## MathCastle

# AIME

Practice Examination

15 questions • 3 hours

**Name** _____

**School** _____

**Grade** _____

**INSTRUCTIONS**

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

**DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.**

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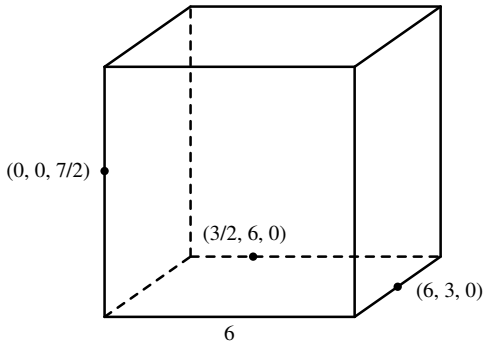
**1.** For how many 4-element subsets of $\{1, 2, 3, \ldots, 20\}$ is the mean of the four elements equal to their median? (The median of four numbers is the average of the two middle numbers.)

**2.** Integers $b$ and $c$ are chosen with $1 \le b \le 20$ and $1 \le c \le 20$. For how many ordered pairs $(b, c)$ do both quadratics $x^2 + bx + c$ and $x^2 + cx + b$ have real roots (a repeated root counts as real)?

**3.** Twelve seats around a round table are numbered 1 to 12 in order, so seat 12 is next to seat 1. Each seat is given a red or a blue cushion. A *block* is a maximal set of consecutive seats (going around the table, possibly passing from seat 12 to seat 1) whose cushions all have the same colour; if all twelve cushions have the same colour, the whole table is a single block of 12 seats. How many of the $2^{12}$ colourings have every block made of an odd number of seats?

**4.** How many ordered pairs $(x, y)$ of integers with $|x| \le 1000$ satisfy $x^2 + y^2 = x^3$?

**5.** A solid cube of edge length 6 occupies the region $0 \le x \le 6$, $0 \le y \le 6$, $0 \le z \le 6$. A plane passes through the three points $(6, 3, 0)$, $\left(\frac{3}{2}, 6, 0\right)$ and $\left(0, 0, \frac{7}{2}\right)$. The intersection of the plane with the solid cube is a polygon whose area is $\dfrac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.




**6.** How many ordered pairs $(A, B)$ are there, where $A$ and $B$ are disjoint subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ (either or both may be empty), such that no element of $A$ differs by exactly 1 from an element of $B$?

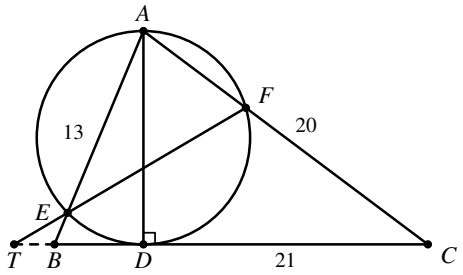
**7.** The number
$\left(1 + 4\cos^2 \dfrac{\pi}{11}\right)\left(1 + 4\cos^2 \dfrac{2\pi}{11}\right)\left(1 + 4\cos^2 \dfrac{3\pi}{11}\right)\left(1 + 4\cos^2 \dfrac{4\pi}{11}\right)\left(1 + 4\cos^2 \dfrac{5\pi}{11}\right)$
is a positive integer. Find it.

**8.** Multiplying a number by 30 can only increase its number of positive divisors. For how many positive integers $n \le 1000$ does $30n$ have exactly four times as many positive divisors as $n$ has?

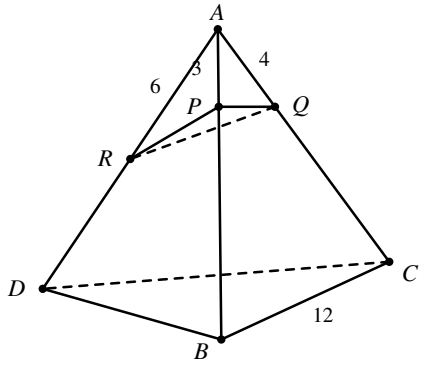
**9.** A *lattice point* is a point whose coordinates are both integers. For how many ordered pairs $(x, y)$ of integers with $1 \le x \le 100$ and $1 \le y \le 100$ does the segment from $(0, 0)$ to $(x, y)$ pass through exactly three lattice points other than its two endpoints?

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**10.** Triangle $ABC$ has $AB = 13$, $BC = 21$ and $CA = 20$. Let $D$ be the foot of the altitude from $A$ to $\overline{BC}$. The circle with diameter $\overline{AD}$ meets side $\overline{AB}$ again at $E$ and side $\overline{AC}$ again at $F$. Lines $EF$ and $BC$ meet at a point $T$. The length $TD$ can be written as $\frac{m}{n}$ in lowest terms. Find $m + n$.




**11.** Regular tetrahedron $ABCD$ has every edge of length 12. Points $P$, $Q$ and $R$ lie on edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ with $AP = 3$, $AQ = 4$ and $AR = 6$. The plane through $P$, $Q$, $R$ cuts off a small tetrahedron $APQR$. The square of the distance from $A$ to the plane $PQR$ can be written as $\frac{m}{n}$ in lowest terms. Find $m + n$.




**12.** The six complex numbers $z$ that satisfy $(z-1)^6 = 64z^6$ are the vertices of a convex hexagon in the complex plane. The area of this hexagon can be written as $\frac{a\sqrt{b}}{c}$, where $a$, $b$, $c$ are positive integers, $a$ and $c$ are relatively prime, and $b$ is not divisible by the square of any prime. Find $a + b + c$.

**13.** How many ordered pairs $(a, b)$ of integers with $|a| \le 20$ and $|b| \le 20$ have the property that the polynomial $x^4 + ax + b$ can be written as the product of two polynomials, each of degree at least 1, with integer coefficients?

