MathCastle

AIME

Hard Edition

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

1. A calculator displays a pair of integers (x, y). Pressing its one button replaces the pair (x, y) by $(3x - y,\ x + y)$. The display starts at $(1, 0)$, and the button is pressed exactly 2026 times. Find the remainder when the first number of the final pair is divided by 1000.

2. Find the sum, in degrees, of all angles x with $0^\circ \le x < 360^\circ$ such that $\sin^3 x + \cos^3 x = \dfrac{\sqrt{2}}{2}$.

3. Starting from a positive integer a, the following operation is repeated until the number becomes 0: replace the current number m by $m - \lfloor \sqrt{m} \rfloor$. For example, starting from 10 the numbers are $10, 7, 5, 3, 2, 1, 0$, so 6 replacements are made. For how many starting values a with $1 \le a \le 1000$ is the number of replacements odd?

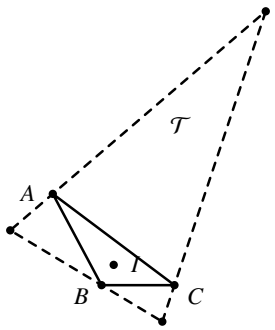
4. Points B and C are fixed in the plane with $BC = 7$. A point A, not on line BC, moves so that triangle ABC always satisfies $\tan \frac{\angle B}{2} \cdot \tan \frac{\angle C}{2} = \frac{1}{3}$. The set of all possible positions of A is a closed curve with two points removed, and the region enclosed by this curve has area $\frac{m\pi\sqrt{n}}{k}$, where m and k are relatively prime positive integers and n is a positive integer not divisible by the square of any prime. Find $m+n+k$.

5. Call a positive integer n *self-ending* if 3^n and n leave the same remainder when divided by 1000, that is, if $3^n - n$ is divisible by 1000. (For instance, $n = 7$ is not self-ending, since $3^7 = 2187$ leaves remainder 187.) Find the least self-ending positive integer n.

6. Nine flags hang in a row, and each flag is red, white or blue. The row is called *lively* if every four consecutive flags (flags 1–4, flags 2–5, ..., flags 6–9) include at least one flag of each of the three colours. How many of the 3^9 possible rows are lively?

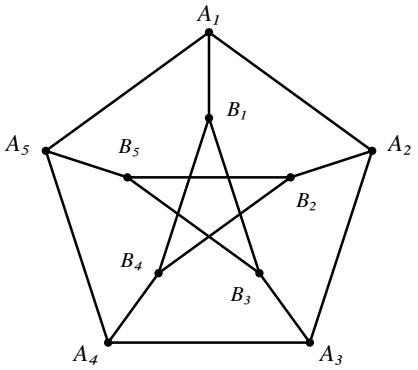
7. The side lengths of triangle ABC are integers, and the triangle is not equilateral. The bisector of angle A, the median from B (the segment from B to the midpoint of $\overline{AC}$), and the altitude from C (the segment from C perpendicular to line AB) all pass through one common point. Find the least possible perimeter of triangle ABC.

8. Triangle ABC has $AB = 17$, $BC = 12$ and $CA = 25$. Its three *excentres* are the centres of its three excircles (each excircle lies outside the triangle and is tangent to one side and to the extensions of the other two sides); they are the vertices of a triangle $\mathcal{T}$. Let I be the incentre of triangle ABC (the centre of its inscribed circle) and let V be the centre of the circle through the three vertices of $\mathcal{T}$. Find IV^2.



9. Triangle ABC has a right angle at A, with $AB = 15$, $AC = 20$ and $BC = 25$. Let $\mathcal{R}$ be the set of points inside the triangle that are closer to the vertex A than to the line BC. The area of $\mathcal{R}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. Ten dots are joined by 15 strings. Five outer dots A_1, A_2, A_3, A_4, A_5 form a ring: A_i is joined to A_{i+1} for each i (with $A_6 = A_1$). Five inner dots $B_1, \ldots, B_5$ form a five-pointed star: B_i is joined to B_{i+2} for each i (indices taken modulo 5, so B_4 is joined to B_1 and B_5 to B_2). Finally each A_i is joined to B_i by a spoke. In how many ways can each of the ten dots be painted red, white or blue so that the two ends of every string have different colours?



11. Six couples (12 people) arrive at a party one person at a time, with all 12! arrival orders equally likely. A person whose partner has not yet arrived waits in the lobby. When a person arrives whose partner is waiting in the lobby, the two of them immediately go into the party together (so the newcomer never waits). The probability that there are never more than 3 people waiting in the lobby at the same time is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Triangle ABC has $AB = 13$, $BC = 14$ and $CA = 15$. A point P inside the triangle has the property that the circumcircles of triangles PAB, PBC and PCA all have the same radius. Then $PA + PB + PC = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. For how many integers n with $1 \le n \le 200$ can n be written as $n = a + b + c$, where a, b, c are positive integers (not necessarily different) such that $a^2 + b^2 + c^2$ is divisible by n?

14. An ant starts at a vertex S of a cube and walks along the cube's edges. Each minute it moves from its current vertex along one of the 3 edges there, each chosen with probability $\frac{1}{3}$, independently of earlier moves. It stops the first time it arrives back at S. The expected number of different vertices, other than S itself, that the ant visits before it stops is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Ten points are labelled $1, 2, \ldots, 10$. A *tree* on these points is a set of 9 segments, each joining two of the points, such that any two of the ten points are connected by a path made of these segments. (Two trees are different if their sets of segments are different; there are 10^8 trees in all.) The *degree* of a point is the number of segments of the tree that end at it. Let N be the number of trees in which every one of the ten points has odd degree. Find the remainder when N is divided by 1000.

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Answer Key — AIME — Hard Edition Practice Contest 4

1. 96

2. 495

3. 504

4. 54

5. 387

6. 816

7. 40

8. 425

9. 593

10. 120

11. 474

12. 101

13. 148

14. 467

15. 80

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Solutions — AIME — Hard Edition Practice Contest 4

1. Answer: 96. **The idea.** One press multiplies the column vector (x, y) by the matrix A with rows $(3, -1)$ and $(1, 1)$. Its characteristic polynomial is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, a double eigenvalue, so A is not a pure scaling: $A = 2I + N$ with N the matrix with rows $(1, -1)$ and $(1, -1)$, and $N^2 = 0$. Because $2I$ commutes with N, the binomial theorem stops after two terms: $A^n = 2^n I + n\,2^{n-1}N$.

