MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

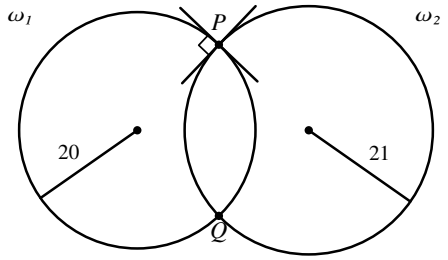
1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A row of 21 seats is numbered $1, 2, \ldots, 21$ from left to right. Three guests will sit in three of the seats, and the choice must satisfy two rules: no two of the guests sit in adjacent seats, and the sum of the three seat numbers is divisible by 3. How many sets of three seats satisfy both rules?

2. Circle ω_1 has radius 20 and circle ω_2 has radius 21. The two circles meet at two points P and Q, and the line tangent to ω_1 at P is perpendicular to the line tangent to ω_2 at P. The length PQ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



3. How many integers n with $1 \le n \le 1000$ can be written as $n = a^3 - b^3$ for some positive integers a and b? (For example, $7 = 2^3 - 1^3$ can be written this way.)

4. The arithmetic mean of two positive numbers a and b is $\frac{a+b}{2}$, and their harmonic mean is $\frac{2ab}{a+b}$. How many pairs (a, b) of integers with $1 \le a < b \le 1000$ have an arithmetic mean that is exactly 12 larger than their harmonic mean?

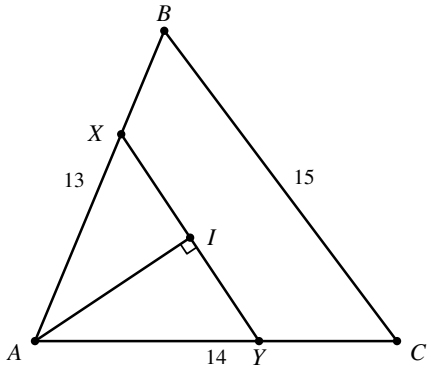
5. For a real number t, let $\lfloor t \rfloor$ denote the greatest integer less than or equal to t. How many integers n with $1 \le n \le 1000$ can be written as $\lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor$ for some real number x?

6. A sequence is defined by $a_1 = a_2 = 2 - \sqrt{3}$ and $a_{n+1} = \dfrac{a_n + a_{n-1}}{1 - a_n a_{n-1}}$ for every $n \ge 2$. (It turns out that the denominator is never 0, so every term is defined.) For how many integers n with $1 \le n \le 1000$ is $a_n = 1$?

7. For a real number x, let $\{x\} = x - \lfloor x \rfloor$ denote its fractional part. Find the least positive integer n such that $\{n\sqrt{2}\} < \dfrac{1}{100}$.

8. Triangle ABC has $AB = 13$, $AC = 14$ and $BC = 15$, and I is its incenter. The line through I perpendicular to $\overline{AI}$ meets side $\overline{AB}$ at X and side $\overline{AC}$ at Y. The product $BX \cdot CY$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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9. How many triples (a, b, c) of positive integers with $a < b < c \leq 100$ have the property that a^2, b^2, c^2 form an arithmetic progression, that is, $b^2 - a^2 = c^2 - b^2$? (For example, $(1, 5, 7)$ works, since $1, 25, 49$ go up by 24 each time.)

10. The integers $1, 2, 3, \ldots, 20$ are written on a board. A move consists of choosing one of the numbers still on the board, uniformly at random, and erasing it together with every divisor of it that is still on the board. Moves are made until the board is empty. The expected number of moves is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. How many ordered pairs (x, y) of real numbers satisfy $x^2 + y^2 \leq 50$ and have the property that both $x + y$ and xy are integers?

12. For a complex number c let $f(z) = z^2 + c$, and write f^k for f applied k times, so $f^1(0) = c$, $f^2(0) = c^2 + c$, and so on. Find the sum of c^2 over all complex numbers c such that $f^7(0) = 0$ but $f^k(0) \neq 0$ for $k = 1, 2, \ldots, 6$.

13. Let $M = 6^8$. Let N be the number of ordered pairs (a, b) of integers with $0 \leq a < M$ and $0 \leq b < M$ such that $a^2 - b^3$ is divisible by M. Find the remainder when N is divided by 1000.

14. The 16 vertices of a four-dimensional cube are the strings of four binary digits, and two vertices are adjacent when they differ in exactly one digit, so every vertex has exactly four neighbours. Each vertex is coloured black or white. In how many of the 2^{16} colourings does every vertex have at least one black neighbour and at least one white neighbour? Find the remainder when this number is divided by 1000.

15. Call a permutation σ of $\{1, 2, \ldots, 12\}$ connected if there is no k with $1 \leq k \leq 11$ for which $\sigma(1), \sigma(2), \ldots, \sigma(k)$ are the numbers $1, 2, \ldots, k$ in some order. Let N be the number of connected permutations σ with $|\sigma(1) - 1| + |\sigma(2) - 2| + \cdots + |\sigma(12) - 12| = 26$. Find the remainder when N is divided by 1000.

