MathCastle

AIME

Hard Edition

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

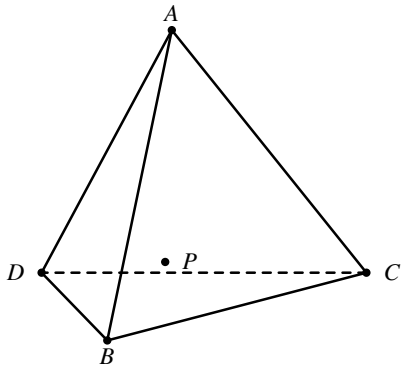
1. For a positive integer n, let $\sigma(n)$ be the sum of the positive divisors of n. For example, $\sigma(12) = 1+2+3+4+6+12 = 28$. For how many integers n with $1 \le n \le 100$ is $\sigma(n)$ divisible by 3?

2. The complex number z ranges over every point of the closed disk $|z-3| \le 1$. Arguments are measured in the interval $(-\pi, \pi]$. Let θ be the largest value that the argument of z^3 takes as z ranges over the disk. Then $\tan^2\theta = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3. Each of the 16 cells of a 4×4 grid is either shaded or left blank. Call a shading *lopsided* if the four rows contain four different numbers of shaded cells and the four columns also contain four different numbers of shaded cells. Let N be the number of lopsided shadings. Find the remainder when N is divided by 1000.

4. Over all nondegenerate triangles ABC, the largest possible value of $3\cos A + 4\cos B + 5\cos C$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

5. A regular tetrahedron $ABCD$ has edge length 10. A point P inside it has distances to the faces BCD, ACD, ABD, ABC in the ratio $1:2:3:4$. Find $PA^2+PB^2+PC^2+PD^2$.



6. As x ranges over all real numbers, the largest value of $(17+8\cos x+15\sin x)(13+12\cos x+5\sin x)$ can be written as $a+b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a+b+c$.

7. Nonnegative real numbers a, b, c satisfy $a^2+b^2+c^2 = 18$. Let M be the largest possible value of $(a-b)(b-c)(c-a)$. Find M^2.

8. For a positive integer m, let $s(m)$ be the number of digits equal to 1 in the base-2 representation of m. For example, $s(13) = 3$ because $13 = 1101_2$. For how many integers n with $1 \le n \le 1000$ is $s(3n) = 2\,s(n)$?

9. Sixty stepping stones form a ring around a pond and are numbered $0, 1, 2, \ldots, 59$ clockwise. A frog starts on stone 0. Each jump carries it clockwise either to the next stone or eleven stones on, that is, from stone s to stone $s+1$ or to stone $s+11$, with numbers taken modulo 60. The frog makes 60 jumps: during the first 59 it lands on each of the stones $1, 2, \ldots, 59$ exactly once, and its 60th jump brings it back to stone 0. How many different sequences of 60 jumps do this?

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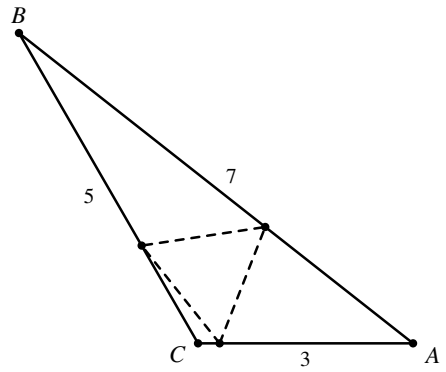
10. Triangle ABC has $AB = 13$ and $BC = 21$. The circle with diameter $\overline{BC}$ is tangent to the incircle of triangle ABC. Find AC.

11. Each of the 24 vertices of a regular 24-gon is coloured red or blue. Let N be the number of colourings in which no four vertices of the same colour are the vertices of a rectangle, and no three vertices of the same colour are the vertices of an equilateral triangle. Find the remainder when N is divided by 1000.

12. Two piles of stones hold a and b stones, where a and b are integers with $1 \le a \le 300$ and $1 \le b \le 300$. Two players move alternately. A move consists of choosing one of the piles and removing at least one stone from it, but no more than half of the stones that pile currently holds. (So a pile of 1 stone can never be touched, and from a pile of 7 stones a player may remove 1, 2 or 3.) A player who cannot move loses. For how many ordered pairs (a, b) can the player who moves second force a win?

13. Each cell of a grid with 2 rows and 20 columns of unit squares is coloured red or blue, independently, each colour with probability $\frac{1}{2}$. A *region* is a maximal set of cells of the same colour that is connected through shared edges (cells touching only at a corner are not connected). The expected number of regions is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. Triangle ABC has $CA = 3$, $CB = 5$ and $AB = 7$. An equilateral triangle is *inscribed* in ABC if each of its three vertices lies on a different side of ABC (a vertex may be an endpoint of that side). Among all inscribed equilateral triangles, let s be the smallest possible side length. Then $s^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



15. A square card lies on a table. A *move* is one of six operations: turn the card 90° clockwise, turn it 90° counterclockwise, or flip it over one of its four axes of symmetry (the two lines through the midpoints of opposite sides and the two diagonals); the axes and turning directions are fixed relative to the table. Say the card is *home* when it lies exactly as it started: same face up and every corner in its original place. Let N be the number of sequences of 2026 moves (each chosen freely from the six) after which the card is home for the first time, that is, it is home after the 2026th move but was not home after any of the moves $1, 2, \ldots, 2025$. Find the remainder when N is divided by 1000.

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Answer Key — AIME — Hard Edition Practice Contest 3

1. 65

2. 729

3. 152

4. 889

5. 160

6. 666

7. 216

8. 143

9. 422

10. 20

11. 584

12. 876

13. 115

14. 931

15. 400

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Solutions — AIME — Hard Edition Practice Contest 3

1. Answer: 65**. The idea.** σ is multiplicative: if $n = p_1^{a_1} \cdots p_k^{a_k}$ then $\sigma(n) = \sigma(p_1^{a_1}) \cdots \sigma(p_k^{a_k})$. Since 3 is prime, $3 \mid \sigma(n)$ exactly when 3 divides one of the factors $\sigma(p^a) = 1 + p + \cdots + p^a$, and that factor modulo 3 depends only on p modulo 3 and on a. It is easier to count the n for which no factor is divisible by 3 and subtract.