**14.** For a permutation $\sigma = (a_1, a_2, \ldots, a_8)$ of $1, 2, \ldots, 8$, a *descent* is an index $i$ with $1 \le i \le 7$ and $a_i > a_{i+1}$; let $d(\sigma)$ be the number of descents of $\sigma$. For example, $d(3, 1, 2, 8, 7, 4, 5, 6) = 3$. Let $N$ be the sum of $2^{d(\sigma)}$ over all $8! = 40320$ permutations $\sigma$. Find the remainder when $N$ is divided by 1000.

**15.** Let $N$ be the number of ordered 10-tuples $(x_1, x_2, \ldots, x_{10})$ with every $x_i \in \{1, 2, 3\}$ such that $x_1x_2 + x_2x_3 + x_3x_4 + \cdots + x_9x_{10} + x_{10}x_1$ is divisible by 3. Find the remainder when $N$ is divided by 1000.

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# Answer Key — AIME Practice Contest 1

**1.** 525

**2.** 183

**3.** 640

**4.** 64

**5.** 281

**6.** 577

**7.** 89

**8.** 68

**9.** 399

**10.** 91

**11.** 323

**12.** 34

**13.** 170

**14.** 835

**15.** 845

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# Solutions — AIME Practice Contest 1

**1. Answer:** 525. **The idea.** Write the subset as $a < b < c < d$. The mean is $\frac{a+b+c+d}{4}$ and the median is $\frac{b+c}{2}$, so the condition is $a + d = b + c$: the outer pair and the inner pair have the same sum. Now turn it around: any two different pairs with the same sum are automatically nested like this. So we only need to count, for each possible sum, how many pairs have that sum.

**Work it out.** Take two different pairs $\{x, y\}$ and $\{u, v\}$ from $\{1, \ldots, 20\}$ with $x + y = u + v = s$. They cannot share an element (the other elements would then be equal too), so together they form a 4-element set. Its smallest element and its largest element lie in the same pair (the partner of the smallest is $s$ minus the smallest, which is the largest), so the set is $a < b < c < d$ with $a + d = b + c$. Conversely a set with $a + d = b + c$ comes from exactly one such split, since the other two splits have $a + b < c + d$ and $a + c < b + d$. So the answer is

$\sum_s \binom{p_s}{2}$, where $p_s$ is the number of pairs in $\{1, \ldots, 20\}$ with sum $s$.

For $s = 3, 4, \ldots, 20$ the pairs are $\{1, s-1\}, \{2, s-2\}, \ldots$, so $p_s = \lfloor \frac{s-1}{2} \rfloor$: the values $1, 1, 2, 2, \ldots, 9, 9$. For $s = 21$, $p_{21} = 10$. Replacing each number $x$ by $21-x$ turns sum $s$ into $42-s$, so $s = 22, \ldots, 39$ repeat the values $9, 9, \ldots, 1, 1$.

Hence the total is $4\left[\binom{2}{2} + \binom{3}{2} + \cdots + \binom{9}{2}\right] + \binom{10}{2} = 4\binom{10}{3} + 45 = 480 + 45 = 525$, using the hockey-stick identity $\binom{2}{2} + \cdots + \binom{9}{2} = \binom{10}{3} = 120$.

**Check.** Try $\{1, \ldots, 6\}$. By the formula, the sums $3, \ldots, 11$ have $p_s = 1, 1, 2, 2, 3, 2, 2, 1, 1$, giving $1 + 1 + 3 + 1 + 1 = 7$. Directly, the sets with $a + d = b + c$ are $\{1, 2, 3, 4\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 2, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 5, 6\}$ and $\{3, 4, 5, 6\}$: seven of them. A computer check over all $\binom{20}{4} = 4845$ subsets also gives 525.

**2. Answer:** 183. **The idea.** The two conditions are $b^2 \geq 4c$ and $c^2 \geq 4b$: the region between two parabolas that cross at $(4, 4)$. First find where the region lives, then notice that for larger $b$ one of the two conditions is automatically true inside the box, so the count becomes a simple column-by-column sum.

**Work it out.** Both conditions force $b, c \geq 4$. Indeed, if $b \leq 3$ then $c \leq b^2/4 \leq 9/4$, so $c \leq 2$ (and $b = 1$ leaves no $c$ at all); then $c^2 \leq 4 < 4b$ because $b \geq 2$. So no pair has $b \leq 3$, and by symmetry none has $c \leq 3$.

For a fixed $b$, the conditions say $2\sqrt{b} \leq c \leq b^2/4$ (and $c \leq 20$). Column by column:

$b = 4$: $c = 4$ only (1). $b = 5$: $c \in \{5, 6\}$ (2). $b = 6$: $5 \leq c \leq 9$ (5). $b = 7$: $6 \leq c \leq 12$ (7). $b = 8$: $6 \leq c \leq 16$ (11). These give 26.

For $b \geq 9$ we have $b^2/4 \geq 81/4 > 20$, so the only condition is $c \geq 2\sqrt{b}$, i.e. $c \geq \lceil 2\sqrt{b} \rceil$: this is 6 for $b = 9$, 7 for $b = 10, 11, 12$, 8 for $b = 13, \ldots, 16$ and 9 for $b = 17, \ldots, 20$. The column sizes are 15, then 14 three times, 13 four times and 12 four times: $15 + 42 + 52 + 48 = 157$.

Total: $26 + 157 = 183$.

**Check.** The region is symmetric in $b$ and $c$. On the diagonal $b = c$ the condition is $b^2 \geq 4b$, i.e. $b \geq 4$: 17 pairs. So the off-diagonal pairs number $183 - 17 = 166$, which is even, as the symmetry $(b, c) \leftrightarrow (c, b)$ requires. Counting only the pairs with $b < c$ (there $c^2 \geq 4b$ is automatic, since $c > b \geq 4$ gives $c^2 > 4c > 4b$, so only $b^2 \geq 4c$ needs testing, i.e. $b < c \leq b^2/4$) gives

$0+1+3+5+8+11+10+9+8+7+6+5+4+3+2+1 = 83$ for $b = 4, \ldots, 19$, and $2 \cdot 83 + 17 = 183$.

