MathCastle

AIME

Practice Examination

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

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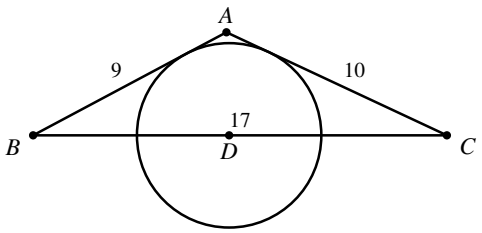
1. Let $r(x)$ be the remainder when $x^{100} + x^{77} + x^{51} + x^8 + 3$ is divided by $x^3 - x$ (so $r(x)$ has degree less than 3). Find $r(10)$.

2. Real numbers a, b, c (none equal to 0) satisfy $\frac{1}{a}+\frac{1}{b} = \frac{1}{3}$, $\frac{1}{b}+\frac{1}{c} = \frac{1}{5}$, and $\frac{1}{c}+\frac{1}{a} = \frac{1}{7}$. Then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

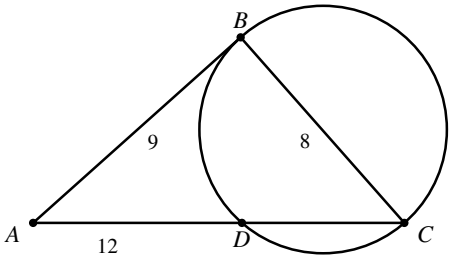
3. Find the number of ordered pairs of integers (a, b) with $|a| \le 10$ and $|b| \le 10$ such that every root of $z^2 + az + b = 0$ (counted with multiplicity) satisfies $|z| = 1$.

4. In triangle ABC, $3\sin A = 4\sin B$ and $\angle C = 60^\circ$. If the area of the triangle is $15\sqrt{3}$, find c^2.

5. Triangle ABC has $AB = 9$, $AC = 10$, $BC = 17$. The bisector of $\angle A$ meets $\overline{BC}$ at D. A circle centered at D is tangent to both line AB and line AC (its two points of tangency land on the segments themselves). Its radius is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.



6. In triangle ABC, $AB = 9$, $BC = 8$, and $CA = 12$. A circle passes through B and C and is tangent to line AB at B. This circle meets segment $\overline{AC}$ again at a point $D \neq C$. Find BD.



7. Find the number of ways to write 2465 as a sum of two positive perfect squares, where two representations that differ only in the order of the two squares are considered the same.

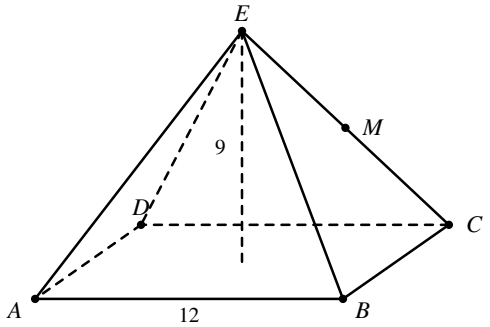
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8. A complex number z satisfies both $|z-3| = |z+1|$ and $|z+1| = 2|z-i|$. The sum of $|z|^2$ over all complex numbers z satisfying both equations is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

9. How many subsets of $\{1, 2, \ldots, 10\}$, including the empty set, have the property that the bitwise XOR of their elements equals 15? (The XOR of the empty set is defined to be 0.)

10. The Stern–Brocot tree is built as follows. Begin with the two boundary fractions $\frac{0}{1}$ and $\frac{1}{0}$ (the latter standing for infinity), and form their mediant $\frac{0+1}{1+0} = \frac{1}{1}$; this is the root of the tree. In general, a fraction in the tree was produced as the mediant of its two bounding fractions, one currently labeled "low" and one "high"; its left child is the mediant of its low bound with itself (which becomes the new high bound), and its right child is the mediant of itself with its high bound (which becomes the new low bound). Every positive rational number in lowest terms occurs exactly once in this tree. Find the number of edges on the path from the root $\frac{1}{1}$ down to $\frac{97}{41}$.

11. A right pyramid has a square base $ABCD$ (vertices in order around the square) with side length 12, and apex E directly above the center of the base at height 9. Let M be the midpoint of edge $\overline{EC}$. The plane through A, B, and M cuts the pyramid in a planar cross-section. The area of this cross-section is $\dfrac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is not divisible by the square of any prime. Find $a+b+c$.



12. A particle starts at position $S_0 = 0$ on the integer number line. Each second, independently, it moves one unit in the positive direction or one unit in the negative direction, each with probability $\frac{1}{2}$; write $S_1, S_2, \ldots, S_{12}$ for its positions after $1, 2, \ldots, 12$ seconds. Find the probability that $S_{12} = 0$ and, in addition, $-3 \le S_k \le 3$ for every $k = 0, 1, \ldots, 12$ (that is, the particle never strays outside the window from -3 to 3 at any time, including the end). The answer is $\dfrac{m}{n}$ in lowest terms; find $m+n$.

