MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

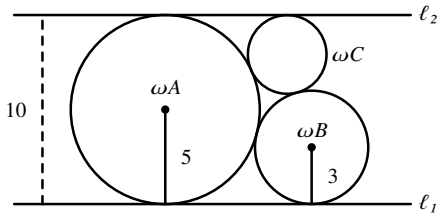
INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

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1. Two parallel lines ℓ_1 and ℓ_2 are 10 units apart, with ℓ_2 above ℓ_1. Circle ω_A has radius 5 and is tangent to both lines. Circle ω_B has radius 3, lies between the two lines, is tangent to ℓ_1, and is externally tangent to ω_A. Circle ω_C lies between the two lines, on the same side of ω_A as ω_B; it is tangent to ℓ_2 and externally tangent to both ω_A and ω_B. The radius of ω_C is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



2. How many positive integers n satisfy $\left\lfloor \frac{n}{20} \right\rfloor = \left\lfloor \frac{n}{21} \right\rfloor$? (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

3. A four-digit positive integer $\overline{abcd}$ has units digit $d \neq 0$. Its reversal is the four-digit integer $\overline{dcba}$. How many four-digit positive integers with nonzero units digit are divisible by 7 and also have their reversal divisible by 7?

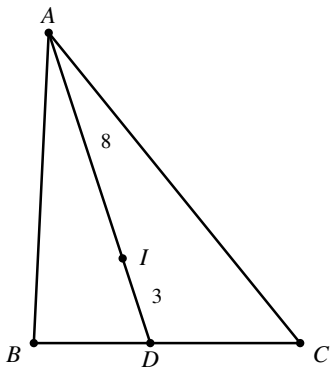
4. Four fair six-sided dice, with faces numbered 1 through 6, are rolled. The probability that the sum of the four numbers rolled is divisible by 7 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. Call an ordered pair (x, y) of integers *balanced* if $x^3 - x^2y - xy^2 + y^3 = 129600$. The integers x and y may be positive, negative, or zero, and (x, y) and (y, x) count as different pairs when $x \neq y$. How many balanced ordered pairs are there?

6. Points A, B, C, D lie on a line in that order, with $AB = 2$, $BC = 3$, and $CD = 9$. A point P not on the line satisfies $\angle APB = \angle BPC = \angle CPD$. Find $PC^2 + PD^2$.

7. In triangle ABC, let I be the incenter, and let the bisector of $\angle BAC$ meet side $\overline{BC}$ at D. Suppose $AI = 8$ and $ID = 3$. These two lengths do not determine the triangle, but they do determine the product $AB \cdot AC$, which equals $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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8. A sphere has radius 13, and P is a point inside it at distance 5 from its center. Three chords of the sphere pass through P, and their directions are mutually perpendicular. Two of these chords have length 25. Find the square of the length of the third chord.

9. A computer stores whole numbers in binary, and a programmer notices that the binary numeral of $n!$ always ends in a long run of zeros: fewer than n of them, but often not many fewer. For how many integers n with $5 \le n \le 1500$ does the binary numeral of $n!$ end in exactly $n - 5$ zeros?

10. A coin is flipped 11 times and the results are written in order as a string of 11 letters, each H or T. Of the $2^{11} = 2048$ possible strings, how many contain the block HTTH, that is, four consecutive letters reading H, T, T, H in that order?

11. Real numbers $x_1, x_2, \ldots, x_{10}$ satisfy $x_1 + x_2 + \cdots + x_{10} = 1$ and $x_1^2 + x_2^2 + \cdots + x_{10}^2 = 1$. Let $S = \sum_{1 \le i < j \le 10} |x_i - x_j|$, the sum over all 45 pairs. Find the square of the largest possible value of S.

12. A novelty clock has three hands, each of length 1, all pivoted at the centre O of the dial. The first hand is fixed and points at 12. The second hand turns clockwise at a steady one full turn per hour, and the third turns clockwise at a steady 250 full turns per hour. At noon all three hands point at 12. Think of each hand as the vector from O to its tip. At how many instants from noon up to (but not including) 1 p.m. does the sum of the three vectors have length exactly 1?

13. A sequence is defined by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 5a_n - a_{n-1}$ for $n \ge 1$, so it begins $0, 1, 5, 24, 115, 551, 2640, \ldots$. Find the least positive integer n for which the sum $a_1 + a_2 + \cdots + a_n$ is divisible by 1000.

14. All 20 diagonals of a regular octagon are drawn. Into how many regions do they divide the interior of the octagon?

15. The 12 edges of a cube are streets, and each street is made one-way in one of its two directions. In how many of the 2^{12} ways of doing this can one drive from every corner of the cube to every other corner, obeying the directions?

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Answer Key — AIME Practice Contest 2

1. 37

2. 209

3. 166

4. 247

5. 24

6. 261

7. 709

8. 578

9. 377

10. 870

11. 297

12. 499

13. 299

14. 80

15. 426

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Solutions — AIME Practice Contest 2

1. Answer: 37**. The idea.** Put ℓ_1 on the x-axis. Tangency to a line fixes the height of a centre, and external tangency of two circles says their centres are the sum of the radii apart. Writing the two tangency conditions for ω_C gives two equations that each contain r^2, and subtracting them removes every quadratic term at once.