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Answer Key — AIME Practice Contest 3

1. 327

2. 869

3. 56

4. 15

5. 982

6. 84

7. 99

8. 217

9. 37

10. 141

11. 681

12. 32

13. 384

14. 924

15. 512

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Solutions — AIME Practice Contest 3

1. Answer: 327**. The idea.** Squeeze out the forced gaps. If the chosen seats are $a < b < c$ with $b \geq a + 2$ and $c \geq b + 2$, then $a < b - 1 < c - 2$ are three different numbers from 1 to 19, and every 3-element subset of $\{1, \ldots, 19\}$ arises from exactly one spaced-out choice. The key point is that this shift lowers the sum by exactly 3, so it does not change whether the sum is divisible by 3. So we just count 3-element subsets of $\{1, \ldots, 19\}$ with sum divisible by 3.

Work it out. Sort $1, \ldots, 19$ by remainder mod 3: remainder 1 has 7 members $(1, 4, \ldots, 19)$, remainder 2 has 6 $(2, \ldots, 17)$ and remainder 0 has 6 $(3, \ldots, 18)$. Three remainders add to a multiple of 3 exactly when they are all equal or all different. All equal: $\binom{7}{3}+\binom{6}{3}+\binom{6}{3} = 35+20+20 = 75$. All different: $7 \cdot 6 \cdot 6 = 252$. The total is $75 + 252 = 327$.

Check. The same method on 6 seats predicts: the numbers $1, \ldots, 4$ have remainders $1, 2, 0, 1$, so only "all different" works, $2 \cdot 1 \cdot 1 = 2$ ways. Directly, the spaced-out triples from 6 seats are $\{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{2, 4, 6\}$ with sums 9, 10, 11, 12: exactly two, as predicted. Also, 327 is close to a third of the $\binom{19}{3} = 969$ spaced-out triples, as it should be.

2. Answer: 869**. The idea.** A tangent line is perpendicular to the radius at its point of tangency. Turning both tangent lines at P by 90° turns them into the two radii O_1P and O_2P, so the tangents are perpendicular exactly when the radii are: triangle O_1PO_2 has a right angle at P. The common chord PQ is perpendicular to the line of centres O_1O_2 and is bisected by it, so half of PQ is the altitude from the right angle.

Work it out. In the right triangle O_1PO_2 the legs are $O_1P = 20$ and $O_2P = 21$, so the hypotenuse is $O_1O_2 = \sqrt{400 + 441} = 29$. Let M be the point where PQ crosses O_1O_2; then PM is the altitude to the hypotenuse, and computing the area two ways gives $PM \cdot 29 = 20 \cdot 21$, so $PM = \frac{420}{29}$. By symmetry in the line O_1O_2, $MQ = PM$, so $PQ = \frac{840}{29}$. Since 29 is prime and does not divide 840, the fraction is in lowest terms, and $m + n = 840 + 29 = 869$.

Check. Put $O_1 = (0, 0)$ and $O_2 = (29, 0)$. Subtracting the circle equations $x^2 + y^2 = 400$ and $(x - 29)^2 + y^2 = 441$ gives $58x = 841 - 41 = 800$, so $x = \frac{400}{29}$ and $y^2 = 400 - \frac{160000}{841} = \frac{176400}{841}$, i.e. $y = \pm\frac{420}{29}$. The two intersection points are $\frac{840}{29}$ apart, and $P = \left(\frac{400}{29}, \frac{420}{29}\right)$ satisfies $\vec{O_1P} \cdot \vec{O_2P} = \frac{400 \cdot (400-841)+420^2}{841} = \frac{-176400+176400}{841} = 0$, confirming the right angle.

3. Answer: 56**. The idea.** Put $d = a - b \geq 1$. Then $a^3 - b^3 = (a - b)(a^2 + ab + b^2) = d\,(3b^2 + 3bd + d^2)$, which increases with both d and b. So for each gap d only a few b keep the value at most 1000, and we can simply list every value and then look for numbers that appear twice.

Work it out. Since $d \cdot d^2 < a^3 - b^3 \leq 1000$ we need $d \leq 9$. Listing $d(3b^2 + 3bd + d^2)$ for $b = 1, 2, \ldots$ until it passes 1000: $d = 1$: 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, 469, 547, 631, 721, 817, 919 (17 values); $d = 2$: 26, 56, 98, 152, 218, 296, 386, 488, 602, 728, 866 (11); $d = 3$: 63, 117, 189, 279, 387, 513, 657, (9); $d = 4$: 124, 208, 316, 448, 604, 784, 988 (7); $d = 5$: 215, 335, 485, 665, 875 (5); $d = 6$: 342, 504, 702, 936 (4); $d = 7$: 511, 721, 973 (3); $d = 8$: 728, 992 (2); $d = 9$: 999 (1). That is $17 + 11 + 9 + 7 + 5 + 4 + 3 + 2 + 1 = 59$ entries. Comparing the lists, exactly three numbers appear twice: $721 = 16^3 - 15^3 = 9^3 - 2^3$, $728 = 12^3 - 10^3 = 9^3 - 1^3$ and $999 = 12^3 - 9^3 = 10^3 - 1^3$. So there are $59 - 3 = 56$ different values.

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Check. A quick filter confirms that no other repeats were missed: every value with $d = 1$ is $3b(b+1)+1 \equiv 1 \pmod 6$, and every value with $d = 5$ is $\equiv 5 \pmod 6$, so those two rows cannot share a number; the only coincidences across rows are the three found above, each verified by computing both cube differences. Also, the last entry of each row is the largest b allowed: for instance $d = 1$, $b = 18$ gives $19^3 - 18^3 = 1027 > 1000$.