13. Find the number of ordered triples of integers (a, b, c) with $0 \le a, b, c \le 6$ such that the congruence $x^3 + ax^2 + bx + c \equiv 0 \pmod 7$ has no solution x with $0 \le x \le 6$.

14. A fair coin is flipped 8 times. Let X be the length of the longest run of consecutive heads appearing among the 8 flips (so $X = 0$ if no heads appear at all). Find the expected value of X, expressed as $\dfrac{m}{n}$ in lowest terms; find $m+n$.

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15. The six vertices $1, 2, \ldots, 6$ determine $\binom{6}{2} = 15$ possible edges (each edge joins an unordered pair of distinct vertices, and at most one edge may join any particular pair). A graph on these six vertices is formed by choosing exactly 6 of these 15 edges. Find the number of such graphs in which every vertex is an endpoint of at least one chosen edge (that is, no vertex is isolated). Then find the remainder when this number is divided by 1000.

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Answer Key — AIME Practice Contest 10

1. 223

2. 281

3. 6

4. 65

5. 91

6. 6

7. 4

8. 67

9. 64

10. 10

11. 88

12. 611

13. 112

14. 899

15. 760

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Solutions — AIME Practice Contest 10

1. Answer: 223**. The idea.** We don't need to do the long division. Since $x^3 - x = x(x-1)(x+1)$ has the three roots 0, 1, −1, the dividend $P(x)$ and the remainder $r(x) = Ax^2 + Bx + C$ agree at each of those three points (their difference is a multiple of $x^3 - x$, which vanishes there). Unlike a dividend built only from odd powers, this one mixes even and odd exponents, so each of the three values has to be computed on its own — there's no shortcut from oddness alone.

Work it out. At $x = 0$: every term except the constant vanishes, so $P(0) = 3$, giving $r(0) = C = 3$.

At $x = 1$: every power of 1 is 1, so $P(1) = 1 + 1 + 1 + 1 + 3 = 7$, giving $r(1) = A + B + C = 7$, i.e. $A + B = 4$.

At $x = -1$: $(-1)^{100} = 1$ (even), $(-1)^{77} = -1$ (odd), $(-1)^{51} = -1$ (odd), $(-1)^8 = 1$ (even), so $P(-1) = 1 - 1 - 1 + 1 + 3 = 3$, giving $r(-1) = A - B + C = 3$, i.e. $A - B = 0$.

From $A+B = 4$ and $A-B = 0$: $A = B = 2$. So $r(x) = 2x^2+2x+3$, and $r(10) = 200+20+3 = 223$.

Check. A direct polynomial division of $x^{100}+x^{77}+x^{51}+x^8+3$ by x^3-x, carried out by computer, also ends with remainder $2x^2 + 2x + 3$. As a smaller check, $r(0) = 3$ matches $P(0) = 3$ exactly (no reduction needed, since $0^3 - 0 = 0$), and $r(1) = 2 + 2 + 3 = 7$ matches $P(1) = 7$.

2. Answer: 281**. The idea.** Each of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ appears in exactly two of the three given sums. Adding all three equations therefore counts each of $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ exactly twice, turning the sum of the three left sides into twice the quantity we want.

Work it out. Let $A = \frac{1}{a}$, $B = \frac{1}{b}$, $C = \frac{1}{c}$, so $A + B = \frac{1}{3}$, $B + C = \frac{1}{5}$, $C + A = \frac{1}{7}$. Adding: $2(A + B + C) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$.

With common denominator 105: $\frac{1}{3} = \frac{35}{105}$, $\frac{1}{5} = \frac{21}{105}$, $\frac{1}{7} = \frac{15}{105}$, so the sum is $\frac{71}{105}$.

Hence $A + B + C = \frac{71}{210}$. Since 71 is prime and does not divide 210, this is already in lowest terms.

So $p + q = 71 + 210 = 281$.

Check. Solving fully: $C = (A + B + C) - (A + B) = \frac{71}{210} - \frac{70}{210} = \frac{1}{210}$, $A = \frac{71}{210} - \frac{42}{210} = \frac{29}{210}$, $B = \frac{71}{210} - \frac{30}{210} = \frac{41}{210}$. Checking: $A + B = \frac{70}{210} = \frac{1}{3}$ ✓, $B + C = \frac{42}{210} = \frac{1}{5}$ ✓, $C + A = \frac{30}{210} = \frac{1}{7}$ ✓. These give genuine real (nonzero) values $a = \frac{210}{29}$, $b = \frac{210}{41}$, $c = 210$, and an independent computer solve of the original 3×3 linear system returns exactly this triple.

3. Answer: 6**. The idea.** Split into cases by whether the roots are real or not. If they are not real, the coefficients being real forces them to be complex conjugates of each other, and a conjugate pair automatically has $|z_1| = |z_2|$. If they are real, each one individually must equal 1 or −1, since those are the only real numbers of absolute value 1.