Work it out. Let ℓ_1 be $y = 0$ and ℓ_2 be $y = 10$. Then ω_A has centre $(0, 5)$. The centre of ω_B is $(b, 3)$ with $b > 0$, and its distance to $(0, 5)$ is $5+3 = 8$, so $b^2+4 = 64$ and $b = 2\sqrt{15}$. Let ω_C have radius r and centre $(x, 10-r)$ with $x > 0$. Tangency to ω_A gives $x^2+(5-r)^2 = (5+r)^2$, so $x^2 = 20r$. Tangency to ω_B gives $(x-2\sqrt{15})^2+(7-r)^2 = (3+r)^2$, so $(x-2\sqrt{15})^2 = 20r-40$. Subtracting the second from the first: $4\sqrt{15}\,x - 60 = 40$, so $x = \frac{25}{\sqrt{15}} = \frac{5\sqrt{15}}{3}$. Then $r = \frac{x^2}{20} = \frac{125/3}{20} = \frac{25}{12}$, and the answer is $25 + 12 = 37$.

Check. The centre of ω_C is $\left(\frac{5\sqrt{15}}{3}, \frac{95}{12}\right)$, with $x \approx 6.45$ sitting between the centres of ω_A ($x = 0$) and ω_B ($x \approx 7.75$), as the picture suggests. Distance to ω_A's centre: $\frac{125}{3} + \left(\frac{35}{12}\right)^2 = \frac{6000+1225}{144} = \left(\frac{85}{12}\right)^2$, and $5 + \frac{25}{12} = \frac{85}{12}$. Distance to ω_B's centre: $\left(\frac{\sqrt{15}}{3}\right)^2 + \left(\frac{59}{12}\right)^2 = \frac{240+3481}{144} = \left(\frac{61}{12}\right)^2$, and $3 + \frac{25}{12} = \frac{61}{12}$. Its diameter $\frac{25}{6}$ is less than 10, so it really fits between the lines.

2. Answer: 209**. The idea.** Name the common value q. Then n must lie in the block $[20q, 20q+19]$ and also in the block $[21q, 21q+20]$. The second block starts q steps later, so the overlap shrinks by one each time q grows, and eventually it vanishes. That is why only finitely many n work.

Work it out. The conditions are $20q \le n \le 20q + 19$ and $21q \le n \le 21q + 20$. Together these give $21q \le n \le 20q + 19$, which is nonempty exactly when $q \le 19$. It then contains $(20q+19)-21q+1 = 20-q$ integers. Summing over $q = 0, 1, \ldots, 19$ gives $20+19+\cdots+1 = 210$ integers $n \ge 0$. The value $n = 0$ (from $q = 0$) is not positive, so the count is $210 - 1 = 209$.

Check. The largest solution is $n = 20 \cdot 19 + 19 = 399$, where $\lfloor 399/20 \rfloor = 19 = \lfloor 399/21 \rfloor$. Then $n = 400$ fails because $\lfloor 400/20 \rfloor = 20$ but $\lfloor 400/21 \rfloor = 19$. Also, for $q = 1$ the allowed range is $21 \le n \le 39$: that is $19 = 20 - 1$ values, as predicted.

3. Answer: 166**. The idea.** Work modulo 7. The place values $1000, 100, 10, 1$ are $\equiv -1, 2, 3, 1 \pmod 7$. Writing both the number and its reversal in these terms gives two linear congruences in the digits. Adding them wipes out a and d and ties b to c.

Work it out. We have $\overline{abcd} \equiv -a+2b+3c+d$ and $\overline{dcba} \equiv -d+2c+3b+a \pmod 7$. If both are 0, their sum $5(b+c) \equiv 0$, so $c \equiv -b \pmod 7$. Substituting into the first gives $-a+2b-3b+d \equiv 0$, that is, $d \equiv a + b \pmod 7$. Conversely, these two conditions make both numbers divisible by 7 (the second expression becomes $-(a + b) - 2b + 3b + a \equiv 0$). So we choose $a \in \{1, \ldots, 9\}$ and $b \in \{0, \ldots, 9\}$ freely. Then we count the digits $c \in \{0, \ldots, 9\}$ with $c \equiv -b$ and the digits $d \in \{1, \ldots, 9\}$ with $d \equiv a + b$.

Among 0 to 9, the residues $0, 1, 2$ occur twice (via $7, 8, 9$) and the others once, so the number of choices for c is $f(b) = 2$ when $-b \bmod 7 \in \{0, 1, 2\}$, i.e. $b \in \{0, 5, 6, 7\}$, and $f(b) = 1$ otherwise. Among 1 to 9, the residues $1, 2$ occur twice (via $8, 9$), so the number of choices for d is $g(r) = 2$ when $r \bmod 7 \in \{1, 2\}$ and 1 otherwise. We need $\sum_{a,b} f(b)\, g(a + b)$.

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Fix b and let $G(b) = \sum_{a=1}^{9} g(a+b)$. The nine values $b+1, \ldots, b+9$ run once through all seven residues (contributing $1 \cdot 5 + 2 \cdot 2 = 9$) and then repeat $b+1, b+2$. So $G(b) = 9 + g(b+1) + g(b+2)$, which gives $G = 13, 12, 11, 11, 11, 11, 12$ for $b \equiv 0, 1, \ldots, 6$. Summing $f(b)G(b)$ over $b = 0, \ldots, 9$: $2 \cdot 13 + 12 + 11 + 11 + 11 + 2 \cdot 11 + 2 \cdot 12 + 2 \cdot 13 + 12 + 11 = 166$.

Check. Direct enumeration of all four-digit multiples of 7 with nonzero last digit, testing each reversal, also gives 166. A quick example: $1001 = 7 \cdot 143$ is its own reversal. Here $a = 1$, $b = 0$, $c = 0 \equiv -0$ and $d = 1 \equiv 1 + 0$, as the conditions require.