4. Answer: 15. **The idea.** The gap between the two means is a perfect square over a simple denominator: $\frac{a+b}{2} - \frac{2ab}{a+b} = \frac{(a+b)^2-4ab}{2(a+b)} = \frac{(b-a)^2}{2(a+b)}$. So the condition is $(b-a)^2 = 24(a+b)$. Writing $d = b - a$ turns this into a formula for a in terms of d, and the question becomes: which d make that formula a positive integer?

Work it out. With $b = a+d$ the condition reads $d^2 = 24(2a+d)$, so $48a = d^2 - 24d = d(d-24)$. We need $a \geq 1$, so $d > 24$, and we need $48 \mid d(d-24)$.

Divisibility by 3*:* $d(d-24) \equiv d^2 \pmod 3$, so $3 \mid d$. *Divisibility by* 16*:* $d(d-24)$ must be even, and d and $d-24$ have the same parity, so $d = 2f$ is even; then $d(d-24) = 4f(f-12)$, and $4 \mid f(f-12)$ forces f even (if f is odd, so is $f-12$). So $4 \mid d$, and together $12 \mid d$. Conversely, if $d = 12m$ then $48a = 12m(12m-24) = 144m(m-2)$, so $a = 3m(m-2)$ and $b = a + 12m = 3m(m+2)$, both integers.

So the pairs are exactly $(a, b) = (3m(m-2), 3m(m+2))$ with $m \geq 3$ (for $a \geq 1$). The bound $b \leq 1000$ means $m(m+2) \leq 333$; since $17 \cdot 19 = 323$ and $18 \cdot 20 = 360$, m runs over $3, 4, \ldots, 17$. The answer is 15.

Check. For $m = 3$: $(a, b) = (9, 45)$, with arithmetic mean 27 and harmonic mean $\frac{2\cdot 405}{54} = 15$, a gap of 12. For $m = 17$: $(765, 969)$, with arithmetic mean 867 and harmonic mean $\frac{2\cdot 765\cdot 969}{1734} = 855$, again a gap of 12. A computer search over all $a < b \leq 1000$ finds exactly these 15 pairs.

5. Answer: 982. **The idea.** Let $f(x) = \lfloor x \rfloor + \lfloor x^2 \rfloor + \lfloor x^3 \rfloor$. For $x \geq 0$ it never decreases, and it only changes by jumps upward, at the points where one of the three floors increases. It hits every integer on its way up except where it jumps by more than 1. So the whole problem is to find where several floors jump at the same moment.

Work it out. *Negative x gives nothing.* For $x < 0$ we have $f(x) \leq x + x^2 + x^3 = x(1 + x + x^2) < 0$, since $1 + x + x^2 > 0$. So only $x \geq 0$ matters.

Where the jumps are. On $x \geq 0$: $\lfloor x \rfloor$ jumps by 1 at each positive integer, $\lfloor x^2 \rfloor$ jumps by 1 at each $x = \sqrt{m}$, and $\lfloor x^3 \rfloor$ jumps by 1 at each $x = \sqrt[3]{m'}$ (m, m' positive integers). Can two of these happen at the same non-integer point? If $\sqrt{m} = \sqrt[3]{m'}$ then $m^3 = m'^2$, so m^3 is a perfect square and hence m is a perfect square, which makes the point an integer. So at a non-integer point exactly one floor jumps, by 1, and no value is skipped. At a positive integer k all three jump together: just before k, $f = (k-1) + (k^2-1) + (k^3-1)$, and at k, $f(k) = k + k^2 + k^3$. So the two values $f(k) - 1$ and $f(k) - 2$ are skipped, and every other nonnegative integer is attained (since $f(0) = 0$ and f grows without bound).

Count. $f(k) = k + k^2 + k^3$ takes the values $3, 14, 39, 84, 155, 258, 399, 584, 819$ for $k = 1, \ldots, 9$, and $f(10) = 1110$, whose skipped values $1108, 1109$ exceed 1000. So exactly $2 \cdot 9 = 18$ of the integers $1, \ldots, 1000$ are missed, and $1000 - 18 = 982$ are attained.

Check. The smallest cases by hand: on $[0, 1)$, $f = 0$, and $f(1) = 3$, so 1 and 2 are missed. On $[1, 2)$ the jump points in order are $\sqrt[3]{2} \approx 1.26$, $\sqrt{2} \approx 1.41$, $\sqrt[3]{3} \approx 1.44$, $\sqrt[3]{4} \approx 1.59$, $\sqrt[3]{5} \approx 1.71$, $\sqrt{3} \approx 1.73$, $\sqrt[3]{6} \approx 1.82$, $\sqrt[3]{7} \approx 1.91$, and f steps through $4, 5, 6, 7, 8, 9, 10, 11$ there, one at a time. At $x = 2$ it jumps to $2 + 4 + 8 = 14$, so 12 and 13 are missed, as predicted.

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6. Answer: 84. **The idea.** The rule is the tangent addition formula $\tan(x+y) = \frac{\tan x+\tan y}{1-\tan x\tan y}$, and $2-\sqrt{3} = \tan 15°$. So if $a_{n-1} = \tan\theta_{n-1}$ and $a_n = \tan\theta_n$, then $a_{n+1} = \tan(\theta_n+\theta_{n-1})$: the angles add like Fibonacci numbers.