Work it out. The first number after n presses is the top-left entry of A^n (we start from $(1, 0)$), namely $x_n = 2^n + n\,2^{n-1} = 2^{n-1}(n + 2)$. For $n = 2026$ this is $x = 2028 \cdot 2^{2025}$. Modulo 8 it is 0. Modulo 125: the order of 2 divides 100, so $2^{2025} \equiv 2^{25}$; since $2^7 = 128 \equiv 3$, $2^{25} = 2^{21} \cdot 2^4 \equiv 27 \cdot 16 = 432 \equiv 57$. Then $x \equiv 28 \cdot 57 = 1596 \equiv 96 \pmod{125}$. The number 96 is also $\equiv 0 \pmod 8$, so by the Chinese Remainder Theorem $x \equiv 96 \pmod{1000}$.

Check. The first few pairs are $(1, 0)$, $(3, 1)$, $(8, 4)$, $(20, 12)$, $(48, 32)$, $(112, 80)$, and $x_n = 2^{n-1}(n+2)$ gives $1, 3, 8, 20, 48, 112$. Equivalently, x_n satisfies $x_{n+2} = 4x_{n+1} - 4x_n$ (from the characteristic polynomial), whose double root 2 forces the form $(\alpha + \beta n)2^n$. The answer is **096**.

2. Answer: 495. **The idea.** The left side is symmetric in $\sin x$ and $\cos x$, so use $t = \sin x + \cos x = \sqrt{2}\sin(x + 45^\circ)$, which ranges over $[-\sqrt{2}, \sqrt{2}]$, and $\sin x \cos x = \frac{t^2-1}{2}$. The equation becomes a cubic in t. One root gives an ugly angle, but the two solutions it produces are symmetric, so their sum is clean.

Work it out. $\sin^3 x + \cos^3 x = t\left(1 - \frac{t^2-1}{2}\right) = \frac{t(3-t^2)}{2}$. Setting this equal to $\frac{\sqrt{2}}{2}$ gives $t^3 - 3t + \sqrt{2} = 0$, and $t = \sqrt{2}$ is a root: $(t - \sqrt{2})(t^2 + \sqrt{2}\,t - 1) = 0$. The quadratic has roots $t = \frac{-\sqrt{2} \pm \sqrt{6}}{2}$.

• $t = \sqrt{2}$: $\sin(x + 45^\circ) = 1$, so $x = 45^\circ$ only.

• $t = \frac{\sqrt{6}-\sqrt{2}}{2} \approx 0.518$, inside the range: $\sin(x + 45^\circ) = \frac{\sqrt{3}-1}{2}$. With θ the acute angle having this sine, $x + 45^\circ = \theta$ or $180^\circ - \theta$ (up to multiples of 360°). Reduced into $[0^\circ, 360^\circ)$ these are $x_1 = \theta + 315^\circ$ (since $\theta < 45^\circ$) and $x_2 = 135^\circ - \theta$, with sum 450°.

• $t = -\frac{\sqrt{6}+\sqrt{2}}{2} \approx -1.93 < -\sqrt{2}$: impossible.

Total: $45^\circ + 450^\circ = 495^\circ$.

Check. Numerically $\theta \approx 21.47^\circ$, so the solutions are about 45°, 113.53° and 336.47°; for example at 113.53°, $\sin^3 + \cos^3 \approx 0.7707 - 0.0636 = 0.7071$. Note that $x = 45^\circ$ is a double root: $t = \sqrt{2}$ is the largest value of t, and $\frac{t(3-t^2)}{2}$ is decreasing there, so the left side has a local minimum equal to $\frac{\sqrt{2}}{2}$ at 45° and just touches the level; it is still one angle, counted once. The answer is **495**.

3. Answer: 504. **The idea.** Write each number as $m = k^2 + r$ with $k = \lfloor \sqrt{m} \rfloor$ and $0 \le r \le 2k$ (the "band" of k). One replacement subtracts k, which moves you either to a smaller remainder in the same band or into band $k - 1$. Tracking this shows the number of replacements $f(m)$ depends only on which of three parts of the band m sits in: $f(k^2) = 2k - 1$, $f(k^2 + r) = 2k$ for $1 \le r \le k$, and $f(k^2 + r) = 2k + 1$ for $k + 1 \le r \le 2k$.

Work it out. Prove the three formulas together by induction on k. For $k = 1$: $f(1) = 1$, $f(2) = 2$ $(2 \to 1 \to 0)$, $f(3) = 3$. Now let $k \ge 2$ and assume them for $k - 1$.

(i) $k^2 \to k^2 - k = (k - 1)^2 + (k - 1)$, the top of the middle part of band $k - 1$, which needs $2(k - 1)$ more; so $f(k^2) = 2k - 1$.

(ii) $1 \le r \le k-1$: $k^2+r-k = (k-1)^2+(k-1+r)$ with $k \le k-1+r \le 2(k-1)$, the top part of band $k-1$, which needs $2(k-1)+1$ more; so $f = 2k$. For $r = k$: $k^2+k-k = k^2$, needing $2k-1$ more, so again $f = 2k$.

(iii) $k+1 \le r \le 2k$: $k^2+r-k = k^2+(r-k)$ with $1 \le r-k \le k$, the middle part of band k, needing $2k$ more; so $f = 2k+1$.

So $f(m)$ is odd exactly when $r = 0$ or $r \ge k+1$: that is $1+k$ values in band k. Bands $k = 1, \dots, 30$ lie entirely in $[1, 1000]$ (band 30 ends at $960 = 31^2-1$) and contribute $\sum_{k=1}^{30}(k+1) = 465+30 = 495$. Band 31 is cut off: $961 \le m \le 1000$ means $0 \le r \le 39$, and the odd ones are $r = 0$ and $32 \le r \le 39$, that is 9 more. Total $495+9 = 504$.

Check. The three formulas combine into $f(m) = \lceil 2\sqrt{m} \rceil - 1$: for $m = 10$ this is $\lceil 6.32 \rceil - 1 = 6$, matching the example, and for $m = 1, 2, 3, 4$ it gives $1, 2, 3, 3$ ($4 \to 2 \to 1 \to 0$). The answer is **504**.