4. Answer: 247. **The idea.** Look at the last die. The six faces $1, \ldots, 6$ hit every nonzero residue mod 7 exactly once. So whatever the first dice show, the last die can bring the total to a multiple of 7 in exactly one way. The one exception is when the earlier dice already sum to a multiple of 7: no face is $\equiv 0$, so then the last die cannot fix it.

Work it out. Let N_k be the number of outcomes of k dice whose sum is divisible by 7. Among the 6^{k-1} outcomes of the first $k-1$ dice, the N_{k-1} with sum $\equiv 0$ admit no good last die. Each of the other $6^{k-1} - N_{k-1}$ outcomes admits exactly one. So $N_k = 6^{k-1} - N_{k-1}$. Starting from $N_1 = 0$, we get $N_2 = 6$, $N_3 = 36 - 6 = 30$, and $N_4 = 216 - 30 = 186$. The probability is $\frac{186}{1296} = \frac{31}{216}$, in lowest terms since 31 is prime and does not divide 216. The answer is $31 + 216 = 247$.

Check. For two dice the pairs summing to 7 are $(1,6), (2,5), \ldots, (6,1)$: that is 6 pairs, matching N_2. For three dice, the sums 7 and 14 occur in 15 ways each (the distributions are mirror images under $x \mapsto 7 - x$), so $N_3 = 30$. Also, the answer $\frac{31}{216} \approx 0.1435$ sits close to $\frac{1}{7} \approx 0.1429$, as it should.

5. Answer: 24. **The idea.** Group the terms: $x^3 - x^2y - xy^2 + y^3 = x^2(x-y) - y^2(x-y) = (x-y)^2(x+y)$. So the equation says that the square of $d = x - y$ times $t = x + y$ is 129600. The only hidden constraint is parity: $x = \frac{t+d}{2}$ and $y = \frac{t-d}{2}$ are integers exactly when t and d have the same parity.

Work it out. Factor $129600 = 2^6 \cdot 3^4 \cdot 5^2$. We need $d \neq 0$ with $d^2 \mid 129600$. So $|d| = 2^i 3^j 5^k$ with $0 \le i \le 3$, $0 \le j \le 2$, $0 \le k \le 1$, and then $t = 129600/d^2$, which is positive. The parity of t depends only on i. If $i = 0$, then d is odd but t is divisible by 2^6, so this fails. If $i = 3$, then d is even but t is odd, so this also fails. If $i = 1$ or $i = 2$, both are even, so this works. That leaves $2 \cdot 3 \cdot 2 = 12$ values of $|d|$. Each sign of d gives a different ordered pair: swapping x and y flips d and keeps t. So there are 24 ordered pairs.

Check. Take $d = 2$: then $t = 32400$, which gives $x = 16201$ and $y = 16199$. Indeed $(x-y)^2(x+y) = 4 \cdot 32400 = 129600$. Now take $d = 8$ ($i = 3$): then $t = 2025$ is odd while d is even, so $x = \frac{2033}{2}$ is not an integer, as predicted. A computer search over every d with $|d| \le 360$ finds exactly 24 pairs.

6. Answer: 261. **The idea.** The equal angles mean two angle bisectors. PB bisects $\angle APC$, and PC bisects $\angle BPD$. By the angle bisector theorem, each one fixes a ratio of distances from P. Two such ratio conditions pin P down.

Work it out. In $\triangle APC$, PB bisects the angle at P and meets AC at B, so $\frac{PA}{PC} = \frac{AB}{BC} = \frac{2}{3}$. In $\triangle BPD$, PC bisects the angle at P, so $\frac{PB}{PD} = \frac{BC}{CD} = \frac{3}{9} = \frac{1}{3}$. Place the line on the x-axis with $A = 0$, $B = 2$, $C = 5$, $D = 14$, and let $P = (x, y)$ with $y \neq 0$.

From $9\,PA^2 = 4\,PC^2$: $9(x^2 + y^2) = 4((x-5)^2 + y^2)$, so $5(x^2 + y^2) + 40x - 100 = 0$.

From $9\,PB^2 = PD^2$: $9((x-2)^2 + y^2) = (x-14)^2 + y^2$, so $8(x^2 + y^2) - 8x - 160 = 0$, that is, $x^2 + y^2 = x + 20$.

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Substituting into the first equation gives $5(x+20)+40x-100=0$, so $45x=0$ and $x=0$. Then $y^2=20$. So P sits directly above A with $PA^2=20$, and $PC^2=25+20=45$ and $PD^2=196+20=216$. The answer is $45+216=261$.

Check. Now the ratios can be checked. $PB^2=4+20=24$, so $\frac{PA}{PC}=\sqrt{20/45}=\frac{2}{3}$ and $\frac{PB}{PD}=\sqrt{24/216}=\frac{1}{3}$. By the converse of the angle bisector theorem, these give the equal angles. Directly: $\tan\angle APB=\frac{2}{\sqrt{20}}=\frac{1}{\sqrt{5}}$, and $\tan\angle APC=\frac{5}{\sqrt{20}}=\frac{\sqrt{5}}{2}$. By the double-angle formula, $\frac{2\cdot(1/\sqrt{5})}{1-1/5}=\frac{\sqrt{5}}{2}$, so $\angle APC=2\angle APB$, as required. The reflection $(0,-\sqrt{20})$ gives the same answer.