Work it out. With $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$, induction gives $a_n = \tan(15° \cdot F_n)$. Tangent has period $180° = 15° \cdot 12$, so only F_n mod 12 matters. Modulo 12 the Fibonacci numbers run $1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, 5, 10, 3, 1, 4, 5, 9, 2, 11, 1, 0$, and then $1, 1, \ldots$ again, so they repeat with period 24. The residue 6 never appears, so no angle is an odd multiple of 90°: every a_n exists, and $1 - a_n a_{n-1} \neq 0$ because $\theta_n + \theta_{n-1} = \theta_{n+1}$ is never 90° modulo 180°. (This justifies the induction.)

Now $a_n = 1$ exactly when $15F_n \equiv 45 \pmod{180}$, i.e. $F_n \equiv 3 \pmod{12}$. In the period this happens at $n = 4$ and $n = 16$, so $a_n = 1$ exactly when $n \equiv 4$ or $16 \pmod{24}$. Since $1000 = 41 \cdot 24 + 16$, the values $n = 4, 28, \ldots, 988$ give 42 terms and $n = 16, 40, \ldots, 1000$ give 42 more. The answer is 84.

Check. Directly: $a_3 = \frac{2(2-\sqrt{3})}{1-(7-4\sqrt{3})} = \frac{2-\sqrt{3}}{2\sqrt{3}-3} = \frac{1}{\sqrt{3}}$ $(= \tan 30°)$, and $a_4 = \frac{(2-\sqrt{3})+\frac{1}{\sqrt{3}}}{1-\frac{2-\sqrt{3}}{\sqrt{3}}} = \frac{2\sqrt{3}-2}{2\sqrt{3}-2} = 1$ $(= \tan 45°)$, matching $F_3 = 2$ and $F_4 = 3$. Also $F_{16} = 987 \equiv 3 \pmod{12}$ and $F_{28} = 317811 \equiv 3 \pmod{12}$, as the period predicts.

7. Answer: 99. **The idea.** Let $m = \lfloor n\sqrt{2} \rfloor$. Then $k = 2n^2 - m^2$ is a positive integer (positive since $m < n\sqrt{2}$), and $\{n\sqrt{2}\} = n\sqrt{2} - m = \frac{2n^2-m^2}{n\sqrt{2}+m} = \frac{k}{n\sqrt{2}+m}$. The fractional part is tiny only if the integer k is small compared with n. That turns the search into solving $2n^2 - m^2 = k$ for small k, which are Pell-type equations with very sparse solutions.

Work it out. Since $m < n\sqrt{2}$, the denominator is less than $2\sqrt{2}\,n$, so $\{n\sqrt{2}\} > \frac{k}{2\sqrt{2}\,n}$. For this to be below $\frac{1}{100}$ we need $n > 25\sqrt{2}\,k \approx 35.4k$.

$k = 1$: $2n^2 - m^2 = 1$. The solutions come from the powers of $1+\sqrt{2}$: $(n, m) = (1, 1), (5, 7), (29, 41), (169, 239), \ldots$ (each next pair is $(3n + 2m, 4n + 3m)$). We need $n > 35$, so the first candidate is 169.

$k = 2$: m must be even, $m = 2j$, and then $n^2 - 2j^2 = 1$, with solutions $n = 1, 3, 17, 99, 577, \ldots$. We need $n > 70.7$, so the first candidate is $n = 99$ (with $m = 140$).

$k = 3$: m would be odd, so $m^2 \equiv 1 \pmod 8$ and $2n^2 \equiv 4 \pmod 8$, i.e. $n^2 \equiv 2 \pmod 4$, impossible.

$k \geq 4$: we would need $n > 141$.

So no $n < 99$ works, and $n = 99$ does: $2 \cdot 99^2 - 140^2 = 19602 - 19600 = 2$, so $\{99\sqrt{2}\} = \frac{2}{99\sqrt{2}+140} < \frac{2}{280} = \frac{1}{140} < \frac{1}{100}$. The answer is 99.

Check. Numerically $99\sqrt{2} = 140.00714\ldots$, while the earlier near-misses are $29\sqrt{2} = 41.0121\ldots$ and $17\sqrt{2} = 24.0416\ldots$, both with fractional parts above $\frac{1}{100}$. Note that the good approximations to $\sqrt{2}$ from *below* ($k > 0$) are the ones that matter here; $70\sqrt{2} = 98.9949\ldots$ is very close to an integer but just under it, so its fractional part is about 0.995.

8. Answer: 217. **The idea.** Since AI bisects $\angle A$ and $XY \perp AI$, triangle AXY is isosceles with $AX = AY$ and $IX = IY$. An angle chase then shows $\triangle BXI \sim \triangle IYC$, so $BX \cdot CY = XI \cdot YI = XI^2$, and XI is easy to get from the inradius.

