, and so on. So the first walk produces $W_1 m_1 W_2 m_2 \cdots W_k m_k$.

Counting starting rows. Every row q produced this way ends with 10. Conversely, if q ends with 10, a starting row that produces q is determined by which students of q play the parts of $m_1, \ldots, m_k$: each must be a leader of q (it is taller than everything before it in q), and 10 must be one of them. Any set of leaders of q that includes 10 works: cut q just after each chosen student and move each chosen student to the front of its piece. In the resulting row the chosen students are exactly the leaders, because each is taller than its own block. So q has exactly $2^{L(q)-1}$ starting rows, where $L(q)$ is the number of leaders of q. Removing the final 10 leaves a row q' of $1, \ldots, 9$ with the same values of r and one leader fewer, so $N = \sum 2^{L(q')}$, summed over the rows q' of $1, \ldots, 9$ in which every student has at most one shorter student to the right.

Two states. Build q' from left to right. The condition says that each student placed is the shortest or the second shortest of those not yet placed. After j placements the placed set is either $\{1, \ldots, j\}$ (state A) or $\{1, \ldots, j+1\}$ minus one student s who is still waiting as the shortest remaining (state B). From A we place $j+1$ (staying in A) or $j+2$ (moving to B); both are leaders, so each move carries weight 2. From B we place s (back to A; not a leader, weight 1) or $j+2$ (staying in B; a leader, weight 2). If a_j and b_j are the total weights after j placements, then $a_{j+1} = 2a_j + b_j$ and $b_{j+1} = 2a_j + 2b_j$, with $(a_0, b_0) = (1, 0)$: $(2, 2)$, $(6, 8)$, $(20, 28)$, $(68, 96)$, $(232, 328)$, $(792, 1120)$, $(2704, 3824)$, $(9232,$ for $j = 1, \ldots, 8$. The ninth student is forced: from A it is 9, a leader (weight 2); from B it is s (weight 1). So $N = 2 \cdot 9232 + 13056 = 31520$, and the remainder is 520.

Check. With 4 students the same recursion gives $2a_2 + b_2 = 12 + 8 = 20$ of the 24 rows. The four failures are 3412, 3421, 4312 and 4321: for instance 4321 becomes 3214 after the first walk, where 3 still has two shorter students to its right, and the second walk only gets to 1324. A computer run of both walks on all 10! rows also finds 31520. The answer is $\boxed{520}$.