7. Answer: 709**. The idea.** Two classical facts about the bisector AD each turn one given length into information about the sides. First, the incenter cuts the bisector in the ratio $\frac{AI}{ID}=\frac{AB+AC}{BC}$. Second, the length of a bisector is $AD^2=AB\cdot AC\left(1-\frac{BC^2}{(AB+AC)^2}\right)$. The first fact gives exactly the ratio that the second one needs, and what is left is $AB\cdot AC$.

Work it out. Write $a=BC$, $b=CA$, $c=AB$. By the angle bisector theorem $BD=\frac{ac}{b+c}$. In $\triangle ABD$ the segment BI bisects $\angle B$, so $\frac{AI}{ID}=\frac{AB}{BD}=\frac{c(b+c)}{ac}=\frac{b+c}{a}$. Hence $\frac{a}{b+c}=\frac{ID}{AI}=\frac{3}{8}$.

For the bisector length, Stewart's theorem on cevian AD (with $BD=\frac{ac}{b+c}$, $DC=\frac{ab}{b+c}$) gives $AD^2=bc-BD\cdot DC=bc\left(1-\frac{a^2}{(b+c)^2}\right)$. Here $AD=AI+ID=11$, so $121=bc\left(1-\frac{9}{64}\right)=\frac{55}{64}\,bc$, $\qquad bc=\frac{121\cdot 64}{55}=\frac{704}{5}$. Since $704=2^6\cdot 11$ is coprime to 5, the answer is $704+5=709$.

Check. A second route through the circumcircle. Extend AD to meet the circumcircle again at E, the midpoint of arc BC. Then $\angle EBI=\frac{A}{2}+\frac{B}{2}=\angle EIB$, so $EI=EB$; and $\triangle EBD\sim\triangle EAB$ (common angle E, and $\angle EBD=\angle EAB=\frac{A}{2}$), so $EB^2=ED\cdot EA$. With $x=DE$: $(3+x)^2=x(11+x)$, so $9=5x$ and $x=\frac{9}{5}$, making $AE=\frac{64}{5}$. Finally $\triangle ABE\sim\triangle ADC$ gives $AB\cdot AC=AD\cdot AE=11\cdot\frac{64}{5}=\frac{704}{5}$, the same value. Such triangles exist: for example $a=9$, $b+c=24$, $bc=\frac{704}{5}$ gives $b,c=12\pm\frac{4}{\sqrt{5}}$, a genuine triangle, and other values of a near 9 work too, all with the same product.

8. Answer: 578**. The idea.** A chord through P is determined by its direction, and its length depends only on the angle that direction makes with $\overline{OP}$, where O is the center. For three mutually perpendicular directions, the squared cosines of those angles add up to 1, so the sum of the three squared chord lengths is fixed.

Work it out. Let $\vec{d}=\overrightarrow{OP}$, so $|\vec{d}|=5$, and let $\vec{u}$ be a unit vector along a chord. The chord's endpoints are $P+t\vec{u}$ with $|\vec{d}+t\vec{u}|^2=169$, that is $t^2+2(\vec{d}\cdot\vec{u})t+(25-169)=0$. The chord length is the difference of the roots, so $L^2=(t_1-t_2)^2=4(\vec{d}\cdot\vec{u})^2+4\cdot 144=576+4(\vec{d}\cdot\vec{u})^2$.

For three mutually perpendicular unit vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3$, the numbers $\vec{d}\cdot\vec{u}_k$ are the coordinates of $\vec{d}$ in that frame, so $\sum(\vec{d}\cdot\vec{u}_k)^2=|\vec{d}|^2=25$. Hence $L_1^2+L_2^2+L_3^2=3\cdot 576+4\cdot 25=1828$, whatever the orientation.

With $L_1=L_2=25$ we get $L_3^2=1828-625-625=578$.

Check. The configuration exists: $625=576+4(\vec{d}\cdot\vec{u})^2$ gives $(\vec{d}\cdot\vec{u}_1)^2=(\vec{d}\cdot\vec{u}_2)^2=\frac{49}{4}$, and then $(\vec{d}\cdot\vec{u}_3)^2=25-\frac{49}{2}=\frac{1}{2}\geq 0$. So take $\vec{d}=\left(\frac{7}{2},\frac{7}{2},\frac{\sqrt{2}}{2}\right)$ in the frame of the chords; the third chord has $L^2=576+4\cdot\frac{1}{2}=578$, which lies between the shortest possible chord (576) and the diameter (676).

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9. Answer: 377**. The idea.** Legendre's formula for the prime 2 has a beautiful closed form: the exponent of 2 in $n!$ is $n - s(n)$, where $s(n)$ is the number of 1s in the binary representation of n. So the condition says exactly that n has five 1s in binary, and the problem becomes a count of binary strings below a bound.

Work it out. The exponent of 2 in $n!$ is $\lfloor n/2 \rfloor + \lfloor n/4 \rfloor + \lfloor n/8 \rfloor + \cdots$. If $n = \sum 2^{e_i}$ in binary, a single power 2^e contributes $2^{e-1} + 2^{e-2} + \cdots + 1 = 2^e - 1$ to this sum, and the floors split over the binary digits because no carries occur. Adding up gives $n - s(n)$. The number of zeros at the end of the binary numeral of $n!$ is exactly this exponent, so we need $n - s(n) = n - 5$, i.e. $s(n) = 5$.

Numbers below $1024 = 2^{10}$ are 10-bit strings; those with five 1s number $\binom{10}{5} = 252$, and all of them lie between $11111_2 = 31$ and $1111100000_2 = 992$, inside the range.