4. Answer: 54**. The idea.** Half-angle tangents are ratios of incircle lengths: with inradius r and semiperimeter s, the tangent lengths from B and C are $s-b$ and $s-c$, so $\tan\frac{B}{2} = \frac{r}{s-b}$ and $\tan\frac{C}{2} = \frac{r}{s-c}$. Their product simplifies with Heron's formula to a ratio involving only s and a, turning the condition into a statement about $AB+AC$: an ellipse.

Work it out. By Heron, $r^2s^2 = K^2 = s(s-a)(s-b)(s-c)$, so $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$ and $\tan\frac{B}{2}\tan\frac{C}{2} = \frac{r^2}{(s-b)(s-c)} = \frac{s-a}{s}$. Setting $\frac{s-a}{s} = \frac{1}{3}$ gives $s = \frac{3a}{2}$, i.e. $a+b+c = 3a$, i.e. $AB+AC = 2\cdot BC = 14$. Conversely every non-collinear A with $AB+AC = 14$ satisfies the condition. So the locus is the ellipse with foci B and C and major axis 14, minus its two points on line BC.

Semi-major axis 7, half focal distance $\frac{7}{2}$, so the semi-minor axis is $\sqrt{49-\frac{49}{4}} = \frac{7\sqrt{3}}{2}$. Area $= \pi\cdot 7\cdot\frac{7\sqrt{3}}{2} = \frac{49\pi\sqrt{3}}{2}$, so $m+n+k = 49+3+2 = 54$.

Check. Take A at the top of the ellipse: $AB = AC = 7$, an equilateral triangle, and $\tan^2 30^\circ = \frac{1}{3}$. Take A with $AB = 5$, $AC = 9$: $s = \frac{21}{2}$, $s-a = \frac{7}{2}$, ratio $\frac{1}{3}$ again. The answer is **054**.

5. Answer: 387**. The idea.** Split $1000 = 8\cdot 125$. The trouble is that n appears both as the exponent and as the value. But $3^n \bmod 125$ depends only on $n \bmod 100$ (the order of 3 modulo 125 divides $\varphi(125) = 100$), and $3^n \bmod 8$ only on the parity of n, so we can solve for the last two digits of n first and then read off $n \bmod 1000$.

Work it out. Modulo 8: $3^n \equiv 1$ for even n and $\equiv 3$ for odd n. If n were even we would need $n \equiv 1 \pmod 8$, which is odd; so n is odd and $n \equiv 3 \pmod 8$.

Let $t = n \bmod 100$. Then $3^n \equiv 3^t \pmod{125}$, so we need $n \equiv 3^t \pmod{125}$. Reducing to modulo 25 and 4: $t \equiv n \equiv 3^t \pmod{25}$ and $t \equiv n \equiv 3 \pmod 4$. Modulo 25 the powers of 3 repeat with period 20, so write $t = 20j+u$ with $0 \le j \le 4$ and $u \in \{3, 7, 11, 15, 19\}$ (as $t \equiv 3 \pmod 4$). The values $3^u \bmod 25$ are $2, 12, 22, 7, 17$, and we need $20j+u \equiv 3^u \pmod{25}$. Since $20j$ is a multiple of 5, this needs $u \equiv 3^u \pmod 5$, true only for $u = 7$ ($7 \equiv 12$); then $20j \equiv 5 \pmod{25}$ gives $j = 4$, so $t = 87$.

Now $n \equiv 3^{87} \pmod{125}$. Compute: $3^5 = 243 \equiv -7$, $3^{10} \equiv 49$, $3^{20} \equiv 2401 \equiv 26$, $3^{40} \equiv 676 \equiv 51$, $3^{80} \equiv 2601 \equiv 101 \equiv -24$, so $3^{87} = 3^{80}\cdot 3^5\cdot 3^2 \equiv (-24)(-7)(9) = 1512 \equiv 12$. So $n \equiv 12 \pmod{125}$ and $n \equiv 3 \pmod 8$: among $12, 137, 262, 387, \dots$ the first that is 3 mod 8 is 387. Every solution is $\equiv 387 \pmod{1000}$, and conversely $n = 387$ works (it ends in 87, is 3 mod 8, and is $\equiv 3^{87} \pmod{125}$), so the least is 387.

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Check. $3^{387} \equiv 3^{87} \equiv 12 \equiv 387 \pmod{125}$ and $3^{387} \equiv 3 \equiv 387 \pmod{8}$. The answer is **387**.

6. Answer: 816**. The idea.** Whether the next flag may be added depends only on the last three flags, and only on their *pattern*, not on which colours they are. Three consecutive flags can never all have the same colour (the window of four containing them would miss a colour), so the last three flags (p, q, r) always have one of four patterns: D, all different; A, $p = q \neq r$; B, $p = r \neq q$; C, $q = r \neq p$. Track how many lively rows end in each pattern.

Work it out. The new flag s must make $\{p, q, r, s\}$ contain all three colours, and the new last three flags are (q, r, s).

• From D: any s works. $s = p$ gives (q, r, p), pattern D; $s = q$ gives (q, r, q), pattern B; $s = r$ gives (q, r, r), pattern C.

• From $A = (x, x, y)$, $B = (x, y, x)$ or $C = (x, y, y)$: only two colours are present, so s must be the third colour z, giving (x, y, z), (y, x, z), (y, y, z): patterns D, D, A.

So if (D, A, B, C) count the lively rows of some length by final pattern, adding a flag gives $(D + A + B,\ C,\ D,\ D)$. For three flags each pattern occurs $3 \cdot 2 = 6$ times (the 3 one-colour rows cannot be extended). Iterating:

$n = 3$: $(6, 6, 6, 6)$; $n = 4$: $(18, 6, 6, 6)$; $n = 5$: $(30, 6, 18, 18)$; $n = 6$: $(54, 18, 30, 30)$; $n = 7$: $(102, 30, 54, 54)$; $n = 8$: $(186, 54, 102, 102)$; $n = 9$: $(342, 102, 186, 186)$.

The number of lively rows of nine flags is $342 + 102 + 186 + 186 = 816$.