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Work it out. *Non-real roots.* Write the roots as $u \pm iv$ with $v \neq 0$. Then $b = (u+iv)(u-iv) = u^2+v^2 = |z|^2$, so $|z| = 1$ means exactly $b = 1$. The roots are non-real exactly when the discriminant $a^2 - 4b < 0$, i.e. $a^2 < 4$, i.e. $a \in \{-1, 0, 1\}$. That gives three pairs: $(-1, 1)$, $(0, 1)$, $(1, 1)$.

Real roots. Both roots lie in $\{1, -1\}$. The three multisets are $\{1, 1\}$, $\{1, -1\}$, $\{-1, -1\}$, giving (by Vieta, $a = -(\text{sum})$, $b = \text{product}$) the pairs $(-2, 1)$, $(0, -1)$, $(2, 1)$.

These six pairs are all different, so the total is $3 + 3 = 6$.

Check. A computer check of all 441 pairs with $|a|, |b| \leq 10$, factoring $z^2 + az + b$ and testing every root's modulus exactly (not numerically), finds precisely these six: $(-2, 1)$ gives the double root -1; $(-1, 1)$ gives $\frac{1 \pm i\sqrt{3}}{2}$, each of modulus 1; and so on. No other integer pair in the box works, since $a^2 - 4b \geq 4$ already pushes one real root's magnitude above 1 once $|a| \geq 3$, and smaller $|a|$ with $b \neq \pm 1$ fails the modulus condition directly.

4. Answer: 65. **The idea.** By the law of sines, $\sin A$ and $\sin B$ are proportional to the opposite sides a and b. So $3 \sin A = 4 \sin B$ pins down the ratio $a : b$, and from there one unknown scale factor is enough to use both the area and the law of cosines.

Work it out. From $3 \sin A = 4 \sin B$ we get $\dfrac{a}{b} = \dfrac{\sin A}{\sin B} = \dfrac{4}{3}$, so write $a = 4k$, $b = 3k$ for some $k > 0$.

The area is $\dfrac{1}{2}ab \sin C = \dfrac{1}{2}(4k)(3k) \sin 60^\circ = 6k^2 \cdot \dfrac{\sqrt{3}}{2} = 3\sqrt{3}\, k^2$. Setting this equal to $15\sqrt{3}$ gives $k^2 = 5$.

By the law of cosines, $c^2 = a^2 + b^2 - 2ab \cos C = 16k^2 + 9k^2 - 2(12k^2) \cdot \dfrac{1}{2} = 25k^2 - 12k^2 = 13k^2 = 13 \cdot 5 = 65$.

Check. With $k = \sqrt{5}$: $a = 4\sqrt{5} \approx 8.944$, $b = 3\sqrt{5} \approx 6.708$. The area $\frac{1}{2}ab \sin 60^\circ \approx \frac{1}{2}(8.944)(6.708)(0.8660) \approx 25.98 \approx 15\sqrt{3} \approx 25.98$ ✓. And $c = \sqrt{65} \approx 8.062$, which with a, b satisfies the triangle inequality comfortably, and an independent build of the triangle from $k^2 = 5$ via the law of cosines (solved fresh from the area equation, not assumed) returns the same $c^2 = 65$.

5. Answer: 91. **The idea.** Any point on the bisector of $\angle A$ is, by definition, equally far from lines AB and AC — so a circle centered at D is automatically tangent to both lines at once, with radius equal to that common distance. The angle bisector theorem locates D on BC, and then the distance from D to line AB is just twice the area of triangle ABD divided by AB.

Work it out. First, $[ABC]$: by Heron's formula with sides $9, 10, 17$ and $s = 18$, $[ABC] = \sqrt{18 \cdot 9 \cdot 8 \cdot 1} = \sqrt{1296} = 36$.

By the angle bisector theorem, $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{9}{10}$, so $BD = 17 \cdot \dfrac{9}{19} = \dfrac{153}{19}$.

Triangle ABD shares its height from A with triangle ABC, so $[ABD] = [ABC] \cdot \dfrac{BD}{BC} = 36 \cdot \dfrac{9}{19} = \dfrac{324}{19}$.

The distance r from D to line AB satisfies $[ABD] = \dfrac{1}{2} \cdot AB \cdot r$, so $r = \dfrac{2[ABD]}{AB} = \dfrac{2 \cdot 324/19}{9} = \dfrac{648/19}{9} = \dfrac{72}{19}$.

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Since $\gcd(72, 19) = 1$, $p + q = 72 + 19 = 91$.

Check. Placing $B = (0, 0)$, $C = (17, 0)$ and solving for A from $AB = 9$, $AC = 10$ gives $A = \left(\frac{135}{17}, \frac{72}{17}\right)$ exactly (no radicals needed, since $9, 10, 17$ form a Heronian triangle). Computing the perpendicular distance from $D = \left(\frac{153}{19}, 0\right)$ to line AB directly, by projecting onto the normal of $\overrightarrow{BA}$, again gives exactly $\frac{72}{19}$. A third route: in right triangle (foot of perpendicular from D), the distance equals $BD \sin B$, and $\cos B = \frac{BC^2+AB^2-AC^2}{2 \cdot BC \cdot AB} = \frac{289+81-100}{306} = \frac{15}{17}$, so $\sin B = \frac{8}{17}$, giving $\frac{153}{19} \cdot \frac{8}{17} = \frac{72}{19}$ once more.