Work it out. Reduce $\sigma(p^a) = 1 + p + \cdots + p^a$ modulo 3: $p = 3$*:* every term after the first is a multiple of 3, so $\sigma(3^a) \equiv 1$. Never divisible. $p \equiv 1 \pmod 3$*:* each of the $a + 1$ terms is $\equiv 1$, so $\sigma(p^a) \equiv a + 1$, divisible by 3 exactly when $a \equiv 2 \pmod 3$. $p \equiv 2 \equiv -1 \pmod 3$*:* the terms alternate $1, -1, 1, \ldots$, so $\sigma(p^a) \equiv 1$ if a is even and $\equiv 0$ if a is odd. So $3 \nmid \sigma(n)$ exactly when every prime $p \equiv 2 \pmod 3$ occurs in n to an even power and every prime $p \equiv 1 \pmod 3$ occurs to a power $\not\equiv 2 \pmod 3$. Up to 100 the only square of a prime $\equiv 1 \pmod 3$ is $7^2 = 49$ (a bad exponent), and $7^3 > 100$; so such a prime can occur only to the first power. Hence the "bad" $n \le 100$ (with $3 \nmid \sigma(n)$) are exactly $n = 3^e \cdot s^2 \cdot m$, where s is a product of primes $\equiv 2 \pmod 3$ and m is a product of distinct primes $\equiv 1 \pmod 3$. The primes $\equiv 1 \pmod 3$ below 100 are $7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97$, and the only product of two of them that is at most 100 is $7 \cdot 13 = 91$. So $m \in \{1, 7, 13, \ldots, 97, 91\}$, 13 values. The possible $s^2 \le 100$ are $1, 4, 16, 25, 64, 100$. Let $G(X)$ count the numbers $s^2 m \le X$. Then $G(100)$: $s^2 = 1$ gives 13; $s^2 = 4$ needs $m \le 25$: $m = 1, 7, 13, 19$, giving 4; $s^2 = 16, 25, 64, 100$ need $m = 1$: 4 more. Total 21. $G(33)$: $m \le 33$: $1, 7, 13, 19, 31$ (5); $4m \le 33$: $m = 1, 7$ (2); 16 and 25 (2). Total 9. $G(11) = 3$ (namely $1, 7, 4$), $G(3) = 1$, $G(1) = 1$. With $e = 0, 1, 2, 3, 4$ the number of bad n is $G(100) + G(33) + G(11) + G(3) + G(1) = 21 + 9 + 3 + 1 + 1 = 35$, so the answer is $100 - 35 = 65$.

Check. Spot checks: $\sigma(2) = 3$ and $\sigma(5) = 6$ are divisible by 3 (odd power of a prime $\equiv 2$); $\sigma(49) = 57 = 3 \cdot 19$ is too (exponent 2 on a prime $\equiv 1$); but $\sigma(91) = 8 \cdot 14 = 112$ and $\sigma(36) = 7 \cdot 13 = 91$ are not, matching $91 = 7 \cdot 13$ and $36 = 3^2 \cdot 2^2$ in the bad list. Adding up the divisors of every $n \le 100$ by computer also gives 65. Answer $\boxed{65}$.

2. Answer: 729**. The idea.** Cubing triples arguments, so first find the widest angle at which the disk is seen from the origin (along a tangent line), then triple it, checking that the tripled angle does not wrap past π.

Work it out. The disk has centre 3 and radius 1, so from the origin it is seen inside the angle $-\varphi \le \arg z \le \varphi$, where the ray at angle φ is tangent to the circle. In the right triangle formed by 0, the centre 3 and the point of tangency, the hypotenuse is 3 and the side opposite φ is the radius 1: $\sin\varphi = \frac{1}{3}$, $\cos\varphi = \frac{2\sqrt{2}}{3}$, $\tan\varphi = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$. Every argument $\alpha \in [-\varphi, \varphi]$ is attained, and $\arg z^3$ is 3α reduced into $(-\pi, \pi]$. Since $\sin\varphi = \frac{1}{3} < \frac{1}{2}$, $\varphi < 30°$, so 3α lies in $(-90°, 90°)$ and needs no reduction. Hence $\theta = 3\varphi$. Triple-angle formula with $t = \tan\varphi$, $t^2 = \frac{1}{8}$, $t^3 = \frac{\sqrt{2}}{32}$: $\tan 3\varphi = \frac{3t - t^3}{1 - 3t^2} = \frac{\frac{3\sqrt{2}}{4} - \frac{\sqrt{2}}{32}}{1 - \frac{3}{8}} = \frac{23\sqrt{2}}{32} \cdot \frac{8}{5} = \frac{23\sqrt{2}}{20}$. So $\tan^2\theta = \frac{529 \cdot 2}{400} = \frac{529}{200}$, and $m + n = 729$.

Check. Via sines: $\sin 3\varphi = 3\sin\varphi - 4\sin^3\varphi = 1 - \frac{4}{27} = \frac{23}{27}$, so $\cos^2 3\varphi = 1 - \frac{529}{729} = \frac{200}{729}$ and $\tan^2 3\varphi = \frac{529}{200}$. Numerically $\varphi \approx 19.47°$ and $\theta \approx 58.41°$, $\tan^2 58.41° \approx 2.645 = \frac{529}{200}$. Answer $\boxed{729}$.

3. Answer: 152**. The idea.** A line holds between 0 and 4 shaded cells, so four different row counts use four of the five values $0, 1, 2, 3, 4$ and miss exactly one; the same goes for the columns. Both

lists add up to the total number of shaded cells, which forces the same missing value, and a full line cannot live alongside an empty line in the other direction. After that the whole grid turns out to be forced.