**3. Answer:** 640**. The idea.** A colouring is decided by where the blocks start and by the colour of one seat: the blocks must alternate in colour. So we count the ways to cut the circle of 12 labelled seats into arcs of odd length. Look at the arc containing seat 1: once it is placed, the rest of the table is a straight row, and splitting a row into odd pieces is a Fibonacci count.

**Work it out.** Let $c(m)$ be the number of ways to write $m$ as an ordered sum of odd positive integers (a row of $m$ seats cut into odd blocks). Either the first part is 1, leaving $c(m-1)$ ways, or it is at least 3, and subtracting 2 from it gives a composition of $m-2$: $c(m) = c(m-1)+c(m-2)$. With $c(1) = c(2) = 1$ this is the Fibonacci sequence: $c(m) = F_m$, namely $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$ for $m = 1, \ldots, 11$.

The whole table cannot be one block (12 is even), so there are at least two blocks. Suppose the block containing seat 1 has odd length $\ell \le 11$. Seat 1 can be in any of $\ell$ positions inside it, which fixes the block; the remaining $12 - \ell$ seats form a row between its two ends, to be cut into odd blocks in $F_{12-\ell}$ ways. The number of ways to cut the circle is

$$\sum_{\ell \text{ odd}} \ell\, F_{12-\ell} = 1 \cdot 89 + 3 \cdot 34 + 5 \cdot 13 + 7 \cdot 5 + 9 \cdot 2 + 11 \cdot 1 = 89 + 102 + 65 + 35 + 18 + 11 = 320.$$

An even number of odd blocks is needed to make 12, so the blocks can indeed alternate red, blue, red, blue around the table, and there are exactly 2 ways to colour them (choose the colour of seat 1). The answer is $2 \cdot 320 = 640$.

**Check.** Try 4 seats. The formula gives $1 \cdot F_3 + 3 \cdot F_1 = 2 + 3 = 5$ ways to cut: all four seats as single blocks, or one single seat and a block of three (4 ways). So 10 colourings: RBRB, BRBR and the 8 with exactly one seat differing from the other three. Listing all 16 colourings of 4 seats confirms 10. (In general the cut count is $L_n - 2$ for even $n$, where $L_n$ is the Lucas number; $L_{12} - 2 = 322 - 2 = 320$.)

**4. Answer:** 64**. The idea.** Move $x^2$ across: $y^2 = x^3 - x^2 = x^2(x-1)$. When $x \neq 0$ this says $x - 1 = \left(\frac{y}{x}\right)^2$, so the integer $x - 1$ is the square of a rational number, and therefore the square of an integer. That turns the equation into a one-parameter family.

**Work it out.** If $x = 0$ the equation reads $y^2 = 0$: the single pair $(0, 0)$.

If $x < 0$ then $x^2(x-1) < 0$, which cannot equal $y^2$: no pairs.

If $x \ge 1$, then $x - 1 = \left(\frac{y}{x}\right)^2$. An integer that is the square of a fraction $\frac{p}{q}$ in lowest terms must have $q = 1$ (otherwise $\frac{p^2}{q^2}$ is again in lowest terms with denominator $q^2 > 1$), so $x - 1 = t^2$ for an integer $t \ge 0$, and $y = \pm xt = \pm t(t^2+1)$. Conversely every $t \ge 0$ works: $x = t^2 + 1$, $y = \pm t(t^2+1)$ gives $y^2 = t^2x^2 = (x-1)x^2$.

The bound $x = t^2 + 1 \le 1000$ means $t^2 \le 999$, i.e. $t \le 31$ (since $31^2 = 961$ and $32^2 = 1024$). For $t = 0$ we get $x = 1$, $y = 0$: one pair. For each $t = 1, \ldots, 31$ we get two pairs (the two signs of $y$): 62 pairs.

Total: $1 + 1 + 62 = 64$.

**Check.** $t = 2$ gives $(5, \pm 10)$: $25 + 100 = 125 = 5^3$. The largest, $t = 31$, gives $(962, \pm 29822)$, and $962^2 + 29822^2 = 962^2(1 + 961) = 962^3$. A computer search over $|x| \le 1000$ (testing whether $x^3 - x^2$ is a perfect square) finds the same 64 pairs. The answer is $\boxed{064}$.

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**5. Answer:** 281**. The idea.** Rather than finding the polygon's vertices and splitting it into triangles, project it straight down onto the base $z = 0$. The shadow is easy to describe, and a flat region's area is multiplied by $\cos\theta$ under projection, where $\theta$ is the angle between the plane and the base, i.e. between their normals.

**Work it out.** Try a plane $px+qy+rz = d$. The points $(6, 3, 0)$ and $(\frac{3}{2}, 6, 0)$ give $6p+3q = \frac{3}{2}p+6q$, so $\frac{9}{2}p = 3q$ and $q = \frac{3}{2}p$; taking $p = 2$, $q = 3$ gives $d = 12+9 = 21$, and then $(0, 0, \frac{7}{2})$ gives $r = 6$. The plane is $2x + 3y + 6z = 21$.

Above a base point $(x, y)$ the plane has height $z = \dfrac{21 - 2x - 3y}{6} \le \dfrac{21}{6} < 6$. So it passes through the cube exactly above the part of the base square where $2x + 3y \le 21$, and never leaves through the top face. That part is the square with one corner cut off: the line $2x + 3y = 21$ meets the edge $x = 6$ at $y = 3$ and the edge $y = 6$ at $x = \frac{3}{2}$, cutting off a right triangle with legs 3 and $\frac{9}{2}$ at the corner $(6, 6)$. The shadow is a pentagon of area $36 - \frac{1}{2} \cdot 3 \cdot \frac{9}{2} = 36 - \frac{27}{4} = \frac{117}{4}$.