6. Answer: 6. **The idea.** Since line AB touches the circle only at B, it is tangent there, so A's power with respect to the circle equals AB^2. But A also sees the circle along the secant through D and C, so that same power equals $AD \cdot AC$. That pins down AD (and hence DC) without any coordinates. From there, BD is just a cevian of triangle ABC cutting side AC at D, so Stewart's Theorem finishes the job.

Work it out. Power of the point A: $AB^2 = AD \cdot AC$, so $81 = AD \cdot 12$, giving $AD = \dfrac{27}{4}$. Then $DC = AC - AD = 12 - \dfrac{27}{4} = \dfrac{21}{4}$.

Now apply Stewart's Theorem to cevian BD of triangle ABC, where D splits AC into $AD = \dfrac{27}{4}$ (next to A) and $DC = \dfrac{21}{4}$ (next to C): $AB^2 \cdot DC + BC^2 \cdot AD - BD^2 \cdot AC = AC \cdot AD \cdot DC$.

Substituting: $81 \cdot \dfrac{21}{4} + 64 \cdot \dfrac{27}{4} - BD^2 \cdot 12 = 12 \cdot \dfrac{27}{4} \cdot \dfrac{21}{4}$, i.e. $\dfrac{1701}{4} + \dfrac{1728}{4} - 12\,BD^2 = \dfrac{1701}{4}$.

So $12\,BD^2 = \dfrac{1728}{4} = 432$, giving $BD^2 = 36$ and $BD = 6$.

Check. Build the triangle by coordinates: $A = (0, 0)$, $C = (12, 0)$, and B from $AB = 9$, $BC = 8$ gives $B = \left(\dfrac{161}{24}, \dfrac{\sqrt{20735}}{24}\right)$. The circle through B, C tangent to line AB at B has its center where the perpendicular to AB at B meets the perpendicular bisector of BC; solving those two linear conditions and intersecting the resulting circle with the x-axis (line AC) gives the two points $x = 12$ (that's C) and $x = \dfrac{27}{4}$ (that's D) — matching $AD = \dfrac{27}{4}$ exactly. The distance from B to this D comes out to exactly 6, and as a side check, A's power with respect to this circle (computed from the actual center and radius) equals exactly $81 = AB^2$, confirming the tangency.

7. Answer: 4. **The idea.** Factor $2465 = 5 \cdot 17 \cdot 29$. Each of these three primes is 1 more than a multiple of 4 (that is, $\equiv 1 \pmod 4$), and a classical fact (provable with Gaussian integers, or just Brahmagupta–Fibonacci) says that a product of primes each $\equiv 1 \pmod 4$, all to the first power, can be written as a sum of two squares in 2^{k-1} essentially different ways, where k is the number of distinct such prime factors. Here $k = 3$, so we expect $2^2 = 4$ ways — and we can find them explicitly using each prime's own representation.

Work it out. Each factor is itself a sum of two squares: $5 = 1^2 + 2^2$, $17 = 1^2 + 4^2$, $29 = 2^2 + 5^2$. Writing each as a Gaussian integer norm, $5 = |1{+}2i|^2$, $17 = |1{+}4i|^2$, $29 = |2{+}5i|^2$, and multiplying any two of the three Gaussian integers (and then by the third, or its conjugate) produces a Gaussian integer whose norm is the product 2465; its real and imaginary parts (in absolute value) are a representation of 2465 as a sum of two squares.

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Carrying this out for the four sign choices (equivalently, conjugating an even number of the three factors) gives the sums $2465 = 8^2 + 49^2 = 16^2 + 47^2 = 23^2 + 44^2 = 28^2 + 41^2$.

Each of these four pairs is genuinely different (no pair is a reordering of another), so the answer is 4.

Check. Directly searching every a from 1 to $\lfloor\sqrt{2465/2}\rfloor = 35$ and testing whether $2465 - a^2$ is a perfect square finds exactly the four pairs above and no others: $8^2 + 49^2 = 64 + 2401 = 2465$ ✓, $16^2+47^2 = 256+2209 = 2465$ ✓, $23^2+44^2 = 529+1936 = 2465$ ✓, $28^2+41^2 = 784+1681 = 2465$ ✓.

8. Answer: 67**. The idea.** The equation $|z - 3| = |z + 1|$ says z is equidistant from 3 and -1, so it places z on the perpendicular bisector of the segment joining them — the vertical line $\mathrm{Re}(z) = 1$. That collapses the problem to one real unknown, which the second equation (an Apollonius-circle condition) then pins down.