Work it out. Say the rows miss the value r and the columns miss c. The row counts add to $10-r$ and the column counts to $10-c$; both equal the number of shaded cells, so $r=c$. If r were 1, 2 or 3, some row would have all 4 cells shaded, so every column would contain at least one shaded cell, yet some column has count 0. Contradiction. So the missing value is 4 (counts $0,1,2,3$ both ways) or 0 (counts $1,2,3,4$ both ways). Swapping shaded and blank everywhere turns every count k into $4-k$ and matches the two cases one to one, so count the first and double. Counts $0,1,2,3$: choose which row gets which count (4! ways) and which column gets which count (4! ways). We show the grid is then determined. The row with 3 shaded cells must avoid the empty column, so it fills the other three columns. That already gives the column of count 1 its cell, so the rows of counts 1 and 2 must avoid both that column and the empty one: they only use the columns of counts 2 and 3, which still need 1 and 2 more cells. The row of count 2 must take both of these cells, and then the column of count 3 still needs one more, which is the row of count 1. Every count now checks out, so there is exactly one grid. Hence $N=2\cdot 4!\cdot 4!=2\cdot 576=1152$, and the remainder is 152.

Check. The same argument for a 3×3 grid gives $2\cdot 3!\cdot 3!=72$ shadings, and a direct computer count of all 2^9 shadings finds 72. For the 4×4 grid a direct count of all $2^{16}=65{,}536$ shadings finds 1152. Answer $\boxed{152}$.

4. Answer: 889**. The idea.** Cosines of the angles are (minus) dot products of unit vectors along the sides. A squared length $|u\mathbf{e}_1+v\mathbf{e}_2+w\mathbf{e}_3|^2\ge 0$ produces exactly a weighted cosine sum, and choosing u,v,w to match the weights $3,4,5$ gives a sharp upper bound. Equality needs a triangle whose sides are proportional to u,v,w.

Work it out. Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be unit vectors in the directions $\overrightarrow{BC},\overrightarrow{CA},\overrightarrow{AB}$. The angle between $\overrightarrow{CA}$ and $\overrightarrow{AB}$ is $180^\circ-A$, so $\mathbf{e}_2\cdot\mathbf{e}_3=-\cos A$; likewise $\mathbf{e}_3\cdot\mathbf{e}_1=-\cos B$ and $\mathbf{e}_1\cdot\mathbf{e}_2=-\cos C$. For any reals u,v,w: $0\le|u\mathbf{e}_1+v\mathbf{e}_2+w\mathbf{e}_3|^2=u^2+v^2+w^2-2vw\cos A-2wu\cos B-2uv\cos C$. Pick positive u,v,w with $vw=3k$, $wu=4k$, $uv=5k$. Then $u^2=\frac{(wu)(uv)}{vw}=\frac{20k}{3}$, $v^2=\frac{15k}{4}$, $w^2=\frac{12k}{5}$, and the inequality reads $2k(3\cos A+4\cos B+5\cos C)\le k\left(\frac{20}{3}+\frac{15}{4}+\frac{12}{5}\right)$, so $3\cos A+4\cos B+5\cos C\le\frac{1}{2}\cdot\frac{400+225+144}{60}=\frac{769}{120}$. Equality needs $u\mathbf{e}_1+v\mathbf{e}_2+w\mathbf{e}_3=\mathbf{0}$. Since $a\mathbf{e}_1+b\mathbf{e}_2+c\mathbf{e}_3=\mathbf{0}$ (walking around the triangle), and $\mathbf{e}_1,\mathbf{e}_2$ are independent, this happens exactly when $a:b:c=u:v:w=\sqrt{400}:\sqrt{225}:\sqrt{144}=20:15:12$. Since $12+15>20$, that triangle exists, so the maximum is $\frac{769}{120}$ and $m+n=889$.

Check. In the triangle with $a=20$, $b=15$, $c=12$: $\cos A=\frac{225+144-400}{360}=-\frac{31}{360}$, $\cos B=\frac{400+144-225}{480}=\frac{319}{480}$, $\cos C=\frac{400+225-144}{600}=\frac{481}{600}$. Then $3\cos A+4\cos B+5\cos C=-\frac{93}{360}+\frac{1276}{480}+\frac{2405}{600}=\frac{-93+957+1443}{360}=\frac{2307}{360}=\frac{769}{120}$. The bound is attained. Answer $\boxed{889}$.

5. Answer: 160**. The idea.** Cutting the tetrahedron into four pyramids with apex P shows that the four face distances add up to the height, and that each distance, as a fraction of the height, is the weight of the opposite vertex in P's barycentric coordinates. So $P=\frac{1}{10}(A+2B+3C+4D)$. Then the sum of squared distances follows from its distance to the centroid.

Work it out. Let the faces have area S, the height be h and the distances be d_A,d_B,d_C,d_D (to the faces opposite A,B,C,D). The four pyramids $PBCD$, $PACD$, $PABD$, $PABC$ fill the tetrahedron,

so $\frac{1}{3}S(d_A+d_B+d_C+d_D)=\frac{1}{3}Sh$: the distances sum to h, and they are $\frac{h}{10},\frac{2h}{10},\frac{3h}{10},\frac{4h}{10}$. Along the line from A the distance to face BCD changes linearly, so writing $P=\alpha A+\beta B+\gamma C+\delta D$ with $\alpha+\beta+\gamma+\delta=1$ gives $d_A=\alpha h$, and so on. Thus $(\alpha,\beta,\gamma,\delta)=\left(\frac{1}{10},\frac{2}{10},\frac{3}{10},\frac{4}{10}\right)$. Let G be the centroid. For any point P, $\sum_X PX^2=\sum_X GX^2+4\,PG^2$ (expand $|(P-G)+(G-X)|^2$; the cross terms cancel because $\sum_X(G-X)=\mathbf{0}$). The circumradius of a regular tetrahedron with edge 10 satisfies $GX^2=\frac{3}{8}\cdot 100=\frac{75}{2}$, so $\sum GX^2=150$. Next $P-G=\sum c_X X$ with $c=\left(-\frac{3}{20},-\frac{1}{20},\frac{1}{20},\frac{3}{20}\right)$, which sums to 0. For coefficients summing to zero, $|\sum c_X X|^2=-\sum_{X<Y}c_X c_Y\,XY^2$, and every edge is 10, so $PG^2=-100\sum_{X<Y}c_X c_Y=100\cdot\frac{\sum c_X^2}{2}=50\cdot\frac{9+1+1+9}{400}=\frac{5}{2}$. Hence the sum is $150+4\cdot\frac{5}{2}=160$.