Check. For four flags the count 36 is right directly: a window of four with all three colours has one colour twice, so there are 3 choices of the doubled colour and $\frac{4!}{2!} = 12$ arrangements, 36 in all. The rule also gives $D_{n+1} = D_n + D_{n-1} + D_{n-2}$ (since $A_n = C_{n-1} = D_{n-2}$ and $B_n = D_{n-1}$), and indeed $342 = 186 + 102 + 54$. The answer is **816**.

7. Answer: 40**. The idea.** Three concurrent cevians mean Ceva's theorem, and each ratio is easy to write down: the median splits $\overline{AC}$ evenly, the bisector splits $\overline{BC}$ as $AB : AC$, and the altitude splits $\overline{AB}$ via $\cos A$. Ceva pins down $\cos A$, the Law of Cosines turns that into a formula for BC^2, and integrality becomes a divisibility condition that leaves only a short search.

Work it out. Write $a = BC$, $b = CA$, $c = AB$. The altitude from C can pass through an interior point only if its foot F lies inside $\overline{AB}$ (otherwise A and B are on the same side of line CF and the triangle touches that line only at C), so $AF = b\cos A$ and $FB = c - b\cos A$. With D the foot of the bisector on $\overline{BC}$ and E the midpoint of $\overline{AC}$, Ceva gives $\frac{b\cos A}{c - b\cos A} \cdot \frac{c}{b} \cdot 1 = 1$, so $\cos A = \frac{c}{b+c}$. The Law of Cosines then gives $a^2 = b^2 + c^2 - \frac{2bc^2}{b+c} = (b - c)^2 + \frac{2b^2c}{b+c}$.

Let $g = \gcd(b, c)$, $b = gp$, $c = gq$ with $\gcd(p, q) = 1$, and $s = p + q$. Then $a^2 = \frac{g^2 N}{s}$ with $N = s(p - q)^2 + 2p^2q$. Since s shares no factor with p or q, a^2 is an integer only if $s \mid 2g^2$. If $s = 2$ then $p = q = 1$ and $a = g = b = c$, the equilateral case; so $s \geq 3$. The perimeter exceeds $b + c = gs$, so for a perimeter below 40 we need $gs < 40$ with $s \mid 2g^2$, which leaves only these cases (for $s \geq 10$ the least allowed g already makes $gs \geq 50$):

• $s = 3$, $g = 3k$: $a^2 = 3k^2N$, so $3N$ must be a square; $(p, q) = (1, 2), (2, 1)$ give $N = 7, 11$. No.

• $s = 4$, $g = 2k$: $a^2 = k^2N$; $(1, 3), (3, 1)$ give $N = 22, 34$. No.

• $s = 5$, $g = 5$: $a^2 = 5N$; $N = 53, 29, 41, 77$. No.

• $s = 6$, $g = 3k$: $a^2 = \frac{3}{2}k^2N$; $(1, 5), (5, 1)$ give $N = 106, 146$, and $\frac{3}{2}N = 159, 219$. No.

• $s = 8$, $g = 2k$: $a^2 = \frac{1}{2}k^2N$; $N = 302, 122, 182, 386$, halves $151, 61, 91, 193$. No.

• $s = 9$, $g = 3$: $a^2 = N$, and $(p, q) = (1, 8), (2, 7), (4, 5), (5, 4), (7, 2), (8, 1)$ give $N = 457, 281, 169, 209, 421, 569$. Only $169 = 13^2$ works: $b = 12$, $c = 15$, $a = 13$, perimeter 40.

So no perimeter below 40 is possible, and 40 is attained.

Check. For $a = 13$, $b = 12$, $c = 15$: $\cos A = \frac{144+225-169}{2\cdot12\cdot15} = \frac{200}{360} = \frac{5}{9} = \frac{15}{27}$, as Ceva requires; $\cos B = \frac{169+225-144}{2\cdot13\cdot15} > 0$, so the foot F is inside $\overline{AB}$ with $AF = \frac{20}{3}$, $FB = \frac{25}{3}$, and $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{4}{5} \cdot \frac{15}{12} \cdot 1 = 1$. The answer is **040**.

8. Answer: 425**. The idea.** Look at ABC from the point of view of the big triangle $\mathcal{T} = I_AI_BI_C$. At each vertex the internal and external angle bisectors are perpendicular, and the excentres sit on them: I_B, A, I_C lie on the external bisector at A, while A, I, I_A lie on the internal bisector. So I_AA is an *altitude* of $\mathcal{T}$, and likewise I_BB and I_CC. Hence I is the orthocentre of $\mathcal{T}$ and A, B, C are the feet of its altitudes, so the circumcircle of ABC is the nine-point circle of $\mathcal{T}$. The nine-point centre is the midpoint of the orthocentre and the circumcentre, so the circumcentre O of ABC is the midpoint of IV, and $IV^2 = 4\,OI^2 = 4(R^2 - 2Rr)$ by Euler's formula.

Work it out. With $a = BC = 12$, $b = CA = 25$, $c = AB = 17$: $s = 27$ and Heron gives area $K = \sqrt{27 \cdot 15 \cdot 2 \cdot 10} = \sqrt{8100} = 90$. So $r = \frac{K}{s} = \frac{10}{3}$ and $R = \frac{abc}{4K} = \frac{12\cdot25\cdot17}{360} = \frac{85}{6}$. Then $R^2 - 2Rr = \frac{7225}{36} - 2 \cdot \frac{85}{6} \cdot \frac{10}{3} = \frac{7225}{36} - \frac{3400}{36} = \frac{3825}{36}$, and $IV^2 = 4 \cdot \frac{3825}{36} = \frac{3825}{9} = 425$. (The triangle is obtuse at B, since $25^2 > 12^2 + 17^2$; nothing above needs it acute. The triangle $\mathcal{T}$ itself is always acute, its angles being $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$.)