The normal of the plane is $(2, 3, 6)$, of length $\sqrt{4 + 9 + 36} = 7$, and the normal of the base is $(0, 0, 1)$, so $\cos\theta = \frac{6}{7}$. The cross-section therefore has area $\frac{117}{4} \cdot \frac{7}{6} = \frac{273}{8}$. Since $\gcd(273, 8) = 1$, the answer is $273 + 8 = 281$.

**Check.** The section's vertices are $(0, 0, \frac{7}{2})$, $(6, 0, \frac{3}{2})$, $(6, 3, 0)$, $(\frac{3}{2}, 6, 0)$, $(0, 6, \frac{1}{2})$. Adding the cross products of consecutive position vectors and halving the length of the total gives the area directly: the total is $\frac{39}{4}(2, 3, 6)$, of length $\frac{273}{4}$, and half of that is $\frac{273}{8}$.

**6. Answer:** 577**. The idea.** A pair of disjoint sets is the same thing as a labelling of $1, 2, \ldots, 7$ with three labels: $A$, $B$, or $N$ (in neither). The condition only involves neighbours $i$ and $i + 1$: labels $A$ and $B$ may never sit next to each other. Reading the labels from left to right and remembering the last label gives a recursion.

**Work it out.** For a row $1, \ldots, n$ let $a_n$ be the number of good labellings, $x_n$ the number ending in $N$ and $y_n$ the number ending in $A$ (by symmetry also the number ending in $B$), so $a_n = x_n + 2y_n$. A label $N$ may follow anything: $x_n = a_{n-1}$. A label $A$ may follow $N$ or $A$: $y_n = x_{n-1} + y_{n-1} = a_{n-2} + y_{n-1}$.

Then $a_n = a_{n-1} + 2y_n = a_{n-1} + 2a_{n-2} + 2y_{n-1}$, and $2y_{n-1} = a_{n-1} - x_{n-1} = a_{n-1} - a_{n-2}$. So $a_n = 2a_{n-1} + a_{n-2}$.

Start: $a_1 = 3$, and $a_2 = 9 - 2 = 7$ (all label pairs except $AB$ and $BA$). Then $a_3 = 17$, $a_4 = 41$, $a_5 = 99$, $a_6 = 239$, $a_7 = 2 \cdot 239 + 99 = 577$.

**Check.** For $n = 3$, count directly by the middle label. Middle $N$: the ends are free, $3 \cdot 3 = 9$. Middle $A$: each end is $A$ or $N$, $2 \cdot 2 = 4$; middle $B$: likewise 4. Total 17, as the recursion says. (These numbers satisfy $a_n^2 - 2q^2 = \pm 1$ for suitable $q$, e.g. $577^2 - 2 \cdot 408^2 = 1$: they are the numerators of the continued-fraction approximations of $\sqrt{2}$.)

**7. Answer:** 89**. The idea.** Write each factor as a product of two linear factors in roots of unity. Since $1 + 4\cos^2\theta = 3 + 2\cos 2\theta$, choose $u$ with $u + \frac{1}{u} = 3$; with $\zeta = e^{2i\theta}$ we get $3 + 2\cos 2\theta = u + \frac{1}{u} + \zeta + \frac{1}{\zeta} = \dfrac{(u+\zeta)(u+\zeta^{-1})}{u}$. Over all 11th roots of unity, $\prod(u + \zeta)$ is just $u^{11} + 1$.

**Work it out.** First double the product. Since $\cos^2\frac{k\pi}{11} = \cos^2\frac{(11-k)\pi}{11}$, the product $Q$ of $1 + 4\cos^2\frac{k\pi}{11}$ over $k = 1, \ldots, 10$ is the square of the requested number $P$.

For $k = 1, \ldots, 10$, $\zeta_k = e^{2ik\pi/11}$ runs over the ten 11th roots of unity other than 1. For $\zeta = 1$ the expression $\frac{(u+\zeta)(u+\zeta^{-1})}{u}$ equals $\frac{(u+1)^2}{u} = u + 2 + \frac{1}{u} = 5$. Over all eleven roots, $\prod_\zeta (u + \zeta) = u^{11} + 1$

(the roots of $x^{11} = 1$ are the $\zeta$, so $\prod(x - \zeta) = x^{11} - 1$; put $x = -u$ and multiply by $-1$), and the same holds for the $\zeta^{-1}$, which are the same roots in another order. So

$$5\,Q = \frac{(u^{11}+1)^2}{u^{11}} = u^{11} + 2 + u^{-11}.$$

Let $t_n = u^n + u^{-n}$. Then $t_0 = 2$, $t_1 = 3$ and $t_{n+1} = 3t_n - t_{n-1}$ (multiply $u^n + u^{-n}$ by $u + u^{-1}$). This gives $2, 3, 7, 18, 47, 123, 322, 843, 2207, 5778, 15127, 39603$, so $t_{11} = 39603$ and $5Q = 39605$, $Q = 7921 = 89^2$. Every factor is positive, so $P = 89$.

**Check.** Numerically, $\cos^2\frac{\pi}{11} \approx 0.9206$, $\cos^2\frac{2\pi}{11} \approx 0.7077$, $\cos^2\frac{3\pi}{11} \approx 0.4288$, $\cos^2\frac{4\pi}{11} \approx 0.1726$, $\cos^2\frac{5\pi}{11} \approx 0.0203$; the factors are about 4.683, 3.831, 2.715, 1.690, 1.081, with product about 89.0. (Curiously, 89 is the Fibonacci number $F_{11}$; the same computation with 11 replaced by any odd $n$ gives $F_n$.)