Check. Compute each distance with the barycentric distance formula $PX^2=\sum_Y w_Y\,XY^2-\sum_{Y<Z}w_Y w_Z\,YZ^2$. Here $\sum_{Y<Z}w_Y w_Z=\frac{1-(1+4+9+16)/100}{2}=0.35$, so $PA^2=100(1-0.1)-35=55$, $PB^2=80-35=45$, $PC^2=70-35=35$, $PD^2=60-35=25$. Their sum is 160. Answer $\boxed{160}$.

6. Answer: 666. **The idea.** In each factor the constant equals the amplitude of the wave: $\sqrt{8^2+15^2}=17$ and $\sqrt{12^2+5^2}=13$. So each factor has the form $A(1+\cos\theta)=2A\cos^2\frac{\theta}{2}$, and the product is a perfect square of a product of two cosines, which product-to-sum turns into a constant plus a single cosine.

Work it out. Let α,β be the acute angles with $\cos\alpha=\frac{8}{17}$, $\sin\alpha=\frac{15}{17}$ and $\cos\beta=\frac{12}{13}$, $\sin\beta=\frac{5}{13}$. Then $8\cos x+15\sin x=17\cos(x-\alpha)$ and $12\cos x+5\sin x=13\cos(x-\beta)$, so the product is $221\,(1+\cos(x-\alpha))(1+\cos(x-\beta))=221\,(2\cos p\cos q)^2$, $\quad p=\frac{x-\alpha}{2}$, $q=\frac{x-\beta}{2}$. Now $2\cos p\cos q=\cos(p-q)+\cos(p+q)=\cos\frac{\beta-\alpha}{2}+\cos\left(x-\frac{\alpha+\beta}{2}\right)$. The first term is a constant k, positive because $|\beta-\alpha|<\pi$, and the second takes every value in $[-1,1]$. So $2\cos p\cos q$ ranges over $[k-1,k+1]$, and its square is largest at $(k+1)^2$. To find k: $\cos(\beta-\alpha)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\frac{96+75}{221}=\frac{171}{221}$, so $k^2=\frac{1+\frac{171}{221}}{2}=\frac{196}{221}$ and $k=\frac{14}{\sqrt{221}}$. The maximum is $221\left(1+\frac{14}{\sqrt{221}}\right)^2=221+28\sqrt{221}+196=417+28\sqrt{221}$. Since $221=13\cdot 17$ is squarefree, $a+b+c=417+28+221=666$.

Check. Independently, put $t=\tan\frac{x}{2}$, so $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$. The factors become $\frac{9t^2+30t+25}{1+t^2}=\frac{(3t+5)^2}{1+t^2}$ and $\frac{t^2+10t+25}{1+t^2}=\frac{(t+5)^2}{1+t^2}$, so the product is $g(t)^2$ with $g(t)=\frac{(3t+5)(t+5)}{1+t^2}=3+\frac{20t+22}{1+t^2}$. The value v of $\frac{20t+22}{1+t^2}$ is attained when $vt^2-20t+(v-22)=0$ has a real root, i.e. $100\ge v(v-22)$, so $v\le 11+\sqrt{221}$; the negative values of g are at least $14-\sqrt{221}>-1$, and $x=\pi$ gives only 9. So the maximum is $(14+\sqrt{221})^2=417+28\sqrt{221}\approx 833.2$, which a numerical scan over x confirms. Answer $\boxed{666}$.

7. Answer: 216. **The idea.** The product only sees differences. Sliding all three numbers down by the smallest one keeps the product, keeps them nonnegative, and shrinks $a^2+b^2+c^2$; scaling back up to 18 then multiplies the (positive) product by a factor bigger than 1. So at the maximum the smallest number is 0, and a two-variable problem remains.

Work it out. The product is positive only for suitable orderings, e.g. $a<b<c$; and it is unchanged by cycling (a,b,c), so we may take a smallest. Let $m=a\ge 0$. The triple $(0,b-m,c-m)$ has the same product and, if $m>0$, a strictly smaller sum of squares ($(b-m)^2<b^2$ since $b>m>0$). Multiplying it by $\lambda>1$ to bring the sum of squares back to 18 multiplies the product by $\lambda^3>1$. So a maximum has $a=0$. With $a=0$ the product is $(-b)(b-c)(c)=bc(c-b)$,

where $b^2 + c^2 = 18$ and $0 \le b < c$. Put $p = c - b > 0$; then $p^2 = 18 - 2bc$, so $bc = \frac{18-p^2}{2}$ and the product is $f = \frac{p(18-p^2)}{2}$. Let $x = p^2$. Then $f^2 = \frac{x(18-x)^2}{4} = \frac{(2x)(18-x)(18-x)}{8}$. By AM–GM on $2x$, $18-x$, $18-x$ (sum 36), their product is at most $12^3 = 1728$, so $f^2 \le 216$, with equality at $2x = 18 - x$, i.e. $x = 6$. That case is achievable: $c - b = \sqrt{6}$, $bc = 6$ gives $b = \frac{\sqrt{30}-\sqrt{6}}{2} > 0$, $c = \frac{\sqrt{30}+\sqrt{6}}{2}$. So $M^2 = 216$ ($M = 6\sqrt{6}$).

Check. Without the sign condition the answer is much larger: shifting to mean 0 makes a, b, c roots of $x^3 - 9x - q$, whose discriminant $4 \cdot 9^3 - 27q^2 = (a-b)^2(b-c)^2(c-a)^2$ peaks at 2916. The constraint $a, b, c \ge 0$ is what pins the smallest value at 0. Numerically, $b \approx 1.514$, $c \approx 3.964$: $bc(c-b) \approx 6.000 \cdot 2.449 \approx 14.70 = \sqrt{216}$. Answer $\boxed{216}$.

8. Answer: 143**. The idea.** Write $3n = n + 2n$ and add in binary. Each carry takes two 1s in one column and replaces them by a single 1 in the next column, so it lowers the number of 1s by exactly one: $s(x + y) = s(x) + s(y) -$ (number of carries). Since $2n$ is n shifted one place left, $s(2n) = s(n)$, and the condition $s(3n) = 2s(n)$ says the addition $n + 2n$ has no carries at all.