Check. Coordinates: $B = (0, 0)$, $C = (12, 0)$, $A = (-8, 15)$ (indeed $AB^2 = 64 + 225 = 289$ and $AC^2 = 400 + 225 = 625$). The incentre is $\frac{aA+bB+cC}{a+b+c} = \frac{(-96+204,\ 180)}{54} = (2, \frac{10}{3})$. The excentres are $I_A = \frac{-aA+bB+cC}{-a+b+c} = (10, -6)$, $I_B = \frac{aA-bB+cC}{a-b+c} = (27, 45)$, $I_C = \frac{aA+bB-cC}{a+b-c} = (-15, 9)$. The point $V = (10, \frac{67}{3})$ is at distance $\frac{85}{3}$ from all three: to I_A directly, to I_B since $17^2 + (\frac{68}{3})^2 = \frac{7225}{9}$, to I_C since $25^2 + (\frac{40}{3})^2 = \frac{7225}{9}$. The circumradius $\frac{85}{3} = 2R$, as it must be for a triangle whose nine-point circle has radius R, and the circumcentre of ABC, $O = (6, \frac{77}{6})$, is indeed the midpoint of I and V. Finally $IV^2 = 8^2 + 19^2 = 64 + 361 = 425$. The answer is $\boxed{425}$.

9. Answer: 593**. The idea.** The points equally far from a point and from a line form a parabola, with the point as focus and the line as directrix. So $\mathcal{R}$ is the part of the triangle on the focus side of a parabola: a curved region cut off near A by the parabola. It splits into a triangle and a parabolic segment, and Archimedes' rule (a parabolic segment has $\frac{4}{3}$ the area of the inscribed triangle whose third vertex is where the tangent is parallel to the chord) finishes it without calculus.

Work it out. The altitude from A has length $\frac{15\cdot20}{25} = 12$ and its foot D splits BC into $BD = \frac{15^2}{25} = 9$ and $DC = \frac{20^2}{25} = 16$. Put BC on the x-axis with D at the origin: $B = (-9, 0)$, $C = (16, 0)$, $A = (0, 12)$. Inside the triangle $y > 0$, so the distance to line BC is y, and $x^2+(y-12)^2 < y^2 \iff y > \frac{x^2}{24}+6$. Line AB is $y = 12+\frac{4}{3}x$. Setting $\frac{x^2}{24}+6 = 12+\frac{4}{3}x$ gives $x^2-32x-144 = 0 = (x+4)(x-36)$; on side AB we take $x = -4$, the point $P = (-4, \frac{20}{3})$. Line AC is $y = 12-\frac{3}{4}x$. Setting $\frac{x^2}{24}+6 = 12-\frac{3}{4}x$ gives $x^2 + 18x - 144 = 0 = (x - 6)(x + 24)$; on side AC we take $x = 6$, the point $Q = (6, \frac{15}{2})$. The parabola's lowest point $(0, 6)$ is above BC, so $\mathcal{R}$ is bounded by AP, the parabolic arc from P to Q, and QA. Cut it along the chord PQ. *Triangle APQ.* With $\vec{AP} = (-4, -\frac{16}{3})$ and $\vec{AQ} = (6, -\frac{9}{2})$, the area is $\frac{1}{2}\left|(-4)(-\frac{9}{2}) - (-\frac{16}{3})(6)\right| = \frac{1}{2}(18 + 32) = 25$. *The parabolic segment.* The chord PQ has slope $\frac{15/2-20/3}{10} = \frac{1}{12}$; the parabola has that slope where $\frac{x}{12} = \frac{1}{12}$, at $x = 1$, the midpoint of the x-range $[-4, 6]$ (true for every parabola $y = ax^2 + \dots$). There the parabola is at $\frac{1}{24} + 6 = \frac{145}{24}$

and the chord at $\frac{20}{3} + \frac{5}{12} = \frac{170}{24}$: a vertical gap of $\frac{25}{24}$. The inscribed triangle on the chord has area $\frac{1}{2} \cdot \frac{25}{24} \cdot 10 = \frac{125}{24}$ (a vertical cut of length g over a width w gives area $\frac{1}{2}gw$), so the segment has area $\frac{4}{3} \cdot \frac{125}{24} = \frac{125}{18}$. Total: $25 + \frac{125}{18} = \frac{575}{18}$, already in lowest terms ($575 = 5^2 \cdot 23$). So $m + n = 575 + 18 = 593$.

Check. The segment area also follows from the formula $\frac{a\,w^3}{6}$ for $y = ax^2 + \cdots$ over a chord of width w: $\frac{1}{24} \cdot \frac{1000}{6} = \frac{125}{18}$. Reasonableness: the region has area about 31.9, about a fifth of the triangle's area 150; a fine grid count of points in the triangle closer to A than to BC gives the same value. The answer is $\boxed{593}$.

10. Answer: 120**. The idea.** Paint the outer ring first. Every good painting of a 5-cycle with 3 colours looks the same up to rotating and renaming colours, so the number of ways to finish the inner star is the same for all of them. For one fixed outer ring, each inner dot has only two colours left (it must avoid its spoke partner), and the inner star is again a 5-cycle, which we can finish by hand.

Work it out. *The outer ring.* Going around a 5-cycle, three colours must all be used (two colours cannot alternate around an odd cycle), and no colour can be used 3 times (a 5-cycle has no three pairwise non-adjacent points). So the colour counts are $2, 2, 1$: one dot has a unique colour and the other four alternate the other two colours. That gives 5 places for the unique dot, 3 choices of its colour, and 2 ways to start the alternation: 30 paintings. (The chromatic formula $(3-1)^5 - (3-1) = 30$ agrees.) Rotating the labels ($A_i \to A_{i+1}$, $B_i \to B_{i+1}$ keeps every string a string) and renaming the colours turns any of these 30 into the painting $A_1A_2A_3A_4A_5 = 1\,2\,1\,2\,3$ (colours called $1, 2, 3$). So we count the completions of this one and multiply by 30. *The inner star.* The spokes force $B_1 \in \{2,3\}$, $B_2 \in \{1,3\}$, $B_3 \in \{2,3\}$, $B_4 \in \{1,3\}$, $B_5 \in \{1,2\}$. Around the star the order is $B_1 - B_3 - B_5 - B_2 - B_4 - B_1$. Since B_1, B_3 are joined and both lie in $\{2,3\}$, they are $2, 3$ in some order; likewise B_2, B_4 are $1, 3$ in some order. Case $B_1 = 2, B_3 = 3$: the string B_4B_1 is automatically fine. If $B_2 = 1$ (so $B_4 = 3$), then $B_5 \neq B_2 = 1$ and $B_5 \neq B_3 = 3$ force $B_5 = 2$: 1 way. If $B_2 = 3$ (so $B_4 = 1$), then B_5 may be 1 or 2: 2 ways. Case $B_1 = 3, B_3 = 2$: now $B_4 \neq B_1 = 3$ gives $B_4 = 1$, $B_2 = 3$, and $B_5 \neq B_3 = 2$ gives $B_5 = 1$, which also differs from B_2: 1 way. So there are $1 + 2 + 1 = 4$ completions, and the total is $30 \cdot 4 = 120$.