Work it out. Write $z = 1 + yi$ for a real number y. Then $|z + 1| = |2 + yi| = \sqrt{4 + y^2}$ and $|z - i| = |1 + (y - 1)i| = \sqrt{1 + (y - 1)^2}$.

The condition $|z+1| = 2|z-i|$ becomes $4+y^2 = 4\big(1+(y-1)^2\big) = 4+4(y-1)^2$, so $y^2 = 4(y-1)^2$.

Expanding: $y^2 = 4y^2 - 8y + 4$, i.e. $3y^2 - 8y + 4 = 0$. The quadratic formula gives $y = \dfrac{8 \pm \sqrt{64 - 48}}{6} = \dfrac{8 \pm 4}{6}$, so $y = 2$ or $y = \dfrac{2}{3}$.

For each, $|z|^2 = 1+y^2$: at $y = 2$ this is 5, and at $y = \dfrac{2}{3}$ this is $1+\dfrac{4}{9} = \dfrac{13}{9}$. The sum is $5+\dfrac{13}{9} = \dfrac{58}{9}$, already in lowest terms since $58 = 2 \cdot 29$ shares no factor with 9.

So $m + n = 58 + 9 = 67$.

Check. Solving the original system of two circle/line equations directly for $(x, y) = (\mathrm{Re}\, z, \mathrm{Im}\, z)$ with a computer algebra system, with no assumption about where z lies in advance, returns exactly the two points $(1, 2)$ and $\left(1, \frac{2}{3}\right)$ and the same sum $\frac{58}{9}$. As a sanity check, at $z = 1+2i$: $z+1 = 2+2i$ has $|z+1| = \sqrt{8} = 2\sqrt{2}$, and $z - i = 1 + i$ has $2|z - i| = 2\sqrt{2}$ — these agree, confirming the second equation; also $z - 3 = -2 + 2i$ has $|z - 3| = \sqrt{8} = 2\sqrt{2} = |z + 1|$, confirming the first.

9. Answer: 64**. The idea.** Every integer from 1 to 10 is less than 16, so each can be viewed as a vector of four bits, i.e. an element of the vector space $(\mathbb{F}_2)^4$ over the field with two elements, where addition is exactly bitwise XOR. The map that sends a subset to the XOR of its elements is then linear (over $\mathbb{F}_2$) from a 10-dimensional space (one coordinate per element, in or out) to this 4-dimensional target space. A linear map's fibers (the sets of inputs landing on a given output) all have the same size whenever the map is surjective, so the count we want does not actually depend on the target being 15 rather than some other value.

Work it out. The numbers $1, 2, 4, 8$ already appear in $\{1, \ldots, 10\}$, and their bit vectors are the four standard basis vectors of $(\mathbb{F}_2)^4$. So the image of the linear map already contains a basis of the target space, which means the map is surjective (its image is all of $(\mathbb{F}_2)^4$, all 16 possible XOR values).

A surjective linear map from a 10-dimensional space to a 4-dimensional space has kernel of dimension $10 - 4 = 6$, so the kernel has $2^6 = 64$ elements. Every fiber of a linear map is a translate (coset) of the kernel, hence has the same size as the kernel: 64.

In particular the fiber over the target value 15 — the subsets we want — has exactly 64 elements.

Check. A direct computer search over all $2^{10} = 1024$ subsets, computing the XOR of each one explicitly, finds exactly 64 subsets whose XOR is 15 (for example $\{15\}$ is not available since $15 \notin \{1, \ldots, 10\}$, but $\{1, 14\}$ fails since $14 \notin$ the set either; a genuine example is $\{7, 8\}$, since $7 \oplus 8 = 15$, and so is $\{1, 2, 4, 8\}$, since $1 \oplus 2 \oplus 4 \oplus 8 = 15$). The same search also confirms every one of the other 15 possible XOR targets is hit by exactly 64 subsets, matching the surjectivity argument.

10. Answer: 10**. The idea.** At each node, the current fraction is compared to the target: if the target is smaller, we descend left (and the current fraction becomes the new upper bound); if larger, we descend right (and it becomes the new lower bound). Tracking how many times in a row we go the same direction links this descent to the Euclidean algorithm: a run of k consecutive right-moves (or left-moves) corresponds exactly to one step of the Euclidean algorithm with quotient k. So the total number of edges equals the sum of the partial quotients of the continued fraction of the target, except the very last quotient contributes one less step than its value (since the descent stops as soon as it reaches the target, without needing the final confirming step).

Work it out. Compute the continued fraction of $\frac{97}{41}$ via the Euclidean algorithm: $97 = 2 \cdot 41 + 15$, $41 = 2 \cdot 15 + 11$, $15 = 1 \cdot 11 + 4$, $11 = 2 \cdot 4 + 3$, $4 = 1 \cdot 3 + 1$, $3 = 3 \cdot 1 + 0$. So $\frac{97}{41} = [2, 2, 1, 2, 1, 3]$, with quotients summing to $2 + 2 + 1 + 2 + 1 + 3 = 11$.