Work it out. Write $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$. In the right triangle AIX, $\angle AXI = 90° - \frac{\alpha}{2}$, so $\angle BXI = 90° + \frac{\alpha}{2}$, and in the same way $\angle IYC = 90° + \frac{\alpha}{2}$. In triangle BXI, $\angle XBI = \frac{\beta}{2}$, so

$\angle XIB = 180° - (90° + \frac{\alpha}{2}) - \frac{\beta}{2} = \frac{\gamma}{2} = \angle ICY$. So the triangles BXI and IYC have two equal angles and are similar, with $\frac{BX}{XI} = \frac{IY}{YC}$, i.e. $BX \cdot CY = XI \cdot IY = XI^2$.

The semiperimeter is 21 and the area is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so $r = 4$. Let F be the foot of the perpendicular from I to AB, so $IF = r$. In the right triangle XFI the angle at X is $90° - \frac{\alpha}{2}$, so $\angle FIX = \frac{\alpha}{2}$ and $XI = \frac{r}{\cos(\alpha/2)}$. By the law of cosines $\cos\alpha = \frac{13^2+14^2-15^2}{2\cdot13\cdot14} = \frac{140}{364} = \frac{5}{13}$, so $\cos^2\frac{\alpha}{2} = \frac{1+5/13}{2} = \frac{9}{13}$. Therefore $BX \cdot CY = \frac{16}{9/13} = \frac{208}{9}$, and $m + n = 208 + 9 = 217$.

Check. Directly: $AX = \frac{AI}{\cos(\alpha/2)} = \frac{r}{\sin(\alpha/2)\cos(\alpha/2)} = \frac{2r}{\sin\alpha}$. With $\sin\alpha = \frac{12}{13}$ this is $\frac{8\cdot13}{12} = \frac{26}{3}$, so $BX = 13 - \frac{26}{3} = \frac{13}{3}$ and $CY = 14 - \frac{26}{3} = \frac{16}{3}$, whose product is $\frac{208}{9}$.

9. Answer: 37. **The idea.** The condition is $a^2 + c^2 = 2b^2$. Rotating the pair (a, c) by 45° turns this into a Pythagorean equation: with $x = \frac{c+a}{2}$ and $y = \frac{c-a}{2}$ we get $x^2 + y^2 = \frac{a^2+c^2}{2} = b^2$. So such triples correspond exactly to right triangles with integer sides $x > y$ and hypotenuse b, with $c = x + y$ the sum of the legs.

Work it out. First the correspondence. Since $a^2 + c^2 = 2b^2$ is even, a and c have the same parity, so x and y are positive integers with $x > y$ (as $a < c$) and $x^2 + y^2 = b^2$. Conversely, a Pythagorean triple with legs $x > y$ and hypotenuse b gives $a = x - y > 0$ and $c = x + y$ with $a^2 + c^2 = 2(x^2 + y^2) = 2b^2$; then $a^2 < b^2 < c^2$ because b^2 is the average of the two, so $a < b < c$. The two maps undo each other. So we must count Pythagorean triples whose legs add up to at most 100.

Every such triple is k times a primitive one with legs $m^2 - n^2$ and $2mn$, where $m > n \geq 1$ are coprime of opposite parity, and the leg sum is $s = m^2 - n^2 + 2mn$. Since $s > m^2$, we need $m \leq 9$. The primitive triples with $s \leq 100$, with s and the number $\lfloor 100/s \rfloor$ of allowed multiples k: $(3, 4, 5)$: $s = 7, 14$; $(5, 12, 13)$: 17, 5; $(8, 15, 17)$: 23, 4; $(7, 24, 25)$: 31, 3; $(20, 21, 29)$: 41, 2; $(12, 35, 37)$: 47, 2; $(9, 40, 41)$: 49, 2; $(11, 60, 61)$: 71, 1; $(28, 45, 53)$: 73, 1; $(16, 63, 65)$: 79, 1; $(33, 56, 65)$: 89, 1; $(13, 84, 85)$: 97, 1. The other admissible (m, n) are too big: $(8, 3)$ gives $55+48 = 103$, $(8, 5)$ gives 119, $(8, 7)$ gives 127, and $(9, 2)$, $(9, 4)$, $(9, 8)$ give 113, 137, 161. The total is $14 + 5 + 4 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 37$.

Check. The example $(1, 5, 7)$ comes from $(x, y) = (4, 3)$, the triple $3, 4, 5$ itself. Another: $k = 2$ on $(5, 12, 13)$ gives $x = 24$, $y = 10$, so $(a, b, c) = (14, 26, 34)$, and indeed $196 + 1156 = 1352 = 2 \cdot 676$. At the edge, $(13, 84, 85)$ gives $(71, 85, 97)$: $5041 + 9409 = 14450 = 2 \cdot 7225$.

10. Answer: 141. **The idea.** Count moves by which numbers get chosen: the number of moves is the number of x that are ever chosen, so by linearity the expectation is $\sum_x P(x \text{ is chosen})$. The number x disappears the first time some multiple of x is chosen, and at that moment all multiples of x are still on the board. So the first multiple of x to be chosen is uniformly distributed among the $\lfloor 20/x \rfloor$ multiples, and x is chosen with probability $1/\lfloor 20/x \rfloor$.

Work it out. Why are all multiples of x still present? A number y is erased only when some multiple of y is chosen. If y is a multiple of x, a multiple of y is also a multiple of x. So before any multiple of x has been chosen, no multiple of x has been erased. At the move where a multiple of x is first chosen, the choice is uniform over the board, so conditioned on landing on a multiple of x it is uniform over all $\lfloor 20/x \rfloor$ of them. After that move x is gone, so x is chosen exactly when it is this first multiple, which has probability $\frac{1}{\lfloor 20/x \rfloor}$.