Numbers from 1024 to 1500 are $1024 + m$ with $0 \le m \le 476$ and $s(m) = 4$. In binary, $476 = 111011100_2$ (nine bits). Count $m \le 476$ with four 1s by the first bit where m drops below 476:

first bit 0: the remaining 8 bits carry four 1s: $\binom{8}{4} = 70$. Bits $1, 0$: $\binom{7}{3} = 35$. Bits $1, 1, 0$: $\binom{6}{2} = 15$. Bits $1, 1, 1, 0, 0$ (the fourth bit of 476 is already 0, so m must match it): $\binom{4}{1} = 4$. Bits $1, 1, 1, 0, 1$: four 1s are used, so $m = 111010000_2 = 464$: 1. Every later branch needs more than four 1s.

So there are $70 + 35 + 15 + 4 + 1 = 125$ such m, and the answer is $252 + 125 = 377$.

Check. Small case: $n = 31 = 11111_2$ has $s = 5$, and the exponent of 2 in 31! is $15 + 7 + 3 + 1 = 26 = 31 - 5$. Also $1500 = 10111011100_2$ has seven 1s, so it is correctly excluded.

10. Answer: 870**. The idea.** Count the strings that avoid HTTH and subtract. Let a_n be the number of length-n strings avoiding HTTH, and b_n the number of length-n strings in which HTTH appears for the first time at the very end. Two ways of extending a string give two relations between a and b. The only way HTTH overlaps itself is in the single letter H (it begins and ends with H, but HT $\neq$ TH and HTT $\neq$ TTH), and this is what decides the second relation.

Work it out. First relation: appending one letter to an avoiding string of length n gives $2a_n$ strings of length $n + 1$. Each of them either still avoids HTTH or has its first HTTH at the very end, so $2a_n = a_{n+1} + b_{n+1}$, i.e. $b_m = 2a_{m-1} - a_m$.

Second relation: take an avoiding string w of length n and append the whole block HTTH. The first occurrence of HTTH in wHTTH cannot lie inside w. If it ended at position $n + 2$ or $n + 3$, a proper ending of HTTH (namely TH or TTH) would equal a beginning of HTTH of the same length (HT or HTT), which is false. So the first occurrence ends at position $n + 1$ (this happens when w ends in HTT) or at position $n + 4$. Conversely, every string of length $n + 4$ whose first HTTH is at the end is wHTTH for an avoiding w, and every string y of length $n + 1$ whose first HTTH is at the end becomes yTTH $= w$HTTH with w equal to y minus its last letter. Hence $a_n = b_{n+4} + b_{n+1}$.

Substituting $b_m = 2a_{m-1} - a_m$: $a_n = (2a_{n+3} - a_{n+4}) + (2a_n - a_{n+1})$, so $a_{n+4} = 2a_{n+3} - a_{n+1} + a_n$. Starting from $a_0, a_1, a_2, a_3 = 1, 2, 4, 8$, the sequence continues 15, 28, 52, 97, 181, 338, 631, 1178 for $n = 4, \ldots, 11$. So $a_{11} = 1178$ strings avoid HTTH, and $2048 - 1178 = 870$ contain it.

Check. $a_4 = 15$: of the 16 strings of length 4, only HTTH itself is excluded. $a_5 = 28$: HTTH can start at position 1 or 2 of a length-5 string (2 strings each), and it cannot occur twice, since two occurrences would have to overlap in more than one letter; so $32 - 4 = 28$. A brute-force count over all 2048 strings also gives 870.

11. Answer: 297**. The idea.** Absolute values disappear once the numbers are sorted: then S is a

linear expression. Removing the mean turns the two constraints into a single length condition, and Cauchy–Schwarz finishes, with an equality case that happens to be sorted already.

Work it out. Relabel so that $x_1 \le x_2 \le \cdots \le x_{10}$ (this does not change S or the constraints). Then $|x_i - x_j| = x_j - x_i$ for $i < j$, and x_k appears with a plus sign $k-1$ times and a minus sign $10-k$ times, so $S = \sum_{k=1}^{10} c_k x_k$ with $c_k = 2k - 11$, i.e. $c = (-9,-7,-5,-3,-1,1,3,5,7,9)$.

Write $x_k = \frac{1}{10} + y_k$. Then $\sum y_k = 0$ and $\sum x_k^2 = \frac{1}{10} + \sum y_k^2 = 1$, so $\sum y_k^2 = \frac{9}{10}$. Because $\sum c_k = 0$, $S = \sum c_k y_k$. By Cauchy–Schwarz, $S^2 \le \left(\sum c_k^2\right)\left(\sum y_k^2\right) = 330 \cdot \frac{9}{10} = 297$, since $\sum c_k^2 = 2(1+9+25+49+81) = 330$.

Equality needs $y_k = t c_k$ with $t > 0$ and $t^2 \cdot 330 = \frac{9}{10}$. These y_k sum to 0 and are increasing, so the x_k are sorted as assumed and the constraints hold. The maximum of S^2 is 297.

Check. Without the sum condition the bound would be $\sqrt{330}$, reached by a vector with mean 0; forcing mean $\frac{1}{10}$ spends $\frac{1}{10}$ of the squared length on the constant part, which leaves $\frac{9}{10}$ of it, giving $\frac{9}{10} \cdot 330 = 297$. A quick test with $x = (0,\dots,0,1)$ (sum 1, squares 1) gives $S = 9$, and $81 < 297$.

12. Answer: 499**. The idea.** On the unit circle, $|a+b+c|^2$ for three unit complex numbers depends only on the cosines of the angles between them. The resulting trigonometric equation factors completely by sum-to-product, into three cosines whose zeros can be listed and compared.