Work it out. The addition $n + 2n$ is carry-free exactly when n and $2n$ never have a 1 in the same column (if they share a 1, the lowest shared column produces a carry; if they share none, no column ever holds two 1s). A 1 of $2n$ in column $k + 1$ comes from a 1 of n in column k, so the condition is: the binary representation of n has no two adjacent 1s.

Count such n from 1 to 1000. Let a_L be the number of binary strings of length L with no two adjacent 1s. A string either ends in 0 (then any good string of length $L - 1$ precedes it) or ends in 01, so $a_L = a_{L-1} + a_{L-2}$ with $a_1 = 2$, $a_2 = 3$: $a_L = 2, 3, 5, 8, 13, 21, 34, 55, 89$ for $L = 1, \ldots, 9$.

Numbers below $2^9 = 512$ are strings of length 9 (with leading zeros); excluding the zero string gives $89 - 1 = 88$. Numbers from 512 to 1023 have 10 bits, starting with 1; the next bit must be 0, and the last 8 bits form any good string: $a_8 = 55$ numbers. The largest of them is $1010101010_2 = 682 < 1000$, so all 55 are in range. Total $88 + 55 = 143$.

Check. Small cases: $n = 5 = 101_2$ gives $3n = 15 = 1111_2$, and $s(15) = 4 = 2s(5)$; $n = 3 = 11_2$ gives $9 = 1001_2$, and $s(9) = 2 \ne 4$. Among $1 \le n \le 7$, the good values are $1, 2, 4, 5$, which is $a_3 - 1 = 4$, as the count predicts.

9. Answer: 422**. The idea.** In such a tour every stone is left exactly once and entered exactly once. If the frog enters stone y with a short jump (from $y - 1$), then the stone $y - 11$ may not use a long jump, or y would be entered twice. This "forcing" spreads the choice of jump along steps of 10, so the choice depends only on the last digit of the stone. Both jumps move the last digit forward by 1, which makes the tour easy to follow ten jumps at a time.

Work it out. In a valid tour, let S be the set of stones the frog leaves with a short jump (a +1). If $x \in S$, stone $x + 1$ is entered from x, so it is not entered from $x - 10$: the stone $x - 10$ also jumps short, $x - 10 \in S$. Stepping back by 10 six times returns to x (as $6 \cdot 10 = 60$), so S is a union of whole residue classes modulo 10. Conversely, take any set T of residues mod 10 and let $f(x) = x + 1$ if $x \bmod 10 \in T$ and $f(x) = x + 11$ otherwise. Then f is one-to-one: $x + 1 = y + 11$ would force $y = x - 10$, which lies in the same class as x and therefore makes the same kind of jump. So the valid tours are exactly the maps f of this kind for which, starting from 0, the frog visits all 60 stones before returning, that is, f is a single 60-cycle. Different T give different jump sequences. Both jumps raise the residue mod 10 by 1, so ten jumps from x pass through each residue once: $j = |T|$ short jumps and $10 - j$ long ones, for a total of $j + 11(10 - j) = 10(11 - j)$. Hence $f^{10}(x) = x + 10(11 - j)$ for every x. The orbit of 0 under

f covers all 60 stones exactly when its visits to the class $\{0, 10, \ldots, 50\}$, which come every tenth jump, cover all 6 of those stones, i.e. when adding $10(11-j)$ generates the multiples of 10 modulo 60: $\gcd(11-j, 6) = 1$. With $0 \le j \le 10$, $11-j \in \{1, 5, 7, 11\}$, so $j \in \{10, 6, 4, 0\}$. The number of tours is $\binom{10}{10} + \binom{10}{6} + \binom{10}{4} + \binom{10}{0} = 1 + 210 + 210 + 1 = 422$.

Check. $j = 10$ is the all-short tour $0, 1, 2, \ldots$ and $j = 0$ the all-long tour, which works because $\gcd(11, 60) = 1$. For $j = 5$, ten jumps advance $60 \equiv 0$ and the frog is back at 0 after only 10 jumps, as predicted. On a small ring of 12 stones with jumps +1 and +5 the same reasoning gives $2^4 - \binom{4}{2} = 10$ tours, and a computer search finds 10; a computer search of the 60-stone ring finds 422. Answer $\boxed{422}$.

10. Answer: 20**. The idea.** Put everything on line BC. Let M be the midpoint of BC (the centre of the diameter circle, radius $\frac{a}{2}$), I the incentre, r the inradius and D the point where the incircle touches BC. Then $ID \perp BC$, so $MI^2 = r^2 + MD^2$, and MD is easy in terms of the sides. Tangency becomes an equation in the side lengths.

Work it out. Write $a = BC = 21$, $b = CA$, $c = AB = 13$, $s = \frac{a+b+c}{2}$. The tangent length is $BD = s - b$, so $MD = \left|\frac{a}{2} - (s-b)\right| = \frac{|b-c|}{2}$ and $MI^2 = r^2 + \frac{(b-c)^2}{4}$.

External tangency would need $MI = \frac{a}{2} + r$, i.e. $(b-c)^2 = a^2 + 4ar > a^2$, impossible because $|b-c| < a$. So the incircle is inside the diameter circle and $MI = \frac{a}{2} - r$. Squaring: $r^2 + \frac{(b-c)^2}{4} = \frac{a^2}{4} - ar + r^2$, so $4ar = a^2 - (b-c)^2 = (a-b+c)(a+b-c) = 4(s-b)(s-c)$. Hence $ar = (s-b)(s-c)$.

Square it and use $r^2 = \frac{(s-a)(s-b)(s-c)}{s}$ (Heron with $r = \frac{K}{s}$): $a^2(s-a) = s(s-b)(s-c)$. With $a = 21$, $c = 13$: $s = \frac{34+b}{2}$, $s - a = \frac{b-8}{2}$, $s - b = \frac{34-b}{2}$, $s - c = \frac{b+8}{2}$, so

$441 \cdot \frac{b-8}{2} = \frac{(34+b)(34-b)(b+8)}{8}$, i.e. $1764(b-8) = (1156 - b^2)(b+8)$,

which expands to $b^3 + 8b^2 + 608b - 23360 = 0$. Trying $b = 20$: $8000 + 3200 + 12160 - 23360 = 0$. Factoring, $(b-20)(b^2 + 28b + 1168) = 0$, and the quadratic has discriminant $784 - 4672 < 0$. So $AC = 20$.