Now add. For $11 \leq x \leq 20$, $\lfloor 20/x \rfloor = 1$: total 10. For $7 \leq x \leq 10$ it is 2: total $4 \cdot \frac{1}{2} = 2$. Then $x = 6, 5, 4, 3, 2, 1$ give $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{20}$, which sum to $\frac{20+15+12+10+6+3}{60} = \frac{66}{60} = \frac{11}{10}$. So the

expectation is $12 + \frac{11}{10} = \frac{131}{10}$, and $m + n = 141$.

Check. Try the board $1, 2, 3, 4$, where the formula predicts $\frac{1}{4} + \frac{1}{2} + 1 + 1 = \frac{11}{4}$. Directly: choosing 4 first leaves $\{3\}$ (2 moves in all); choosing 3 leaves $\{2, 4\}$ (then 1 or 2 more moves, equally likely: $\frac{5}{2}$ in all); choosing 2 leaves $\{3, 4\}$ (3 in all); choosing 1 leaves $\{2, 3, 4\}$, from which the next move leads to totals of 2, $\frac{5}{2}$ or 3 moves, averaging $\frac{5}{2}$ more ($\frac{7}{2}$ in all). The average is $\frac{2+5/2+3+7/2}{4} = \frac{11}{4}$, as predicted. A computer run of the exact recursion over every reachable board for $1, \ldots, 20$ also gives $\frac{131}{10}$.

11. Answer: 681**. The idea.** Do not look for x and y directly; look at $s = x + y$ and $p = xy$ instead. The numbers x, y are the roots of $t^2 - st + p$, so they are real exactly when $s^2 \geq 4p$, and $x^2 + y^2 = s^2 - 2p$. So each integer pair (s, p) with $s^2 - 2p \leq 50$ and $s^2 \geq 4p$ gives two ordered pairs (x, y) (the two roots in either order), except that it gives just one when $s^2 = 4p$ (a double root, $x = y$).

Work it out. For a fixed integer s we need $\frac{s^2-50}{2} \leq p \leq \frac{s^2}{4}$. This interval is nonempty only when $s^2 - 50 \leq \frac{s^2}{2}$, i.e. $s^2 \leq 100$, so $-10 \leq s \leq 10$, and the count for s equals the count for $-s$.

If $s = 2m$ is even, p runs from $2m^2 - 25$ to m^2: that is $26 - m^2$ values, one of which ($p = m^2$) is a double root, giving $2(26 - m^2) - 1 = 51 - 2m^2$ pairs. For $m = 0, 1, 2, 3, 4, 5$ this is $51, 49, 43, 33, 19, 1$.

If $s = 2m + 1$ is odd, $s^2 = 4m^2 + 4m + 1$, so p runs from $2m^2 + 2m - 24$ to $m^2 + m$: that is $25 - m^2 - m$ values, never a double root, giving $50 - 2m^2 - 2m$ pairs. For $m = 0, 1, 2, 3, 4$ (that is, $s = 1, 3, 5, 7, 9$) this is $50, 46, 38, 26, 10$.

Total: $51 + 2(49 + 43 + 33 + 19 + 1) + 2(50 + 46 + 38 + 26 + 10) = 51 + 2 \cdot 145 + 2 \cdot 170 = 681$.

Check. Different pairs (s, p) give different pairs (x, y), and each root pair really has $x^2 + y^2 = s^2 - 2p$ for both orders, so nothing is double-counted. At the extreme $s = 10$ the only p is 25, giving $x = y = 5$ with $x^2 + y^2 = 50$, on the boundary, as expected. A computer search over a much larger window of (s, p), testing the actual roots, also finds 681 pairs.

12. Answer: 32**. The idea.** $g_n(c) = f^n(0)$ is a polynomial in c, and the admissible c are its roots with a few removed. Because 7 is prime there is almost nothing to remove, and because we only need the power sum $\sum c^2$, we only need the top three coefficients of g_7, which follow a very simple recursion.

Work it out. We have $g_1 = c$ and $g_{n+1} = g_n^2 + c$, so g_n is monic of degree 2^{n-1}; g_7 has degree 64. If $f^7(0) = 0$ then the orbit of 0 is periodic with some least period d dividing 7, so $d = 1$ (meaning $c = f(0) = 0$) or $d = 7$. Hence the admissible c are exactly the nonzero roots of g_7; since $c = 0$ contributes 0 to $\sum c^2$, we may sum over all roots of g_7.

Are the roots distinct? Yes: $g_n' = 2g_{n-1}g_{n-1}' + 1$, so g_7' has integer coefficients and $g_7' \equiv 1$ modulo 2. At a root c of the monic integer polynomial g_7, c is an algebraic integer, so $g_7'(c) = 1 + 2(\text{algebraic integer})$ is not 0: the algebraic integers are closed under sums and products, so $g_7'(c) = 0$ would make $-\frac{1}{2}$ an algebraic integer, yet a rational number that is a root of a monic integer polynomial must be an integer (rational root theorem). So summing over roots with multiplicity is the same as summing over distinct c.