**8. Answer:** 68**. The idea.** Only the exponents of 2, 3 and 5 change when we multiply by 30, and each of them goes up by exactly one. So the ratio of divisor counts is a product of three factors of the form $\frac{e+2}{e+1}$, and these factors are small: $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$. Getting a product of exactly 4 is very restrictive.

**Work it out.** Write $n = 2^a 3^b 5^c m$ with $\gcd(m, 30) = 1$. Then $\tau(n) = (a+1)(b+1)(c+1)\tau(m)$ and $\tau(30n) = (a+2)(b+2)(c+2)\tau(m)$, so $\frac{\tau(30n)}{\tau(n)} = f(a)f(b)f(c)$, where $f(e) = \frac{e+2}{e+1}$. Here $f(0) = 2$, $f(1) = \frac{3}{2}$, $f(2) = \frac{4}{3}$, and $f$ keeps decreasing toward 1. If none of $a, b, c$ is 0, the product is at most $\left(\frac{3}{2}\right)^3 = \frac{27}{8} < 4$. If two or three of them are 0, the product is $4f(e) > 4$ or 8. So exactly one exponent is 0, and the other two factors multiply to 2. Neither of those two can be 2 (the other factor would have to be 1, impossible), so both lie in $\{\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\}$. Two factors of $\frac{3}{2}$ give $\frac{9}{4} \neq 2$, and two factors at most $\frac{4}{3}$ give at most $\frac{16}{9} < 2$; so one factor is $\frac{3}{2}$ and the other is $\frac{4}{3}$, i.e. the two exponents are 1 and 2. So $\{a, b, c\} = \{0, 1, 2\}$, and $n = Km$ where $K$ is one of $3 \cdot 5^2 = 75$, $3^2 \cdot 5 = 45$, $2 \cdot 5^2 = 50$, $2^2 \cdot 5 = 20$, $2 \cdot 3^2 = 18$, $2^2 \cdot 3 = 12$, and $m$ is coprime to 30 with $Km \leq 1000$. Numbers coprime to 30 come 8 to every block of 30: $1, 7, 11, 13, 17, 19, 23, 29$ (plus 30 each time). $K = 75$: $m \leq 13$, giving $1, 7, 11, 13$: 4. $K = 45$: $m \leq 22$: 6. $K = 50$: $m \leq 20$: 6. $K = 20$: $m \leq 50$: 8+6 = 14 (the extra ones are $31, 37, 41, 43, 47, 49$). $K = 18$: $m \leq 55$: 8+7 = 15 (add 53). $K = 12$: $m \leq 83$: $16 + 7 = 23$ (the extra ones are $61, 67, 71, 73, 77, 79, 83$). These six families do not overlap (the exponents of 2, 3, 5 differ), so the total is $4+6+6+14+15+23 = 68$.

**Check.** Take $n = 12 = 2^2 \cdot 3$: $\tau(12) = 6$ and $\tau(360) = 24 = 4 \cdot 6$. And $n = 60$ (all three primes present) gives $\tau(60) = 12$, $\tau(1800) = 36$, only three times as many, as the bound $\frac{27}{8}$ predicts. The answer is $\boxed{068}$.

**9. Answer:** 399**. The idea.** The lattice points on the segment from $(0, 0)$ to $(x, y)$ are evenly spaced: if $g = \gcd(x, y)$, they are exactly the points $\frac{k}{g}(x, y)$ for $k = 0, 1, \ldots, g$. So there are $g - 1$ of them strictly between the endpoints, and we need $g = 4$. Dividing by 4 leaves a count of relatively prime pairs in a $25 \times 25$ square, which Euler's totient function handles.

**Work it out.** *The lattice points.* Write $x = ga$, $y = gb$ with $\gcd(a, b) = 1$. A point of the segment is $(sa, sb)$ with $0 \leq s \leq g$ real. If $sa$ and $sb$ are integers, then choosing integers $u, v$ with $ua + vb = 1$ (possible since $\gcd(a, b) = 1$) shows $s = u(sa) + v(sb)$ is an integer. Conversely every integer $s$ gives a lattice point. So the lattice points are $s = 0, 1, \ldots, g$, and $g - 1$ of them lie strictly between the endpoints.

*The count.* We need $\gcd(x, y) = 4$, i.e. $x = 4a$, $y = 4b$ with $1 \leq a, b \leq 25$ and $\gcd(a, b) = 1$.

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Sort these pairs by $m = \max(a, b)$. For $m = 1$ there is only $(1, 1)$. For $m \geq 2$ the two coordinates cannot both equal $m$ (then the gcd would be $m$), so one of them is $m$ and the other is a number below $m$ relatively prime to $m$: $2\varphi(m)$ pairs. Hence the count is $1+2\sum_{m=2}^{25}\varphi(m) = 2\sum_{m=1}^{25}\varphi(m) - 1$.

The values $\varphi(1), \ldots, \varphi(25)$ are $1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, 12, 10, 22, 8, 20$, in groups of five summing to $10, 22, 40, 56, 72$, so $\sum_{m=1}^{25}\varphi(m) = 200$ and the answer is $2\cdot 200 - 1 = 399$.

**Check.** Shrink the square to $1 \leq x, y \leq 8$. The formula gives $2(\varphi(1)+\varphi(2)) - 1 = 3$ pairs: $(4, 4)$, $(4, 8)$, $(8, 4)$. Indeed the segment to $(4, 8)$ passes through $(1, 2)$, $(2, 4)$, $(3, 6)$, while the one to $(8, 8)$ passes through seven lattice points. A computer counting $\gcd(x, y) - 1$ for all 10000 pairs also finds 399. The answer is $\boxed{399}$.

**10. Answer:** 91**. The idea.** Two powers of a point do all the work. The circle $\omega$ with diameter $AD$ is tangent to $BC$ at $D$ (its radius $AD$ is perpendicular to $BC$). From $A$: $B, C, F, E$ lie on one circle. From $T$: $TD^2$ equals the power of $T$ with respect to $\omega$ and also, through that second circle, $TB \cdot TC$.