Check. Every good painting uses each colour on an independent set of dots, and a computer run over all $3^{10} = 59049$ paintings finds exactly 120 good ones. Another sanity check: $120 = 3! \cdot 20$, divisible by $3! = 6$ as it must be, since renaming the colours permutes the good paintings in orbits of size 6. The answer is $\boxed{120}$.

11. Answer: 474**. The idea.** Record only the number k of people waiting after each arrival. Each arrival either raises k by 1 (someone from a couple not yet seen) or lowers it by 1 (the partner of someone waiting), so the history is an up-and-down path of 12 steps from 0 back to 0. Count the arrival orders that produce each path: the up-steps always contribute the same factor, and a down-step from height k contributes a factor k. Then a small table counts the paths that stay at height 3 or below.

Work it out. Fix a path. At the u-th up-step, $u - 1$ couples have already appeared, so the arriving person can be any of the $2(7-u)$ people of the unseen couples; over the six up-steps this gives $2^6 \cdot 6!$ choices, whatever the path. At a down-step from height k the arriving person must be the partner of one of the k waiting people: k choices. So the number of orders giving the path is $2^6\,6! \cdot W$, where W is the product of the heights from which the down-steps start. Summing over

all paths must give 12!, so the total of W over all paths is $\frac{12!}{2^6\,6!} = 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1 = 10395$, and the required probability is $\frac{S}{10395}$, where S is the total of W over the paths that never go above 3.

After an even number of arrivals the height is 0 or 2. Let (a, b) be the weighted counts of paths (staying at height at most 3) at heights 0 and 2. The next two arrivals go $0\to 1\to 0$ (weight 1) or $0\to 1\to 2$ (weight 1), and $2\to 1\to 0$ (weight $2\cdot 1$), $2\to 1\to 2$ (weight 2), or $2\to 3\to 2$ (weight 3); the step $2\to 3\to 4$ is forbidden. Hence $(a, b)\mapsto (a+2b,\ a+5b)$. Starting from $(1, 0)$: $(1, 1)$, $(3, 6)$, $(15, 33)$, $(81, 180)$, $(441, 981)$, and after 12 arrivals the height must be 0, which gives $S = 441 + 2\cdot 981 = 2403$.

The probability is $\frac{2403}{10395} = \frac{27\cdot 89}{27\cdot 385} = \frac{89}{385}$, and $m + n = 89 + 385 =$ **474**.

Check. The weights are right on a small case: with two couples the orders $ABAB$-type (height $1, 2, 1, 0$) have $W = 2\cdot 1 = 2$ and the orders $AABB$-type (height $1, 0, 1, 0$) have $W = 1$, total $3 = 3\cdot 1$, matching $\frac{4!}{2^2\cdot 2!} = 3$. With 2 or 3 couples the ceiling cannot be reached, and the recursion indeed gives the full totals 3 and $15 = 5\cdot 3\cdot 1$ (the first coordinates after 4 and 6 arrivals). With 4 couples it gives $81 = 105 - 24$, and the only excluded paths are those climbing straight to height 4, of total weight $4\cdot 3\cdot 2\cdot 1 = 24$. And a direct computer run over all 10395 ways of pairing up the 12 arrival times finds exactly 2403 in which at most 3 people ever wait. The answer is **474**.

12. Answer: 101**. The idea.** By the law of sines, the circumradius of PBC is $\frac{BC}{2\sin\angle BPC}$. So equal radii ρ mean $\sin\angle BPC = \frac{a}{2\rho}$, $\sin\angle CPA = \frac{b}{2\rho}$, $\sin\angle APB = \frac{c}{2\rho}$, where also $\angle BPC + \angle CPA + \angle APB = 360^\circ$. Comparing with the triangle's own circumradius R (where $\sin A = \frac{a}{2R}$) forces $\rho = R$ and $\angle BPC = 180^\circ - A$ and so on, and the one interior point with those angles is the orthocentre H. Then the answer is $HA + HB + HC$, and $HA = 2R\cos A$.

Work it out. Write $x = \angle BPC$, $y = \angle CPA$, $z = \angle APB$, so $x + y + z = 360^\circ$. Since P is strictly inside, $x > A$, $y > B$, $z > C$ (a point inside the triangle sees BC at a larger angle than A does). The triangle 13-14-15 is acute ($15^2 < 13^2 + 14^2$), so $A, B, C < 90^\circ$. If $\rho > R$, then $\sin x = \frac{a}{2\rho} < \sin A$. Because $x > A$ and sin increases up to 90°, this forces $x > 90^\circ$ and then $x > 180^\circ - A$. Likewise $y > 180^\circ - B$ and $z > 180^\circ - C$, so $x + y + z > 540^\circ - 180^\circ = 360^\circ$: impossible. If $\rho < R$, then $\sin x > \sin A$ with $x > A$ gives $A < x < 180^\circ - A$, and similarly, so $x+y+z < 360^\circ$: impossible. So $\rho = R$, $\sin x = \sin A$ with $x > A$, hence $x = 180^\circ - A$, and likewise $y = 180^\circ - B$, $z = 180^\circ - C$. The orthocentre H of an acute triangle has exactly these angles (if E, F are the feet of the altitudes from B, C, the quadrilateral $AFHE$ has right angles at E and F, so $\angle BHC = \angle FHE = 180^\circ - A$), and the points inside the triangle seeing BC at $180^\circ - A$ form one circular arc, the points seeing CA at $180^\circ - B$ another; two different circles through C meet at most once more, so $P = H$. For 13-14-15 the area is 84, so $R = \frac{13\cdot 14\cdot 15}{4\cdot 84} = \frac{65}{8}$. With $a = 14$, $b = 15$, $c = 13$: $\cos A = \frac{15^2+13^2-14^2}{2\cdot 15\cdot 13} = \frac{33}{65}$, $\cos B = \frac{14^2+13^2-15^2}{2\cdot 14\cdot 13} = \frac{5}{13}$, $\cos C = \frac{14^2+15^2-13^2}{2\cdot 14\cdot 15} = \frac{3}{5}$. So $HA = 2R\cos A = \frac{65}{4}\cdot\frac{33}{65} = \frac{33}{4}$, $HB = \frac{65}{4}\cdot\frac{5}{13} = \frac{25}{4}$, $HC = \frac{65}{4}\cdot\frac{3}{5} = \frac{39}{4}$, and $PA + PB + PC = \frac{97}{4}$. So $m + n = 101$.