Write $g_n = c^N + a_n c^{N-1} + b_n c^{N-2} + \cdots$ with $N = 2^{n-1}$. Squaring gives $g_n^2 = c^{2N} + 2a_n c^{2N-1} + (a_n^2 + 2b_n)c^{2N-2} + \cdots$, and adding c does not touch these three coefficients once $2N - 2 > 1$. From $g_3 = c^4 + 2c^3 + c^2 + c$ we get $(a_3, b_3) = (2, 1)$, then $(a_4, b_4) = (4, 6)$, $(a_5, b_5) = (8, 28)$,

$(a_6, b_6) = (16, 120)$, $(a_7, b_7) = (32, 496)$. By Vieta, $\sum c = -32$ and $\sum_{i<j} c_i c_j = 496$, so $\sum c^2 = 32^2 - 2 \cdot 496 = 1024 - 992 = 32$.

Check. The same recipe for period 3: $g_3 = c(c^3 + 2c^2 + c + 1)$, and the cubic's roots have $\sum c^2 = 2^2 - 2 \cdot 1 = 2$, matching $a_3^2 - 2b_3 = 4 - 2 = 2$. Numerically computing all 63 nonzero roots of g_7 and summing their squares also gives 32.

13. Answer: 384**. The idea.** By the Chinese Remainder Theorem, $N = N(2^8) \cdot N(3^8)$, where $N(q)$ counts pairs (a, b) modulo q with $a^2 \equiv b^3$. Modulo a prime power p^k, sort the pairs by the p-adic valuations of a and b. For units there is a clean bijection: the solutions of $u^2 = w^3$ in any finite abelian group are exactly $u = g^3$, $w = g^2$.

Units. If $u^2 = w^3$, put $g = uw^{-1}$. Then $g^2 = u^2 w^{-2} = w$ and $g^3 = g^2 \cdot g = w \cdot uw^{-1} = u$. Conversely (g^3, g^2) is always a solution, and different g give different pairs (as $g = uw^{-1}$). So the unit solutions modulo p^j number exactly $\varphi(p^j)$, even for $p = 2$ where the unit group is not cyclic.

Work it out. Fix p and $k = 8$; write $v(a) = i$, $v(b) = j$. If $a^2 \equiv b^3 \not\equiv 0 \pmod{p^8}$, the two sides have the same valuation below 8, so $2i = 3j < 8$: either $i = j = 0$ or $(i, j) = (3, 2)$. Otherwise both sides are 0 modulo p^8.

Both sides zero: $p^4 \mid a$ (p^4 choices of a) and $p^3 \mid b$ (p^5 choices of b), giving p^9 pairs.

$i = j = 0$: $\varphi(p^8)$ pairs, by the bijection.

$(i, j) = (3, 2)$: write $a = p^3 u$ with u a unit modulo p^5 and $b = p^2 w$ with w a unit modulo p^6. The condition $p^6 u^2 \equiv p^6 w^3 \pmod{p^8}$ says $u^2 \equiv w^3 \pmod{p^2}$, which has $\varphi(p^2)$ solutions modulo p^2; each lifts to p^3 values of u modulo p^5 and p^4 values of w modulo p^6. That gives $\varphi(p^2)p^7$ pairs.

So $N(p^8) = \varphi(p^8) + \varphi(p^2)p^7 + p^9$. For $p = 2$: $128 + 2 \cdot 128 + 512 = 896$. For $p = 3$: $4374 + 6 \cdot 2187 + 19683 = 37179$. Then $N = 896 \cdot 37179 = 33312384$, and the remainder is 384.

Check. The same method modulo 8 gives $\varphi(8) + 2 \cdot 4 = 12$ (only $i = j = 0$ is possible below the top, and both-zero means $4 \mid a$, $2 \mid b$), and listing by hand: $b \equiv 1$ with odd a (4 pairs) and even b with $a \in \{0, 4\}$ (8 pairs) gives 12. A computer count over all residues modulo 6^8 also gives $N \equiv 384$.

14. Answer: 924**. The idea.** Neighbours of a vertex with an odd number of 1s have an even number of 1s, and vice versa. So the conditions at the 8 odd vertices involve only the colours of the 8 even vertices, and the conditions at the even vertices involve only the colours of the odd ones. The answer is K^2, where K counts colourings of the even vertices in which none of the eight neighbourhoods $N(v)$ (v odd) is one colour. (The map flipping the first digit swaps the two halves and carries one problem to the other, so both factors are the same K.)

The structure. Pair each even vertex with its complement: $\{0000, 1111\}$, $\{1100, 0011\}$, $\{1010, 0101\}$, $\{1001, 0110\}$. Complementary vertices are at distance 4 and have no common neighbour, so each block $N(v)$ (four vertices) contains exactly one vertex from each pair: it is a selection, one of the $2^4 = 16$ ways to pick one vertex per pair. Two distinct odd vertices are at distance 2, sharing exactly 2 neighbours, or at distance 4 (complements), sharing none; so two blocks share 2 vertices or 0, so two blocks never differ in exactly one pair. The 16 selections fall into 8 couples differing only in the first pair; each couple holds at most one block, and there are 8 blocks, so each couple holds exactly one. The same holds for any other pair. Also $N(\overline{v})$ is the complementary selection of $N(v)$.