**Work it out.** Since $AD$ is a diameter, $\angle AED = 90°$, so in right triangle $ADB$ the point $E$ is the foot of the altitude from $D$ and $AE \cdot AB = AD^2$. Likewise $AF \cdot AC = AD^2$. Hence $AE \cdot AB = AF \cdot AC$, and by the converse of the power of a point, $B, C, F, E$ are concyclic. Now $T$ lies on line $EF$ and on line $BC$. Its power with respect to $\omega$ is $TE \cdot TF = TD^2$ (tangent at $D$), and its power with respect to circle $BCFE$ is $TE \cdot TF = TB \cdot TC$. So $TD^2 = TB \cdot TC$. Put $B = (0, 0)$, $C = (21, 0)$, $A = (x, y)$. From $x^2 + y^2 = 169$ and $(x - 21)^2 + y^2 = 400$ we get $-42x + 441 = 231$, so $x = 5$, $y = 12$, and $D = (5, 0)$. Let $T = (t, 0)$. Then $(t - 5)^2 = t(t - 21)$, i.e. $-10t + 25 = -21t$, so $t = -\frac{25}{11}$ ($T$ lies beyond $B$). Therefore $TD = 5 + \frac{25}{11} = \frac{80}{11}$ and $m + n = 91$.

**Check.** Directly: $AE = \frac{AD^2}{AB} = \frac{144}{13}$, so $E = A + \frac{144}{169}(B - A) = \left(\frac{125}{169}, \frac{300}{169}\right)$; $AF = \frac{144}{20} = \frac{36}{5}$, so $F = A + \frac{36}{100}(C - A) = \left(\frac{269}{25}, \frac{192}{25}\right)$. The line through these meets $y = 0$ at $x = -\frac{25}{11}$, agreeing. Also $TB \cdot TC = \frac{25}{11} \cdot \frac{256}{11} = \frac{6400}{121} = \left(\frac{80}{11}\right)^2$. The answer is $\boxed{091}$.

**11. Answer:** 323**. The idea.** Compute the volume of $APQR$ two ways. Scaling from the big tetrahedron is easy because the three edges at $A$ are just shrunk; the area of $PQR$ comes from its three side lengths. Then height $= 3V/\text{area}$.

**Work it out.** A regular tetrahedron of edge $s$ has volume $\frac{s^3}{6\sqrt{2}}$, so $V(ABCD) = \frac{1728}{6\sqrt{2}} = 144\sqrt{2}$. Tetrahedra sharing the trihedral angle at $A$ have volumes proportional to the products of the edge lengths along it, so $V(APQR) = \frac{3\cdot 4\cdot 6}{12^3} \cdot 144\sqrt{2} = \frac{1}{24} \cdot 144\sqrt{2} = 6\sqrt{2}$. Every face angle at $A$ is $60°$, so by the law of cosines $PQ^2 = 9+16-12 = 13$, $QR^2 = 16+36-24 = 28$, $RP^2 = 36+9-18 = 27$. Heron's formula in squared-side form: $16K^2 = 2(pq + qr + rp) - (p^2 + q^2 + r^2)$ with $p, q, r = 13, 28, 27$: $16K^2 = 2(364 + 756 + 351) - (169 + 784 + 729) = 2942 - 1682 = 1260$, so $K^2 = \frac{315}{4}$. The distance $h$ from $A$ to the plane satisfies $V = \frac{1}{3}Kh$, so $h^2 = \frac{9V^2}{K^2} = \frac{9\cdot 72}{315/4} = \frac{2592}{315} = \frac{288}{35}$. Thus $m + n = 288 + 35 = 323$.

**Check.** Sanity: if $AP = AQ = AR = 12$ the same method gives $h^2 = \frac{9\cdot(144\sqrt{2})^2}{(36\sqrt{3})^2} = 96$, the square of the true height $12\sqrt{2/3}$ of the tetrahedron. A coordinate computation with $A$ at the origin also gives $h^2 = \frac{288}{35}$. The answer is $\boxed{323}$.

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**12. Answer:** 34. **The idea.** $z = 0$ is not a solution, so we may divide by $z^6$: $\left(\frac{z-1}{z}\right)^6 = 64 = 2^6$. Hence $\frac{z-1}{z} = 2w$ for a sixth root of unity $w$, and $z = \frac{1}{1-2w}$. The six vertices come out explicitly, they are symmetric about the real axis, and the shoelace formula finishes.

**Work it out.** The sixth roots of unity are $\pm 1$ and $\pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$w = 1$: $z = \frac{1}{-1} = -1$. $\quad w = -1$: $z = \frac{1}{3}$.

$w = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$: $1 - 2w = \mp\sqrt{3}\,i$, so $z = \pm\frac{i}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}i$.

$w = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$: $1 - 2w = 2 \mp \sqrt{3}\,i$, so $z = \frac{2\pm\sqrt{3}\,i}{7}$.

All six satisfy $|z - 1| = 2|z|$, a circle (centre $-\frac{1}{3}$, radius $\frac{2}{3}$), so they are in convex position; going around the circle counterclockwise from $\frac{1}{3}$ they are $\frac{1}{3}$, $\frac{2+\sqrt{3}i}{7}$, $\frac{\sqrt{3}}{3}i$, $-1$, $-\frac{\sqrt{3}}{3}i$, $\frac{2-\sqrt{3}i}{7}$. The real axis cuts the hexagon into two mirror-image halves. The upper half is the quadrilateral with vertices $\left(\frac{1}{3}, 0\right)$, $\left(\frac{2}{7}, \frac{\sqrt{3}}{7}\right)$, $\left(0, \frac{\sqrt{3}}{3}\right)$, $(-1, 0)$ in counterclockwise order; its shoelace terms $x_i y_{i+1} - x_{i+1} y_i$ are

$$\tfrac{1}{3} \cdot \tfrac{\sqrt{3}}{7} - 0 = \tfrac{\sqrt{3}}{21}, \quad \tfrac{2}{7} \cdot \tfrac{\sqrt{3}}{3} - 0 = \tfrac{2\sqrt{3}}{21}, \quad 0 - (-1) \cdot \tfrac{\sqrt{3}}{3} = \tfrac{7\sqrt{3}}{21}, \quad 0,$$

summing to $\frac{10\sqrt{3}}{21}$. So the upper half has area $\frac{5\sqrt{3}}{21}$ and the hexagon has area $\frac{10\sqrt{3}}{21}$. The answer is $10 + 3 + 21 = 34$.