Check. In coordinates $B = (0, 0)$, $C = (14, 0)$, $A = (5, 12)$ the altitude from A is $x = 5$ and the altitude from B is perpendicular to CA (direction $(-9, 12)$), i.e. $y = \frac{3}{4}x$; so $H = (5, \frac{15}{4})$. Then $HA = 12 - \frac{15}{4} = \frac{33}{4}$, $HB = \sqrt{25 + \frac{225}{16}} = \frac{25}{4}$, $HC = \sqrt{81 + \frac{225}{16}} = \frac{39}{4}$. Also $HA + HB + HC = 2(R + r) = 2(\frac{65}{8} + 4) = \frac{97}{4}$. The answer is $\boxed{101}$.

13. Answer: 148**. The idea.** Eliminate c: since $c \equiv -(a + b) \pmod n$, the condition is $n \mid 2(a^2 + ab + b^2)$. The quadratic form $a^2 + ab + b^2$ is where the arithmetic lives: $4(a^2 + ab + b^2) =$

$(2a+b)^2+3b^2$, so a prime $p \neq 3$ can divide it with $p \nmid b$ only if -3 is a square mod p, which happens exactly when $p \equiv 1 \pmod 3$. The answer is: n works unless $n = 2^e m$ with $e \leq 2$ and m a product of *distinct* primes, all $\equiv 2 \pmod 3$.

Work it out. *Constructions* (each gives positive a, b, c with $a+b+c = n$). (i) If $3 \mid n$: $a = b = c = \frac{n}{3}$, and $a^2+b^2+c^2 = \frac{n^2}{3} = n \cdot \frac{n}{3}$. (ii) If $8 \mid n$: $a = b = \frac{n}{4}$, $c = \frac{n}{2}$, and $a^2+b^2+c^2 = \frac{3n^2}{8} = n \cdot \frac{3n}{8}$. (iii) If $p^2 \mid n$ for an odd prime p: $a = b = \frac{n}{p}$, $c = n - \frac{2n}{p} \geq \frac{n}{p}$. Then $2(a^2+ab+b^2) = \frac{6n^2}{p^2} = n \cdot \frac{6n}{p^2}$. (iv) If a prime $p \equiv 1 \pmod 3$ divides n: the nonzero residues mod p form a cyclic group of order $p-1$, divisible by 3, so some $\lambda \not\equiv 1$ has $\lambda^3 \equiv 1$, i.e. $\lambda^2+\lambda+1 \equiv 0$; take $1 \leq \lambda \leq p-1$, and note $\lambda \neq p-1$ (that would give $1-1+1=1$). Put $a = \frac{n}{p}$, $b = \lambda\frac{n}{p}$, $c = n - (1+\lambda)\frac{n}{p} \geq \frac{n}{p}$. Then $2(a^2+ab+b^2) = 2\left(\frac{n}{p}\right)^2(1+\lambda+\lambda^2)$, and $p \mid 1+\lambda+\lambda^2$ makes this a multiple of $2 \cdot \frac{n}{p} \cdot n$. *Impossibility for the rest.* Every n not covered is $n = 2^e m$ with $e \in \{0,1,2\}$ and m odd, squarefree, with all prime factors $\equiv 2 \pmod 3$. Take such n and suppose $a+b+c = n$ works. For a prime $p \mid m$ we get $p \mid (2a+b)^2+3b^2$. If $p \nmid b$, then $u = (2a+b)b^{-1}$ has $u^2 \equiv -3$, and $\omega = \frac{u-1}{2}$ satisfies $\omega^2+\omega+1 \equiv 0$, so $\omega^3 \equiv 1$ with $\omega \not\equiv 1$ (as $3 \not\equiv 0$): an element of order 3 mod p, forcing $3 \mid p-1$, a contradiction. So $p \mid b$, then $p \mid a$, then $p \mid c$. Hence m divides a, b, c; write $a = ma'$ etc. Now $a'+b'+c' = 2^e \leq 4$ and $2^e m \mid m^2(a'^2+b'^2+c'^2)$, so $2^e \mid a'^2+b'^2+c'^2$. Three positive integers cannot sum to 1 or 2, and for sum 4 they are $1,1,2$ with $1+1+4 = 6 \not\equiv 0 \pmod 4$. So these n fail. *Count the failures up to* 200. The odd primes $\equiv 2 \pmod 3$ up to 200 are $5, 11, 17, 23, 29, 41, 47$ (up to 50), then $53, 59, 71, 83, 89$ (up to 100), then $101, 107, 113, 131, 137, 149, 167, 173, 179, 191, 197$: 23 in all. Products of two of them up to 200: 55, 85 (up to 100) and 115, 145, 187. Counting $m = 1$ too: $\#\{m \leq 200\} = 1+23+5 = 29$, $\#\{m \leq 100\} = 1+12+2 = 15$, $\#\{m \leq 50\} = 1+7 = 8$. The failures are $n = m, 2m, 4m$, so there are $29+15+8 = 52$ of them, and $200-52 = 148$ values of n work.

Check. Small values: $n = 1, 2, 4, 5$ fail (for 4 the only split is $1+1+2$, sum of squares 6; for 5 the splits are $1+1+3$ and $1+2+2$, with sums of squares 11 and 9), while $n = 3$ $(1+1+1)$, $n = 6$ $(2+2+2 \to 12)$ and $n = 7$ $(1+2+4 \to 21$, using $\lambda = 2)$ work. A computer search over all splits of every $n \leq 200$ confirms 148. The answer is $\boxed{148}$.