Work it out. Call a pair B (both black), W (both white) or M (mixed, 2 orientations). A black block exists only if no pair is W; the black vertices then allow one choice in each M pair and two in each B pair.

Some pair B and some pair W: no one-colour block at all. Weighting B, W, M by 1, 1, 2, the total is 4^4; removing those with no B (3^4) or no W (3^4) and adding back those with neither (2^4) gives $256 - 81 - 81 + 16 = 110$ good colourings.

Some B but no W: the black choices are forced in the M pairs and free in a B pair, and one of the two options in that pair makes a block, so there is an all-black block. All bad; similarly some W but no B.

All four pairs M: the black vertices form one selection and the white vertices the complementary one. Blocks come in complementary pairs, so either both are blocks or neither is. The 16 orientations give all 16 selections once each, 8 of them blocks, so 8 colourings are good.

Hence $K = 110 + 8 = 118$ and the number of colourings is $118^2 = 13924$. The remainder is 924.

Check. Explicitly the blocks are the selections using an even number of second members: $N(1000) = \{0000, 1100, 1010, 1001\}$ uses none, and $N(0100) = \{0000, 1100, 0101, 0110\}$ uses two. A computer check of all 2^{16} colourings finds exactly 13924.

15. Answer: 512**. The idea.** Draw σ as 12 arrows $i \to \sigma(i)$ on a number line and cut the line in the 11 gaps between g and $g + 1$. The term $|\sigma(i) - i|$ is the number of gaps the arrow from i crosses, so the total is the number of (arrow, gap) crossings. If c_g arrows cross gap g to the right, exactly c_g cross it to the left, so the total is $2(c_1 + \cdots + c_{11})$, and being connected says exactly that every $c_g \geq 1$. Then build σ one number at a time and track how c_g can change from one gap to the next: each change of state has a simple weight, and the sum 26 allows only a handful of state sequences.

Crossings. Let c_g be the number of $i \leq g$ with $\sigma(i) > g$. The positions $1, \ldots, g$ send only $g - c_g$ arrows to values in $\{1, \ldots, g\}$, so the other c_g of those values are hit by arrows from positions above g: c_g arrows cross leftward too. Also $c_g = 0$ means $\sigma(1), \ldots, \sigma(g)$ are $1, \ldots, g$ in some order, so σ is connected exactly when $c_1, \ldots, c_{11} \geq 1$. Our condition becomes: every $c_g \geq 1$ and $c_1 + \cdots + c_{11} = 13$.

Building σ. Settle each arrow $i \to j$ at stage $\max(i, j)$. After stage k there are c_k loose tails (positions $\leq k$ whose arrow goes above k) and c_k loose heads (values $\leq k$ whose arrow comes from above k). At stage $k + 1$, position $k + 1$ either points to value $k + 1$, or to one of the c loose heads, or stays loose; value $k + 1$ either receives position $k + 1$, or one of the c loose tails, or stays loose. Counting the options, from $c = c_k$: going to $c + 1$ in 1 way (both stay loose); staying at c in $2c + 1$ ways ($\sigma(k + 1) = k + 1$, or exactly one of the two is joined to an old loose end); dropping to $c - 1$ in c^2 ways (both joined). Every permutation arises from exactly one sequence of choices, so N is the sum, over all admissible sequences $c_0 = 0, c_1, \ldots, c_{11}, c_{12} = 0$, of the product of the step weights.

Work it out. Steps change c by at most 1, so $c_1 = 1$ and $c_{11} = 1$, and the first and last steps (from c_0 and to c_{12}) have weight 1. The sequence $c_1, \ldots, c_{11}$ is all 1s plus 2 extra. A 3 would need 2s on both sides, an excess of at least 4, so the extra is two 2s, in gaps 2 to 10. The weights we need are $1 \to 1$: 3; $1 \to 2$: 1; $2 \to 1$: 4; $2 \to 2$: 5, and there are 10 steps from c_1 to c_{11}.

Two adjacent 2s, in gaps $g, g + 1$ with $2 \leq g \leq 9$: 8 placements, steps $1 \to 2 \to 2 \to 1$ of weight $1 \cdot 5 \cdot 4 = 20$ and 7 other steps of weight 3: $8 \cdot 20 \cdot 3^7 = 349920$.

Two separated 2s: $\binom{9}{2} - 8 = 28$ placements, two bumps $1 \to 2 \to 1$ of weight 4 each and 6 other steps of weight 3: $28 \cdot 16 \cdot 3^6 = 326592$.

So $N = 349920 + 326592 = 676512$, and the remainder is 512.

Check. With all $c_g = 1$ the same method gives 3^{n-2} connected permutations of $\{1, \dots, n\}$ with total $2(n-1)$. For $n = 3$ that is 3: the connected permutations of $\{1, 2, 3\}$ are exactly 231, 312 and 321, and their totals are $1 + 1 + 2$, $2 + 1 + 1$ and $2 + 0 + 2$, that is 4. Replacing 12 by n in the count above gives $(n-4) \cdot 20 \cdot 3^{n-5} + \left(\binom{n-3}{2} - (n-4)\right) 16 \cdot 3^{n-6}$, which is 136 for $n = 6$ and 3024 for $n = 8$; a computer run over all permutations for $n \le 8$ agrees, and a direct search over the permutations of $\{1, \dots, 12\}$ finds 676512.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.