Work it out. Put the dial in the complex plane with $O = 0$ and the fixed hand along 1 (a rotation of the whole picture does not change lengths). When the second hand has turned through angle θ clockwise, it is $z = e^{-i\theta}$, and the third hand, having turned 250 times as far, is z^{250}. As the time runs from noon to just before 1 p.m., θ runs once over $[0, 2\pi)$ and each instant gives a different z on the unit circle, so we are counting the z with $|z| = 1$ and $\left|z^{250} + z + 1\right| = 1$. Replacing z by $\bar{z}$ (i.e. θ by $-\theta$) does not change the count, so write $z = e^{i\theta}$. For unit complex numbers, $|a+b+c|^2 = 3 + 2\,\mathrm{Re}(a\bar{b} + a\bar{c} + b\bar{c})$, so with $a = z^{250}$, $b = z$, $c = 1$ the condition $|a+b+c|^2 = 1$ becomes $\cos 249\theta + \cos 250\theta + \cos\theta + 1 = 0$.

Group the terms: $1 + \cos 250\theta = 2\cos^2 125\theta$ and $\cos 249\theta + \cos\theta = 2\cos 125\theta \cos 124\theta$. So the left side is $2\cos 125\theta\,(\cos 125\theta + \cos 124\theta) = 4\cos 125\theta\,\cos\frac{249\theta}{2}\,\cos\frac{\theta}{2}$.

Count the $\theta \in [0, 2\pi)$ where some factor is 0. $\cos 125\theta = 0$: $\theta = \frac{(2k+1)\pi}{250}$, $k = 0,\dots,249$: 250 values. $\cos\frac{249\theta}{2} = 0$: $\theta = \frac{(2m+1)\pi}{249}$, $m = 0,\dots,248$: 249 values; $m = 124$ gives $\theta = \pi$. $\cos\frac{\theta}{2} = 0$: only $\theta = \pi$, already counted.

The first two families never share a value: $\frac{2k+1}{250} = \frac{2m+1}{249}$ would give $249(2k+1) = 250(2m+1)$, odd equal to even. So there are $250 + 249 = 499$ such z.

Check. $z = -1$: $|(-1)^{250} - 1 + 1| = 1$, correct. Also, clearing denominators turns the condition into a polynomial equation $(z^{250} + z + 1)(z^{250} + z^{249} + 1) = z^{250}$ of degree 500; our count says it has 498 simple roots and a double root at $z = -1$ on the circle, which uses up all 500 roots, consistent with $\theta = \pi$ being a zero of two factors.

13. Answer: 299**. The idea.** The partial sums factor: with $k = \lceil n/2 \rceil$, $S_n = a_1 + \cdots + a_n$ equals $a_k(a_k + a_{k+1})$ for even n and $a_k(a_k + a_{k-1})$ for odd n. The second factor $c_j = a_j + a_{j+1}$ turns out never to be divisible by 5 or by 4, so almost all of the $1000 = 2^3 \cdot 5^3$ has to come from a_k. That becomes a question about which a_k are divisible by 8 and by 125.

Work it out. *Doubling formulas.* Let α, β be the roots of $x^2 - 5x + 1$, so $\alpha\beta = 1$ and $a_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}$. Multiplying out gives $a_{2k} = a_k(a_{k+1} - a_{k-1})$ and $a_{2k+1} = a_{k+1}^2 - a_k^2$ (for the second,

$(\alpha^{k+1} - \beta^{k+1})^2 - (\alpha^k - \beta^k)^2 = \alpha^{2k+2} + \beta^{2k+2} - \alpha^{2k} - \beta^{2k} = (\alpha^{2k+1} - \beta^{2k+1})(\alpha - \beta)$, using $\alpha\beta = 1$). Both are easy to confirm on the listed terms: $a_4 = 115 = 5 \cdot 23 = a_2(a_3 - a_1)$ and $a_5 = 551 = 24^2 - 5^2$.

The factorisation. Claim: $S_{2k-1} = a_k(a_k + a_{k-1})$ and $S_{2k} = a_k(a_k + a_{k+1})$. It holds for $S_1 = 1 = a_1(a_1 + a_0)$. Going from S_{2k-1} to S_{2k} adds $a_k(a_{k+1} - a_{k-1}) = a_{2k}$, and going from S_{2k} to $S_{2k+1} = a_{k+1}(a_{k+1} + a_k)$ adds $(a_{k+1} - a_k)(a_{k+1} + a_k) = a_{2k+1}$. So the claim follows by induction. For example $S_4 = 1 + 5 + 24 + 115 = 145 = a_2(a_2 + a_3) = 5 \cdot 29$.

The second factor. $c_j = a_j + a_{j+1}$ obeys the same recurrence, starting $1, 6, 29, 139, 666, 3191, \ldots$. Mod 5 the recurrence is $c_{j+1} \equiv -c_{j-1}$, so c_j mod 5 runs $1, 1, 4, 4, 1, 1, \ldots$ and is never 0. Mod 4 it runs $1, 2, 1, 3, 2, 3$ and then repeats (the next two are $1, 2$ again), so it is never 0: c_j has at most one factor of 2.