14. Answer: 467**. The idea.** Count vertex by vertex. By linearity of expectation the answer is $\sum_{v \neq S} p_v$, where p_v is the probability that the ant reaches v before it returns to S. The seven vertices fall into three classes by their distance from S (3 at distance 1, 3 at distance 2, 1 at distance 3). For each class, a symmetry of the cube that *swaps* S and v does most of the work: if $h(x)$ is the probability of reaching v before S when starting from x, and σ swaps S and v, then starting from $\sigma(x)$ the roles are exchanged, so $h(x)+h(\sigma(x)) = 1$.

Work it out. Label the vertices by strings of three bits, with $S = 000$; two vertices are joined when they differ in one bit. Always $h(v) = 1$ and $h(S) = 0$, every other $h(x)$ is the average of h over the three neighbours of x, and p_v is the average of h over the three neighbours of S. *Distance* 3*,* $v = 111$. Flipping all three bits swaps S and v and turns a vertex at distance 1 from S into one at distance 2. By the symmetry among the three bits, h is some h_1 at distance 1 and h_2 at distance 2, and $h_1+h_2 = 1$. A vertex at distance 1 has neighbours S and two vertices at distance 2: $h_1 = \frac{2}{3}h_2 = \frac{2}{3}(1-h_1)$, so $h_1 = \frac{2}{5}$ and $p_{111} = \frac{2}{5}$. *Distance* 1*,* $v = 100$. Flipping the first bit swaps S and v. Put $a = h(010) = h(001)$ (swapping the last two bits fixes S and v), $c = h(011)$. The first-bit flip gives $h(110) = 1-a$ and $h(111) = 1-c$. Averaging at 010 (neighbours $000, 110, 011$): $a = \frac{1}{3}(0+(1-a)+c)$, so $c = 4a-1$. Averaging at 011 (neighbours $111, 001, 010$): $c = \frac{1}{3}((1-c)+2a)$, so $4c = 1+2a$. Hence $16a-4 = 1+2a$, $a = \frac{5}{14}$, and $p_{100} = \frac{1}{3}\left(1+\frac{5}{14}+\frac{5}{14}\right) = \frac{4}{7}$.

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Distance 2, $v = 110$. Flipping the first two bits swaps S and v; swapping the first two bits fixes both. So $h(100) = h(010)$ and $h(100) + h(010) = 1$: both are $\frac{1}{2}$. Likewise $h(101) = h(011)$ and they sum to 1: both are $\frac{1}{2}$. At 001 (neighbours 000, 101, 011): $h(001) = \frac{1}{3}(0 + \frac{1}{2} + \frac{1}{2}) = \frac{1}{3}$. So $p_{110} = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3}\right) = \frac{4}{9}$. Adding up: $3 \cdot \frac{4}{7} + 3 \cdot \frac{4}{9} + \frac{2}{5} = \frac{12}{7} + \frac{4}{3} + \frac{2}{5} = \frac{180+140+42}{105} = \frac{362}{105}$. Since $362 = 2 \cdot 181$ shares no factor with $105 = 3 \cdot 5 \cdot 7$, $m + n = 362 + 105 = 467$.

Check. Each p_v must exceed the chance of stepping onto v at once and be below 1: $p_{100} = \frac{4}{7} > \frac{1}{3}$. Electrical networks give the same numbers: with every edge a 1-ohm resistor, $p_v = \frac{1}{3R(S,v)}$, and the cube's effective resistances are $R = \frac{7}{12}, \frac{3}{4}, \frac{5}{6}$ at distances 1, 2, 3, giving $\frac{4}{7}, \frac{4}{9}, \frac{2}{5}$. An exact solve of the seven linear systems agrees, and a simulation of 10^5 walks averages about $3.45 \approx \frac{362}{105}$. The answer is $\boxed{467}$.

15. Answer: 80**. The idea.** Two tools. First, the Prüfer code: repeatedly delete the leaf (degree-1 point) with the smallest label and write down the label of its neighbour, until two points are left. This gives a sequence of length 8 with entries in $\{1, \ldots, 10\}$, it is a one-to-one correspondence between trees and all 10^8 such sequences, and a point of degree d appears exactly $d - 1$ times in the code. So "every degree odd" means "every label appears an *even* number of times." Second, even multiplicities are counted with a ± 1 filter.

Work it out. *Why a point of degree d appears $d - 1$ times.* Each time a leaf is deleted, its neighbour's label is written once and that neighbour loses one segment. A point is written once for each of its segments except the last one: its last segment disappears either when the point itself is deleted as a leaf, or at the end, when it is one of the final two points. (The decoding is also unique: the first deleted leaf is the smallest label not in the code, and so on, so every sequence comes from exactly one tree.) *The filter.* For a sequence $s = (s_1, \ldots, s_8)$ and signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{10}) \in \{\pm 1\}^{10}$, look at $\varepsilon_{s_1}\varepsilon_{s_2} \cdots \varepsilon_{s_8} = \prod_i \varepsilon_i^{c_i}$, where c_i is the number of times i occurs. Summed over all 2^{10} sign vectors this is 2^{10} if every c_i is even and 0 otherwise (if some c_i is odd, flipping ε_i pairs the terms off with opposite signs). Summing over all sequences as well, $2^{10}N = \sum_\varepsilon \left(\sum_{i=1}^{10} \varepsilon_i\right)^8 = \sum_{k=0}^{10} \binom{10}{k}(10 - 2k)^8$, where k counts the minus signs. The terms for k and $10 - k$ are equal and the $k = 5$ term is 0, so $2^{10}N = 2\left(10^8 + 10 \cdot 8^8 + 45 \cdot 6^8 + 120 \cdot 4^8 + 210 \cdot 2^8\right)$. The terms are 100,000,000, 167,772,160, 75,582,720, 7,864,320 and 53,760, with sum 351,272,960. So $N = \frac{2 \cdot 351,272,960}{1024} = \frac{351,272,960}{512} = 686,080$, and the remainder is 80.

Check. Small case: with 4 points, a tree with all degrees odd must have degrees 3, 1, 1, 1 (a star), and there are 4 stars; the formula gives $\frac{2(4^2+4\cdot 2^2)}{2^4} = \frac{64}{16} = 4$. Parity check: the degrees add up to $2 \cdot 9 = 18$, an even total, which ten odd numbers can make. A computer decoding of every Prüfer sequence for 6 and 8 points confirms the method, and a direct count of the length-8 sequences with all multiplicities even gives 686,080. The answer is $\boxed{080}$.

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