**Check.** Each vertex really lies on the circle: $\left|\frac{1}{3} + \frac{1}{3}\right| = \frac{2}{3}$, $|-1 + \frac{1}{3}| = \frac{2}{3}$, $\left|\frac{1}{3} + \frac{\sqrt{3}}{3}i\right| = \sqrt{\frac{1}{9} + \frac{1}{3}} = \frac{2}{3}$, and $\left|\frac{2}{7} + \frac{1}{3} + \frac{\sqrt{3}}{7}i\right|^2 = \frac{169}{441} + \frac{27}{441} = \frac{196}{441} = \frac{4}{9}$. The area $\frac{10\sqrt{3}}{21} \approx 0.825$ is less than the disc's area $\frac{4\pi}{9} \approx 1.396$, as it must be. A computer solving the degree-6 equation and applying the shoelace formula agrees. The answer is $\boxed{034}$.

**13. Answer:** 170. **The idea.** A monic integer quartic that factors over the integers can be written with monic factors (flip signs if both leading coefficients are $-1$). So either it has a linear factor, i.e. an integer root, or it splits as two monic quadratics. The first case gives a handful of lines of pairs $(a, b)$; the second forces a rigid shape because $x^4 + ax + b$ has no $x^3$ and no $x^2$ term. Count each and remove the overlap.

**Work it out.** *Integer roots.* If $s$ is a root, $b = -s^4 - as$. Since $|as| \le 20|s|$ and $|b| \le 20$, we need $s^4 \le 20|s| + 20$, so $|s| \le 2$. The lines are: $s = 0$: $b = 0$ (41 pairs); $s = 1$: $b = -a - 1$ ($a = -20..19$, 40 pairs); $s = -1$: $b = a - 1$ (40 pairs); $s = 2$: $b = -2a - 16$ ($a = -18..2$, 21 pairs); $s = -2$: $b = 2a - 16$ ($a = -2..18$, 21 pairs). The ten pairwise intersections are $(\pm 1, 0)$, $(\pm 8, 0)$, $(0, -1)$, $(0, -16)$, $(\pm 5, -6)$, $(\pm 15, 14)$, all distinct and in the box, so the lines cover $163 - 10 = 153$ pairs.

*Two quadratics.* Write $x^4 + ax + b = (x^2 + px + q)(x^2 - px + r)$ (the $x^3$ coefficients must cancel). The $x^2$ coefficient gives $q + r - p^2 = 0$, so $r = p^2 - q$, and then $a = p(r - q) = p(p^2 - 2q)$ and $b = qr = q(p^2 - q)$. For $|p| \ge 4$, $|a| \le 20$ forces $|p^2 - 2q| \le 5$, so $q$ is near $p^2/2$ and $b = q(p^2 - q) \ge 60$; hence $|p| \le 3$. Replacing $p$ by $-p$ just swaps the factors, so take $p \ge 0$. $p = 0$: $a = 0$, $b = -q^2 \in \{0, -1, -4, -9, -16\}$; new (not on a line): $(0, -4)$, $(0, -9)$: 2. $p = 1$: $a = 1 - 2q$, $b = q(1 - q)$ for $q = -4..5$: the pairs $(\pm 9, -20)$, $(\pm 7, -12)$, $(\pm 5, -6)$, $(\pm 3, -2)$, $(\pm 1, 0)$; new: 6. $p = 2$: $a = 8 - 4q$, $b = q(4 - q)$ for $q = -2..6$: $(\pm 16, -12)$, $(\pm 12, -5)$, $(\pm 8, 0)$, $(\pm 4, 3)$, $(0, 4)$; here $(\pm 4, 3)$ lie on $b = \mp a - 1$ and $(\pm 8, 0)$ on $b = 0$, so new: 5 (including $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$). $p = 3$: $a = 3(9 - 2q)$ with $q = 2..7$, $b = q(9 - q)$: $(\pm 15, 14)$, $(\pm 9, 18)$, $(\pm 3, 20)$; new: 4. Total: $153 + 2 + 6 + 5 + 4 = 170$.

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**Check.** A spot check: $(a, b) = (-9, 18)$ gives $x^4 - 9x + 18 = (x^2 + 3x + 3)(x^2 - 3x + 6)$, which has no rational root, so it really needed the quadratic case. A computer factoring all $41^2$ polynomials finds exactly 170 reducible ones. The answer is $\boxed{170}$.

**14. Answer:** 835**. The idea.** A power of 2 counts yes/no choices, so read $2^{\text{something}}$ as "the number of ways to place optional dividers". Reversing a permutation turns descents into ascents, so $N$ is also the sum of $2^{(\text{number of ascents})}$. Now put a divider in each of the 7 gaps of a permutation where there is a descent (compulsory), and decide freely at each ascent: $2^{\text{ascents}}$ ways. The dividers cut the permutation into increasing blocks, i.e. into an ordered list of groups, which is a ranking of 8 runners in which ties are allowed. Those rankings are easy to count by a recursion.