Check. With sides $21, 20, 13$: $s = 27$, area $K = \sqrt{27 \cdot 6 \cdot 7 \cdot 14} = 126$, $r = \frac{126}{27} = \frac{14}{3}$. Then $MD = \frac{7}{2}$, $MI = \sqrt{\frac{196}{9} + \frac{49}{4}} = \sqrt{\frac{1225}{36}} = \frac{35}{6}$, and $\frac{a}{2} - r = \frac{21}{2} - \frac{14}{3} = \frac{35}{6}$. Tangent. (A pretty reformulation: $ar = (s-b)(s-c) = r\,r_a$, so the condition says the excircle opposite A has radius exactly BC; here $r_a = \frac{K}{s-a} = \frac{126}{6} = 21$.)

11. Answer: 584**. The idea.** A rectangle inscribed in a circle has both diagonals as diameters, so the rectangles are exactly the pairs of the 12 long diagonals (diameters). The equilateral triangles are the 8 triples $\{r, r+8, r+16\}$. Sorting the vertices by their label modulo 8 splits the polygon into four independent blocks. Inside each block the three diameters pair the two triangles off, and the blocks interact only through the number of one-colour diameters.

Work it out. Number the vertices $0, \ldots, 23$ around the polygon. Call a diameter *red* if both its ends are red, *blue* if both are blue, and *mixed* otherwise. Four vertices of one colour forming a rectangle means two diameters of the same colour, so the conditions are: at most one red diameter, at most one blue diameter, and none of the triangles $T_r = \{r, r+8, r+16\}$ $(0 \le r \le 7)$ is one colour. The diameter from vertex v goes to $v + 12$, and $v + 12 \equiv v + 4 \pmod 8$, so the diameters join T_r to T_{r+4}: for example $0 \leftrightarrow 12$, $8 \leftrightarrow 20$, $16 \leftrightarrow 4$ join T_0 to T_4. So the 24 vertices form 4 blocks $T_r \cup T_{r+4}$ $(r = 0, 1, 2, 3)$, each made of 3 diameters. Count the colourings of one block with

neither triangle one colour, sorted by their numbers of red and blue diameters: *No one-colour diameter:* each of the 3 diameters is red at one end, $2^3 = 8$ ways. T_r is all red exactly when every diameter is red at its T_r end, and all blue exactly when T_{r+4} is all red; these are the only failures, leaving $8 - 2 = 6$. *One red diameter, no blue:* choose it (3 ways). Each triangle already has a red vertex, so a triangle fails only if it is all red: for T_r that means both mixed diameters are red on the T_r side, for T_{r+4} both red on the other side. Of the 4 choices for the two mixed diameters, 2 remain: $3 \cdot 2 = 6$. Likewise *one blue, no red:* 6. *One red and one blue diameter:* $3 \cdot 2$ ways to place them and 2 for the mixed one, 12 in all; both triangles then contain both colours. (Blocks with two red or two blue diameters are already forbidden.) So one block contributes the polynomial $g = 6 + 6u + 6v + 12uv$, with u marking a red and v a blue diameter, and N is the sum of the coefficients of 1, u, v, uv in g^4: 1: $6^4 = 1296$. u: one block supplies the red diameter, $4 \cdot 6 \cdot 6^3 = 5184$; the same for v: 5184. uv: the same block supplies both, $4 \cdot 12 \cdot 6^3 = 10368$; or two different blocks do, $4 \cdot 3 \cdot 6 \cdot 6 \cdot 6^2 = 15552$. $N = 1296+5184+5184+10368+15552 = 37584$, and the remainder is 584.

Check. On a regular 12-gon the same analysis has two blocks $T_r \cup T_{r+2}$ with $T_r = \{r, r+4, r+8\}$, and gives $6^2 + 2 \cdot 6\,(6+6+12) + 2 \cdot 6 \cdot 6 = 396$; a direct computer count of the 2^{12} colourings finds 396. For the 24-gon, a computer check of all 2^{24} colourings against every rectangle and every equilateral triangle among the vertices (found from coordinates) gives 37584. Answer $\boxed{584}$.

12. Answer: 876**. The idea.** Give each single pile a label that behaves like a pile in a game of matching: shift everything by one and look at $x = n + 1$ instead of the pile size n. Write $x = 2^j q$ with q odd, and call q the *odd part* of the pile. Two facts about one pile carry the whole game: a move can never keep the odd part the same, and a move can change it to *any* smaller odd number. Then the two-pile game is a race to keep the odd parts equal, and the second player wins exactly when they start equal, that is, when $\frac{a+1}{b+1}$ is a power of 2.

Work it out. A pile of n stones can go to m stones for $n - \lfloor n/2 \rfloor \le m \le n - 1$. In terms of $x = n + 1$ and $y = m + 1$: $y \le x - 1$, and $y \ge n - \frac{n}{2} + 1 = \frac{x+1}{2} > \frac{x}{2}$. So a move replaces x by any integer y with $\frac{x}{2} < y < x$. (Conversely any such y is a legal move: it removes $x - y < \frac{x}{2} = \frac{n+1}{2}$ stones, and a whole number below $\frac{n+1}{2}$ is at most $\lfloor n/2 \rfloor$.)

Fact 1. If x and y have the same odd part, then $\frac{x}{y}$ is a power of 2; but $1 < \frac{x}{y} < 2$. So no move keeps the odd part.

Fact 2. Let x have odd part p and let $q < p$ be odd. The numbers $q, 2q, 4q, \ldots$ each double the one before, so exactly one of them lies in the interval $\frac{x}{2} < y \le x$. It is not x itself (whose odd part is $p \ne q$), so it satisfies $\frac{x}{2} < y < x$ and is a legal move to odd part q.