The number of edges is this sum minus 1 (the final quotient's last unit of descent lands exactly on the target and is not a separate "exploratory" step): $11 - 1 = 10$.

Check. Carrying out the descent directly and mechanically — starting from bounds $\frac{0}{1}, \frac{1}{0}$, repeatedly forming the mediant and comparing it to $\frac{97}{41}$ exactly as fractions, without ever invoking the continued fraction — a computer finds the target after exactly 10 mediant steps, confirming the count. (As a small-scale sanity check, the fraction $\frac{3}{2} = [1, 2]$, with quotient sum 3, is reached in $3 - 1 = 2$ edges: root $\frac{1}{1} \to \frac{2}{1} \to \frac{3}{2}$, which is easy to verify by hand.)

11. Answer: 88**. The idea.** Reflecting the whole pyramid through the vertical plane that perpendicularly bisects AB and CD swaps $A \leftrightarrow B$ and $C \leftrightarrow D$ while fixing E; this is a symmetry of the pyramid. Since the cutting plane contains both A and B, this reflection carries the cutting plane to itself, so it must carry M (the midpoint of EC) to a point also on the cutting plane – namely N, the midpoint of ED. So the cross-section is the quadrilateral $ABMN$, and since MN is a midsegment of triangle ECD, it is parallel to CD, hence parallel to AB as well: $ABMN$ is a trapezoid. Its legs BM and AN are medians of the isosceles triangles BEC and AED, which the median-length formula handles without coordinates.

Work it out. First, $MN = \frac{1}{2}CD = 6$ (midsegment of $\triangle ECD$), while $AB = 12$ is a side of the square.

Next, find the common lateral edge length. The center O of the square base is at distance $\frac{12\sqrt{2}}{2} = 6\sqrt{2}$ from each corner, so each lateral edge, such as BE or CE, has length $\sqrt{OB^2 + OE^2} = \sqrt{72 + 81} = \sqrt{153}$.

Now apply the median length formula in triangle BEC (sides $BE = EC = \sqrt{153}$, $BC = 12$), where M is the midpoint of EC: $BM^2 = \dfrac{2BE^2 + 2BC^2 - EC^2}{4} = \dfrac{2(153) + 2(144) - 153}{4} = \dfrac{441}{4}$, so $BM = \dfrac{21}{2}$. By the same symmetry as before, $AN = \dfrac{21}{2}$ as well, so trapezoid $ABMN$ is isosceles.

For an isosceles trapezoid with parallel sides 12 and 6 and legs $\frac{21}{2}$, the height is found by dropping

perpendiculars from M and N to AB, which land symmetrically, each $\frac{12-6}{2} = 3$ in from an end: height $= \sqrt{\left(\frac{21}{2}\right)^2 - 3^2} = \sqrt{\frac{441}{4} - 9} = \sqrt{\frac{405}{4}} = \frac{9\sqrt{5}}{2}$.

The area is $\frac{1}{2}(AB + MN) \cdot \text{height} = \frac{1}{2}(18) \cdot \frac{9\sqrt{5}}{2} = \frac{81\sqrt{5}}{2}$. Here $a = 81$, $b = 5$, $c = 2$ (and $\gcd(81, 2) = 1$, 5 squarefree), so $a + b + c = 81 + 5 + 2 = 88$.

Check. Placing the pyramid in coordinates (A, B, C, D at $(\pm 6, \pm 6, 0)$ and $E = (0, 0, 9)$), finding where the plane through A, B, M actually meets edge ED by solving directly for the intersection (without assuming in advance that it is the midpoint N), and computing the resulting quadrilateral's area with the 3D cross-product (shoelace) formula also gives exactly $\frac{81\sqrt{5}}{2}$, and the intersection point found this way does turn out to be the midpoint of ED, confirming the symmetry argument. As a further check, $BM = \sqrt{3^2 + \left(\frac{9\sqrt{5}}{2}\right)^2} = \sqrt{9 + \frac{405}{4}} = \sqrt{\frac{441}{4}} = \frac{21}{2}$ matches the median-formula value exactly.

12. Answer: 611**. The idea.** Ignoring the window entirely, the number of 12-step ± 1 paths from 0 back to 0 is a binomial coefficient. To enforce the window, use the reflection principle twice: reflecting a path at the first moment it would cross +4 turns it into an unconstrained path ending somewhere else entirely, and likewise for crossing -4. The key extra observation, needed because the window has two sides, is that a path cannot touch *both* forbidden levels +4 and -4 and still return to 0 in only 12 steps — that would need too many steps — so the two reflected corrections never overlap and can simply be subtracted separately.

Work it out. The total number of 12-step sign sequences returning to 0 (six +1's and six -1's, in any order) is $\binom{12}{6} = 924$.