The first factor. Lucas sequences like this one satisfy $\gcd(a_m, a_j) = a_{\gcd(m,j)}$, so the k with $d \mid a_k$ are exactly the multiples of the least one (the rank of d). Mod 2 the sequence runs $0, 1, 1, 0, 1, 1, \ldots$, so a_k is even exactly when $3 \mid k$, and then $a_3 = 24 \mid a_k$, so $8 \mid a_k$. Mod 5 it runs $0, 1, 0, 4, 0, 1, \ldots$ (again $a_{k+1} \equiv -a_{k-1}$), so the rank of 5 is 2. To climb to 125: if $5 \mid a_m$, then $\frac{a_{5m}}{a_m} = b_{4m} + b_{2m} + 1$ where $b_j = \alpha^j + \beta^j$; since $b_m^2 = 4 + 21a_m^2 \equiv 4 \pmod{25}$, we get $b_{2m} = b_m^2 - 2 \equiv 2$ and $b_{4m} \equiv 2 \pmod{25}$, so the quotient is $\equiv 5 \pmod{25}$ and adds exactly one factor of 5. Thus a_2, a_{10}, a_{50} have exactly $1, 2, 3$ factors of 5, so the rank of 125 divides 50 and is a multiple of 2; it is not 2 or 10, so it is 50. So $125 \mid a_k \iff 50 \mid k$.

Putting it together. $S_n = a_k \cdot c$ with c one of c_k, c_{k-1}. Since $5 \nmid c$, we need $125 \mid a_k$, i.e. $50 \mid k$. Since c has at most one factor of 2, we need $4 \mid a_k$, so a_k is even, so $3 \mid k$ and then $8 \mid a_k$ automatically. Conversely $150 \mid k$ gives $1000 \mid a_k \mid S_n$. So $1000 \mid S_n$ exactly when $k = \lceil n/2 \rceil$ is a multiple of 150, i.e. $n \in \{299, 300, 599, 600, \ldots\}$. The least is $n = 299$, the odd member of the first pair, with $S_{299} = a_{150}(a_{150} + a_{149})$.

Check. The trap is 300: the even case alone gives $n = 2k = 300$, but S_{2k-1} also carries the factor a_k, one term earlier. Small cases of the pattern: $8 \mid S_n$ should hold exactly for $\lceil n/2 \rceil \in \{3, 6, \ldots\}$, first at $n = 5$, and indeed $S_5 = 1+5+24+115+551 = 696 = 8 \cdot 87 = a_3(a_3+a_2) = 24 \cdot 29$, while $S_1, \ldots, S_4 = 1, 6, 30, 145$ are not multiples of 8. Running the partial sums mod 1000 by machine first hits 0 at $n = 299$.

14. Answer: 80**. The idea.** Draw the diagonals one at a time and count how many regions each one adds: this gives a formula in terms of the crossing points and how many diagonals pass through each. If no three diagonals ever met at a point, the answer would be 91. So the real work is to find every point where three or more diagonals meet, and for that there is an exact test: three chords of a circle are concurrent exactly when a certain product of chord lengths balances (trigonometric Ceva).

Work it out. *Counting regions.* When a new diagonal is drawn, the earlier diagonals cut it at some number j of distinct points, splitting it into $j + 1$ pieces, and each piece splits one region in two. So the number of regions is $1 + 20 + \sum_P (m_P - 1)$, summed over the interior points P where diagonals meet, with m_P the number of diagonals through P: at such a point the first diagonal drawn through P gains nothing there and each later one gains 1. Any 4 of the 8 vertices give exactly one crossing pair of diagonals, so there are $\binom{8}{4} = 70$ crossing pairs; with no triple points the answer would be $1 + 20 + 70 = 91$.

The concurrency test. Let A, B, C, D, E, F lie on a circle in this order. In triangle ACE the lines AD, CF, EB are cevians, and by trigonometric Ceva they are concurrent exactly when

$\frac{\sin\angle CAD}{\sin\angle DAE}\cdot\frac{\sin\angle ECF}{\sin\angle FCA}\cdot\frac{\sin\angle AEB}{\sin\angle BEC}=1$. Each of these is an inscribed angle, and a chord equals $2R$ times the sine of an inscribed angle on it, so the condition is $\frac{CD}{DE}\cdot\frac{EF}{FA}\cdot\frac{AB}{BC}=1$, i.e. $AB\cdot CD\cdot EF=BC\cdot DE\cdot FA$.

Which triples meet. Three diagonals through one interior point have six different endpoints (two diagonals with a common endpoint meet only there), and each pair crosses, so the endpoints alternate: they are the three long diagonals AD, BE, CF of a hexagon $ABCDEF$ formed by 6 of the 8 vertices. The six arcs between consecutive chosen vertices are a, b, c, d, e, f steps of 45°, with $a+\cdots+f=8$, and an arc of k steps has chord $2R\sin(22.5°k)$. Write $s_k=\sin(22.5°k)$; the test says $s_as_cs_e=s_bs_ds_f$. With six arcs summing to 8, either one arc is 3 and the rest are 1, or two arcs are 2 and the rest are 1. In the first case the two sides are $s_3s_1^2$ and s_1^3, which differ since $s_3\neq s_1$. In the second case the two sides are equal exactly when one 2 sits among a, c, e and the other among b, d, f, that is, when the two 2-arcs are 1 or 3 places apart.

The two missing vertices produce the two 2-arcs. They are 3 places apart when the missing vertices are opposite (4 such hexagons): then AD, BE, CF are three of the four diameters, which all meet at the center. They are 1 place apart when the missing vertices are 2 steps apart (8 such hexagons). For example, leaving out vertices 0 and 2 (numbering the vertices 0 to 7 around the octagon) gives the concurrent diagonals 1–5, 3–6 and 4–7. Missing vertices 3 steps apart put the 2-arcs 2 places apart, and adjacent missing vertices give an arc of 3; neither is concurrent.