Now call a position *balanced* if the two piles have equal odd parts. From a balanced position every move changes one odd part and not the other (Fact 1), so it produces an unbalanced position. From an unbalanced position, the pile with the larger odd part can be moved to the smaller odd part (Fact 2), producing a balanced position. So whoever moves from an unbalanced position can always hand the opponent a balanced one, while the opponent can only hand back unbalanced ones. The game is finite (the total number of stones drops every move), so the player facing balanced positions is the one who eventually cannot move. (Consistently, the only position with no move, $a = b = 1$, has $x = 2$ in both piles: balanced.) So the second player wins exactly when (a, b) is balanced, i.e. $a + 1$ and $b + 1$ differ by a factor that is a power of 2.

Count pairs (x, y) with $2 \le x, y \le 301$ and $\frac{x}{y} = 2^k$ for some integer k. For $k = 0$ there are 300 pairs. For $k \ge 1$, $x = 2^k y$ needs $2 \le y \le \lfloor 301/2^k \rfloor$, giving $\lfloor 301/2^k \rfloor - 1$ pairs: for

$k = 1, 2, \ldots, 8$ that is 149, 74, 36, 17, 8, 3, 1, 0, adding to 288. The case $\frac{y}{x} = 2^k$ gives another 288. Total $300 + 2 \cdot 288 = 876$.

Check. Piles up to 3 stones: the rule says the second player wins at $(1,1)$, $(2,2)$, $(3,3)$, $(3,1)$, $(1,3)$ (for $(3,1)$, $x = 4 = 2 \cdot 2$), and the count formula gives $3 + 2 \cdot 1 = 5$. Directly: $(2,1)$ and $(3,2)$ are first-player wins (move to $(1,1)$ or $(2,2)$). From $(3,1)$ the only move is to $(2,1)$, after which the opponent moves to $(1,1)$; so $(3,1)$ is a second-player win, as predicted. A larger test: from $(6,2)$, odd parts 7 and 3, the first player should move the 6-pile to $x = 6$ (odd part 3), i.e. remove 1 stone to reach $(5,2)$, which is balanced since $6 = 2 \cdot 3$.

13. Answer: 115. **The idea.** Counting components directly is hard, but counting vertices, edges and cycles is easy. Make a graph G whose vertices are the 40 cells, with an edge between two edge-adjacent cells of the same colour. Regions are the connected components of G, and for any graph, #components $= V - E +$ (number of independent cycles). In a strip only two cells high, the only cycles come from 2×2 blocks of one colour, and each expectation is then a sum of simple probabilities.

Work it out. Draw G with each vertex at the centre of its cell; it is a plane graph, so its number of independent cycles equals its number of bounded faces. A cycle in a two-row ladder must run along the top row from some column i to a column $j > i$, down a vertical edge, back along the bottom row and up again, so all cells in columns i through j share one colour. Then every vertical edge between them is present too, and the region inside the cycle is cut into unit squares. So the bounded faces of G are exactly the 2×2 blocks whose four cells have one colour, and

#regions $= 40 - E + Q$, where E is the number of same-coloured adjacent pairs and Q the number of monochromatic 2×2 blocks.

There are $2 \cdot 19 = 38$ horizontal and 20 vertical adjacent pairs, 58 in all, each same-coloured with probability $\frac{1}{2}$: $E[E] = 29$. There are 19 blocks of 2×2, each monochromatic with probability $\frac{2}{16} = \frac{1}{8}$: $E[Q] = \frac{19}{8}$. So the expected number of regions is $40 - 29 + \frac{19}{8} = \frac{107}{8}$, and $m + n = 115$.

Check. For a 2×1 strip the formula gives $2 - \frac{1}{2} = \frac{3}{2}$: two cells of the same colour make one region, different colours make two. For 2×2 it gives $4 - 2 + \frac{1}{8} = \frac{17}{8}$. Directly over the 16 colourings: 2 all one colour (1 region each), 8 with three of one colour (2 each), 4 with two adjacent pairs (2 each), 2 checkerboards (4 each): $\frac{2+16+8+8}{16} = \frac{34}{16} = \frac{17}{8}$.

14. Answer: 931. **The idea.** The angle at C is 120° (by the law of cosines, $49 = 9 + 25 - 2 \cdot 15 \cos C$ gives $\cos C = -\frac{1}{2}$). Let the inscribed triangle be DEF with D on CB, E on CA, F on AB. Since $\angle DCE + \angle DFE = 120^\circ + 60^\circ = 180^\circ$, the quadrilateral $CDFE$ is cyclic, and this pins F down completely: it must be the point where the bisector of angle C meets AB. That turns a two-parameter problem into minimising $d^2 + de + e^2$ with $d + e$ fixed.

Work it out. Write $d = CD$ and $e = CE$. Segment AB lies on the far side of line DE from C (triangle CDE sits inside CAB at the corner C), so F and C are on opposite sides of DE and C, D, F, E lie on a circle in that order. In that circle the equal chords $FD = FE$ subtend equal angles at C, so CF bisects angle C. Ptolemy's theorem gives $CF \cdot DE = CD \cdot EF + CE \cdot DF$, and since $DE = EF = DF$, $CF = d + e$.

So F is the foot L of the bisector from C, whatever D and E are, and $d + e = CL$. For a 120° angle the bisector length satisfies (areas of the two halves) $\frac{1}{2} \cdot 3 \cdot CL \sin 60^\circ + \frac{1}{2} \cdot 5 \cdot CL \sin 60^\circ = \frac{1}{2} \cdot 3 \cdot 5 \sin 120^\circ$, so $CL = \frac{15}{8}$.

By the law of cosines in triangle CDE, $s^2 = DE^2 = d^2+e^2+de = (d+e)^2-de \geq (d+e)^2-\frac{(d+e)^2}{4} = \frac{3}{4}\cdot\frac{225}{64} = \frac{675}{256}$, with equality at $d = e = \frac{15}{16}$. That choice is allowed: $\frac{15}{16} \leq 3$ and ≤ 5, and $F = L$ lies on AB. Conversely every $d, e \geq 0$ with $d + e = \frac{15}{8}$ gives a genuine inscribed equilateral triangle (take the equilateral triangle on DE away from C; its apex lies on the circle through C, D, E and on the bisector at distance $d + e$, which is L). So $s^2 = \frac{675}{256}$ and $m + n = 675 + 256 = 931$.