Paths touching +4: reflect the portion of the path from 0 up to its first visit to 4 across the line $y = 4$. This turns any path from 0 to 0 that touches 4 into a path from 8 (the reflection of 0 across 4) to 0, with no further restriction, and this reflection is a bijection. The number of unrestricted 12-step paths from 8 to 0 needs net displacement -8 in 12 steps, i.e. 2 steps of +1 and 10 steps of -1: $\binom{12}{2} = 66$ such paths (equivalently $\binom{12}{10}$).

Paths touching -4: by the mirror-image argument, there are likewise $\binom{12}{2} = 66$ such paths.

Overlap: a path touching both +4 and -4 and ending at 0 would need at least 4 steps to first reach 4, at least 8 more to then reach -4, and at least 4 more to return to 0 from -4 — at least 16 steps in total (and symmetrically if it visits -4 before +4). Since we only have 12 steps, no path can touch both forbidden levels and still end at 0, so the two corrections above never double-count the same path.

So the number of good paths is $924 - 66 - 66 = 792$, out of $2^{12} = 4096$ total paths. The probability is $\frac{792}{4096}$. Since $\gcd(792, 4096) = 8$, this reduces to $\frac{99}{512}$ (and $99 = 9 \cdot 11$ shares no factor of 2 with $512 = 2^9$). So $m + n = 99 + 512 = 611$.

Check. A full computer enumeration of all $2^{12} = 4096$ sign sequences, tracking the running position after every single step and discarding any sequence that ever leaves $[-3, 3]$, finds exactly 792 surviving sequences that also end at 0 — matching the reflection-principle count exactly. As a smaller sanity check, the analogous count for an 8-step walk confined to $[-2, 2]$ (forbidden levels ± 3) is $\binom{8}{4} - 2\binom{8}{1} = 70 - 16 = 54$, which a direct enumeration of all 256 length-8 sequences also confirms.

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13. Answer: 112**. The idea.** It's easier to count the complementary set — triples for which the cubic *does* have a root mod 7 — using inclusion–exclusion over which specific residues $r \in \{0, 1, \ldots, 6\}$ are roots, since a cubic can only ever have at most 3 distinct roots modulo a prime (a nonzero cubic mod 7 factors into at most 3 linear factors).

Work it out. For a single residue r, requiring $r^3 + ar^2 + br + c \equiv 0$ is one linear condition on c (given any a, b): for each of the $7 \times 7 = 49$ choices of (a, b), exactly one value of c makes r a root. So there are 49 triples with r as a root, and summing over the 7 choices of r gives $7 \times 49 = 343$ (with multiplicity, i.e. counting a triple once for each root it has).

For a pair of distinct residues $r \neq s$, requiring both to be roots is two independent linear conditions on (a, b, c) (independent because the corresponding 2×2 system in two of the unknowns has a nonzero determinant involving $r - s \not\equiv 0$), leaving one free parameter among the three unknowns, hence 7 triples per pair. There are $\binom{7}{2} = 21$ pairs, contributing $21 \times 7 = 147$.

For a triple of distinct residues r, s, t, requiring all three to be roots is three independent linear conditions on the three unknowns (a, b, c), with a unique solution given directly by Vieta's formulas: $a \equiv -(r + s + t)$, $b \equiv rs + rt + st$, $c \equiv -rst \pmod 7$. There are $\binom{7}{3} = 35$ such triples of residues, each contributing exactly 1.

No four distinct residues can simultaneously be roots (a cubic has at most 3), so inclusion–exclusion stops here: the number of triples (a, b, c) with *some* root is $343 - 147 + 35 = 231$.

Out of all $7^3 = 343$ triples, the number with *no* root is $343 - 231 = 112$.

Check. A direct computer search testing every one of the 343 triples (a, b, c) against every one of the 7 residues $x = 0, \ldots, 6$ finds exactly 112 triples where $x^3 + ax^2 + bx + c \not\equiv 0 \pmod 7$ for all x, confirming the inclusion–exclusion count exactly (and correspondingly finds 231 triples with at least one root).

14. Answer: 899**. The idea.** For a random variable X taking nonnegative integer values, there is a useful identity: $E[X] = \sum_{k=1}^{\infty} P(X \geq k)$ (each unit of X's value is "paid for" by one term of the sum). Here $X \leq 8$, so the sum has 8 terms, and $P(X \geq k)$ — the probability some run of at least k consecutive heads occurs among 8 flips — can be computed one k at a time.