The special points. The center lies on exactly the 4 diameters, which account for the $\binom{4}{3}=4$ concurrent triples of the first kind. Each of the other 8 triples contains exactly one diameter and meets at a point P_v other than the center, and rotating the octagon by 45° carries each triple to the next. The point on 1–5 from leaving out 0, 2 and the point on the same diameter from leaving out 4, 6 are swapped by the half-turn, so they are distinct (neither is the center). Points on different diameters are distinct too. So there are 8 distinct points where exactly 3 diagonals meet: a fourth diagonal through one of them would create further concurrent triples through it, and there are none.

Total. The center uses $\binom{4}{2}=6$ crossing pairs and contributes $4-1=3$. The 8 triple points use 3 pairs each (24) and contribute 2 each (16). The remaining $70-6-24=40$ crossings are simple and contribute 1 each. Regions: $1+20+3+16+40=80$.

Check. The same method on a regular hexagon: its 9 diagonals have $\binom{6}{4}=15$ crossing pairs, and the only multiple point is the center, where the 3 diameters meet (3 pairs, contribution 2). So there are $1+9+2+12=24$ regions, the well-known count. For the octagon, the number of interior crossing points is $1+8+40=49$, and Euler's formula gives the same answer: $V=8+49=57$ vertices, $E=8+\sum(\text{points on a diagonal}+1)=8+(4+3\cdot 8+2\cdot 40)+20=136$ edges, and $E-V+1=80$ inner faces.

15. Answer: 426**. The idea.** If every corner can reach every other, then certainly every corner has a street leading in and a street leading out. So first count the orientations with no "dead end" (every corner has an in-street and an out-street), which is a careful but manageable count. Then find exactly which of those still fail. A failing orientation has a set of corners that traffic can enter but never leave, and in a cube with no dead ends that set is forced to be a single face, driven round as a directed cycle.

Work it out. *Step 1: no dead ends (*450*).* Split the cube into a top square, a bottom square and four vertical edges. Orient the two squares first. Each corner then already has an in-edge and an out-edge (call it M, mixed), or has both square edges pointing in (I) or both pointing out (O). Of

the 16 orientations of a 4-cycle, 2 are directed cycles (MMMM), 2 alternate (IOIO or OIOI), and the other 12 have exactly one I and one O in any two of the 4 places ($4 \cdot 3 = 12$). Each pattern comes from exactly one orientation, except MMMM, which comes from two. A vertical edge joins top corner x to bottom corner y. An I corner needs it to point out, and an O corner needs it to point in. So it has $f = 2$ good directions if both ends are M, $f = 0$ if both are I or both are O, and $f = 1$ otherwise. The count is $\sum w(t)\, w(b) \prod_{k=1}^{4} f(t_k, b_k)$ over top patterns t and bottom patterns b, with weight $w = 2$ for MMMM and 1 otherwise.

If t = MMMM, the product is $2^{\#M(b)}$. Summing over b gives $2 \cdot 16 + 12 \cdot 4 + 2 \cdot 1 = 82$, and times $w(t) = 2$ that is 164. It is also 164 when b = MMMM, and the pair MMMM/MMMM ($2 \cdot 2 \cdot 16 = 64$) was counted twice: $164 + 164 - 64 = 264$.

If both squares have one I and one O, with the top's I at p and O at q and the bottom's at p' and q', we need $p' \neq p$ and $q' \neq q$, and the product is 2 to the number of places that are M in both. For each of the 12 choices of (p, q): $(p', q') = (q, p)$ gives 4; sharing one place with the roles swapped gives 4 choices at 2 each; and (p', q') disjoint from $\{p, q\}$ gives 2 choices at 1 each. That is $12(4 + 8 + 2) = 168$.

If one square alternates and the other has a single I and O, the latter's I must sit under an O and its O under an I: $2 \cdot 2 = 4$ choices for each alternating pattern, and with both orders $4 \cdot 2 \cdot 2 = 16$. If both alternate, the bottom must be the opposite pattern: 2.

So $264 + 168 + 16 + 2 = 450$ orientations have no dead end.

Step 2: the failures. Suppose an orientation has no dead end but some corner u cannot reach every corner. Let S be the set of corners reachable from u and U the rest. No street leads out of S, so every street between S and U points into S. Every corner of S has an out-street, which must stay in S; following out-streets inside S eventually repeats a corner, so S contains a directed cycle. Every corner of U has an in-street, which must come from U; following in-streets backwards shows U contains a directed cycle too. The cube's corners split into two classes by the parity of the coordinate sum, and every edge joins the two, so every cycle has even length at least 4. Hence $|S| \geq 4$ and $|U| \geq 4$, so both are exactly 4 and each contains a 4-cycle of the cube. The 4-cycles of a cube are exactly its 6 faces, so S is a face and U is the opposite face.

Inside S there are 4 streets and each of the 4 corners needs an out-street among them, so each corner has exactly one: the face is a directed cycle (2 ways). Likewise U is a directed cycle (2 ways). The 4 edges joining the faces all point from U to S. Conversely, each such orientation has no dead end (every corner lies on a directed cycle), yet no corner of S can reach U. Different choices give different orientations: S is the face all four connecting edges point into, and no other face qualifies, because the directed cycles on S and U cross between any other pair of opposite faces in both directions. So there are $6 \cdot 2 \cdot 2 = 24$ failures.

Answer: $450 - 24 = 426$.

Check. A small instance of Step 2: take the top face as a clockwise cycle, the bottom face as a clockwise cycle, and all four vertical streets pointing up. Every corner has an in-street and an out-street, but from the top nobody can get down. That is one of the 24. A computer check of all 4096 orientations gives 450 with no dead end and 426 in which every corner reaches every other.

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