**Work it out.** *Symmetry.* The reversal $(a_1, \ldots, a_8) \mapsto (a_8, \ldots, a_1)$ is a bijection of the permutations, and a descent at gap $i$ becomes an ascent at gap $8 - i$. Every one of the 7 gaps is a descent or an ascent, so $N = \sum_\sigma 2^{d(\sigma)} = \sum_\sigma 2^{7-d(\sigma)}$, the sum of $2^{(\text{ascents})}$.

*Dividers.* Count pairs $(\sigma, B)$ where $B$ is a set of gaps of $\sigma$ containing every descent; for each $\sigma$ there are $2^{7-d(\sigma)}$ choices of $B$, so there are $N$ pairs. Cutting $\sigma$ at the gaps in $B$ gives an ordered list of nonempty blocks, each increasing (a block contains no descent). Conversely, from an ordered list of nonempty disjoint blocks covering $\{1, \ldots, 8\}$, writing each block in increasing order and putting the dividers between consecutive blocks rebuilds the only pair $(\sigma, B)$ that produces it. So $N = a_8$, where $a_n$ is the number of ways to split $\{1, \ldots, n\}$ into an ordered list of nonempty groups (a ranking with ties).

*Recursion.* Choosing the first group (of size $k$) and then ranking the rest gives $a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k}$ with $a_0 = 1$. So $a_1 = 1$, $a_2 = 2 + 1 = 3$, $a_3 = 3 \cdot 3 + 3 + 1 = 13$, $a_4 = 4 \cdot 13 + 6 \cdot 3 + 4 + 1 = 75$, $a_5 = 5 \cdot 75 + 10 \cdot 13 + 10 \cdot 3 + 5 + 1 = 541$, $a_6 = 6 \cdot 541 + 15 \cdot 75 + 20 \cdot 13 + 15 \cdot 3 + 6 + 1 = 4683$, $a_7 = 7 \cdot 4683 + 21 \cdot 541 + 35 \cdot 75 + 35 \cdot 13 + 21 \cdot 3 + 7 + 1 = 47293$, and $a_8 = 8 \cdot 47293 + 28 \cdot 4683 + 56 \cdot 541 + 70 \cdot 75 + 56 \cdot 13 + 28 \cdot 3 + 8 + 1 = 378344 + 131124 + 30296 + 5250 + 728 + 84 + 8 + 1 = 545835$. So $N = 545835$, and the remainder is 835.

**Check.** For $n = 3$: 123 has no descent, 132, 213, 231, 312 have one, and 321 has two, so the sum is $1 + 4 \cdot 2 + 4 = 13 = a_3$. A second route for $n = 8$: the numbers of permutations with $0, 1, \ldots, 7$ descents (the Eulerian numbers, built by inserting 8 into permutations of 1..7) are $1, 247, 4293, 15619, 15619, 4293, 247, 1$, and $1 + 2 \cdot 247 + 4 \cdot 4293 + 8 \cdot 15619 + 16 \cdot 15619 + 32 \cdot 4293 + 64 \cdot 247 + 128 = 545835$. A computer summing $2^{d(\sigma)}$ over all 40320 permutations agrees. The answer is $\boxed{835}$.

**15. Answer:** 845**. The idea.** Only residues mod 3 matter, so think of the entries as $0, 1, -1$ (for $3, 1, 2$). Detect "divisible by 3" with $\omega = e^{2\pi i/3}$: the average $\frac{1}{3}(1 + \omega^S + \omega^{2S})$ is 1 when $3 \mid S$ and 0 otherwise. The sum $\sum \omega^S$ over all tuples is a product of factors $\omega^{x_i x_{i+1}}$ around a ring, so it is the trace of the 10th power of the $3 \times 3$ matrix $M = (\omega^{ab})$, and this matrix, the discrete Fourier matrix, has a tiny power: $M^4 = 9I$.

**Work it out.** Let $S(x) = \sum_i x_i x_{i+1}$ (indices mod 10). Then $N = \frac{1}{3} \sum_x \left(1 + \omega^{S(x)} + \omega^{2S(x)}\right) = \frac{1}{3}\left(3^{10} + T + \overline{T}\right)$, where $T = \sum_x \omega^{S(x)}$ (the third sum is the complex conjugate of the second, since $\omega^2 = \overline{\omega}$).

Index rows and columns by $a, b \in \{0, 1, -1\}$ and put $M_{ab} = \omega^{ab}$. Since $\omega^{S(x)} = M_{x_1 x_2} M_{x_2 x_3} \cdots M_{x_{10} x_1}$, summing over all tuples gives $T = \operatorname{tr}(M^{10})$.

Now $(M^2)_{ab} = \sum_c \omega^{c(a+b)}$, which is 3 if $a + b \equiv 0 \pmod 3$ and $1 + \omega + \omega^2 = 0$ otherwise. So $M^2 = 3P$, where $P$ is the permutation matrix of $a \mapsto -a$ (it fixes 0 and swaps 1 and $-1$). As

$P^2 = I$, $M^4 = 9I$ and $M^{10} = (M^4)^2 M^2 = 81 \cdot 3P = 243P$. The trace of $P$ counts its fixed points: just $a = 0$. So $T = 243$, which is real, and $N = \frac{1}{3}(59049 + 243 + 243) = 19683 + 162 = 19845$. The remainder is 845.

**Check.** The same formula for a ring of $n = 4$ gives $M^4 = 9I$, $T = 27$ and $N = \frac{1}{3}(81 + 54) = 45$; for $n = 2$ it gives $T = \operatorname{tr}(3P) = 3$ and $N = \frac{1}{3}(9+6) = 5$, and directly, $2x_1x_2$ is divisible by 3 exactly when $x_1 = 3$ or $x_2 = 3$: $9 - 4 = 5$ pairs. A computer check of all $3^4 = 81$ and all $3^{10} = 59049$ tuples gives 45 and 19845. The answer is $\boxed{845}$.