Work it out. Let $f_k(n)$ be the number of length-n head/tail sequences with *no* run of k or more consecutive heads; this satisfies the recursion $f_k(n) = \sum_{j=0}^{k-1} f_k(n-1-j)$ (condition on the length of the final run of heads, which can be $0, 1, \ldots, k-1$ and must be preceded by a tail or the start), with $f_k(n) = 2^n$ for $n < k$. As a worked example, for $k = 2$ the recursion is the Fibonacci-type $f_2(n) = f_2(n-1) + f_2(n-2)$ with $f_2(0) = 1$, $f_2(1) = 2$: step by step, $f_2(2) = 3$, $f_2(3) = 5$, $f_2(4) = 8$, $f_2(5) = 13$, $f_2(6) = 21$, $f_2(7) = 34$, $f_2(8) = 55$, so $P(X \geq 2) = 1 - \dfrac{55}{256} = \dfrac{201}{256}$. For $k = 3$ the recursion $f_3(n) = f_3(n-1) + f_3(n-2) + f_3(n-3)$ with $f_3(0) = 1$, $f_3(1) = 2$, $f_3(2) = 4$ gives $f_3(3) = 7$, $f_3(4) = 13$, $f_3(5) = 24$, $f_3(6) = 44$, $f_3(7) = 81$, $f_3(8) = 149$, so $P(X \geq 3) = 1 - \dfrac{149}{256} = \dfrac{107}{256}$. The remaining values of k are handled the same way (a run of $k \geq 5$ or so is rare enough that $f_k(8)$ can also just be found by directly subtracting off the few long-run sequences by hand), and computing $P(X \geq k) = 1 - f_k(8)/256$ for $k = 1, \ldots, 8$ this way gives:

$$P(X \geq 1) = \frac{255}{256},\ P(X \geq 2) = \frac{201}{256},\ P(X \geq 3) = \frac{107}{256},\ P(X \geq 4) = \frac{48}{256},\ P(X \geq 5) = \frac{20}{256},$$
$$P(X \geq 6) = \frac{8}{256},\ P(X \geq 7) = \frac{3}{256},\ P(X \geq 8) = \frac{1}{256}.$$

Summing the numerators: $255 + 201 + 107 + 48 + 20 + 8 + 3 + 1 = 643$. So $E[X] = \dfrac{643}{256}$.

Since 643 is odd (in fact prime) and $256 = 2^8$, the fraction is already in lowest terms, so $m + n = 643 + 256 = 899$.

Check. A full computer enumeration of all $2^8 = 256$ head/tail sequences, directly scanning each one for its longest run of consecutive heads and averaging the results, gives exactly $\frac{643}{256}$ as well. As a smaller check, the analogous computation for 3 flips gives $E[X] = \frac{1}{8}(0+1+1+2+1+1+2+3) = \frac{11}{8}$, matching a direct hand count of all 8 sequences (TTT, HTT, THT, HHT, TTH, HTH, THH, HHH).

15. Answer: 760**. The idea.** Count the complementary bad property — having at least one isolated vertex — by inclusion–exclusion over which specific set of vertices is forced to be isolated. If a set S of k vertices must all be isolated, every one of the 6 chosen edges has to avoid S entirely, so all 6 edges must come from the $\binom{6-k}{2}$ edges among the remaining $6 - k$ vertices; this is only possible when $6 \le \binom{6-k}{2}$. Since $\binom{3}{2} = 3 < 6$, it is actually impossible for 3 or more vertices to be simultaneously isolated (there just aren't enough edges left to use up), so the inclusion–exclusion sum collapses to only three nonzero terms.

Work it out. For $k = 0$ (no constraint at all), the number of ways to choose 6 of the 15 edges is $\binom{15}{6} = 5005$.

For $k = 1$: forcing one particular vertex to be isolated leaves $\binom{5}{2} = 10$ usable edges among the other five vertices, so there are $\binom{10}{6} = 210$ graphs avoiding that vertex; there are $\binom{6}{1} = 6$ choices of which vertex to isolate, contributing $6 \times 210 = 1260$ to the inclusion–exclusion sum (with a minus sign).

For $k = 2$: forcing two particular vertices to be isolated leaves only $\binom{4}{2} = 6$ usable edges among the other four vertices, so there is exactly $\binom{6}{6} = 1$ way to use all 6 of them (and no way to use fewer, since we need exactly 6 edges); there are $\binom{6}{2} = 15$ choices of the pair, contributing $15 \times 1 = 15$ (with a plus sign).

For $k \ge 3$, as noted, $\binom{6-k}{2} \le 3 < 6$, so no graph is possible and every such term is 0.

By inclusion–exclusion, the number of graphs with no isolated vertex is $\binom{15}{6} - \binom{6}{1}\binom{10}{6} + \binom{6}{2}\binom{6}{6} = 5005 - 1260 + 15 = 3760$. The remainder when 3760 is divided by 1000 is 760.

Check. A full computer enumeration of all $\binom{15}{6} = 5005$ ways to choose 6 of the 15 edges on 6 labeled vertices, directly computing the degree of every vertex of each resulting graph and discarding any graph with a vertex of degree 0, finds exactly 3760 graphs with no isolated vertex — matching the inclusion–exclusion count exactly, and $3760 \bmod 1000 = 760$. As a smaller independent check, the analogous count for 4 labeled vertices (so $\binom{4}{2} = 6$ possible edges) choosing exactly 3 of them with no isolated vertex is $\binom{6}{3} - \binom{4}{1}\binom{3}{3} = 20 - 4 = 16$ by the same method, which a direct check of all 20 three-edge subsets of the complete graph on 4 vertices also confirms.

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