Check. Coordinates: $C = (0,0)$, $B = (5,0)$, $A = \left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$, $D = \left(\frac{15}{16}, 0\right)$, $E = \frac{15}{16}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Turning E about D by 60° gives $F = \left(\frac{15}{16}, \frac{15\sqrt{3}}{16}\right)$, and F is on AB because $\frac{BF}{BA} = \frac{35/8}{7} = \frac{5}{8}$ with $F = B + \frac{5}{8}(A - B) = \left(5 - \frac{65}{16}, \frac{15\sqrt{3}}{16}\right)$. Also $DE^2 = \frac{225}{256}\cdot 3 = \frac{675}{256}$, so $s \approx 1.62$.

15. Answer: 400**. The idea.** First count *all* sequences of n moves that end at home, call it H_n (with $H_0 = 1$). Any such sequence splits uniquely at its first visit home: a first-return piece of length k followed by an arbitrary home-to-home piece of length $n - k$. So $H_n = \sum_{k=1}^{n} f_k H_{n-k}$ for $n \geq 1$, where f_n counts first returns. In generating functions, $H(z) = 1 + f(z)H(z)$, i.e. $f(z) = 1 - \frac{1}{H(z)}$. Since H_n has a closed form, $f(z)$ turns out to be a simple rational function, and f_n satisfies a two-term recurrence.

Work it out. *Step 1: H_n.* The card has 8 positions, 4 with each face up. Turns keep the face and flips change it, so a sequence ending home has an even number j of flips. If $j \geq 2$, fix which places hold flips, the directions of all turns (2^{n-j} ways) and every flip except the last one (4^{j-1} ways). Just before the last flip the card is in some position; the four flips send it to four different positions, namely all four positions of the other face, and the turns after it act as a fixed one-to-one map. So exactly one choice of the last flip ends at home, giving $\binom{n}{j}2^{n-j}4^{j-1}$ sequences. If $j = 0$, with u clockwise turns the net turn is $(2u - n)\cdot 90°$; this is a multiple of 360° only if n is even and then for half of all 2^n direction patterns. Summing the even $j \geq 2$ with $\sum_{j\text{ even}}\binom{n}{j}2^{n-j}4^j = \frac{6^n+(-2)^n}{2}$:

$$H_n = \frac{1}{4}\left(\frac{6^n+(-2)^n}{2} - 2^n\right) + [n \text{ even}]\, 2^{n-1} = \frac{6^n+3(-2)^n}{8} \text{ for } n \geq 1$$

(check both parities: for even n the correction is $-\frac{2^n}{4} + 2^{n-1} = +\frac{2^n}{4}$, for odd n it is $-\frac{2^n}{4}$). The formula gives $\frac{1}{2}$ at $n = 0$, so $H(z) = \frac{1}{2} + \frac{1}{8}\left(\frac{1}{1-6z} + \frac{3}{1+2z}\right) = \frac{1-4z-6z^2}{(1-6z)(1+2z)}$ (over the denominator $1 - 4z - 12z^2$ the numerator is $\frac{1}{2}(1 - 4z - 12z^2) + \frac{1}{8}(1 + 2z) + \frac{3}{8}(1 - 6z) = 1 - 4z - 6z^2$).

Step 2: first returns. $f(z) = 1 - \frac{1-4z-12z^2}{1-4z-6z^2} = \frac{6z^2}{1-4z-6z^2}$. So $f_1 = 0$, $f_2 = 6$ and $f_n = 4f_{n-1} + 6f_{n-2}$. The roots of $t^2 = 4t+6$ are $2\pm\sqrt{10}$, and matching f_1, f_2 gives $f_n = \frac{3}{\sqrt{10}}\left((2+\sqrt{10})^{n-1} - (2-\sqrt{10})^{n-1}\right)$. Expanding by the binomial theorem, only odd powers of $\sqrt{10}$ survive: with $m = n - 1$,
$f_n = 6\sum_{i\geq 0}\binom{m}{2i+1}2^{m-2i-1}\,10^i$.

Step 3: $m = 2025$ modulo 1000. Modulo 8: the i-th term contains $2^{m-2i-1}\cdot 2^i = 2^{m-i-1}$ with $i \leq 1012$, so every term is divisible by 2^{1012} and $N \equiv 0 \pmod 8$. Modulo 125: terms with $i \geq 3$ contain 10^3. The $i = 1$ term contains $10\binom{2025}{3} = 10\cdot\frac{2025\cdot2024\cdot2023}{6}$, and $2025 = 3^4\cdot 5^2$, so it has three factors of 5. The $i = 2$ term contains $100\binom{2025}{5}$, and $\binom{2025}{5} = \frac{2025\cdot2024\cdot2023\cdot2022\cdot2021}{120}$ keeps one factor 5, so again 5^3 divides it. That leaves $N \equiv 6\cdot 2025\cdot 2^{2024} \equiv 150\cdot 2^{2024} \equiv 25\cdot 2^{2024} \pmod{125}$, and $25x \bmod 125$ depends only on $x \bmod 5$: $2^{2024} = 16^{506} \equiv 1 \pmod 5$. So $N \equiv 25 \pmod{125}$ and $N \equiv 0 \pmod 8$. Among $25, 150, 275, 400, \ldots$ the first multiple of 8 is 400.

Check. $f_2 = 6$ directly: a first return in two moves is a turn followed by the opposite turn (2 ways) or the same flip twice (4 ways). The recurrence gives $f_3 = 24$ and $f_4 = 132$, and the closed form agrees: for $n = 4$ ($m = 3$) it gives $6\left(\binom{3}{1}2^2 + \binom{3}{3}\cdot 10\right) = 6\cdot 22 = 132$. Also $H_2 = \frac{36+12}{8} = 6 = f_2$,

as it must be, since the card cannot be home after one move. A direct count of f_3: the card must leave home and not come back after move 2; for example flip, flip (a different axis), turn works exactly when the two flips compose to the opposite turn.