MathCastle

AIME

Hard Edition

15 questions • 3 hours

Name _____

School _____

Grade _____

INSTRUCTIONS

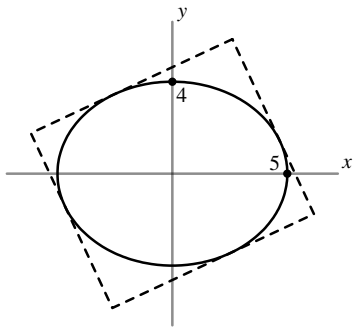
1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

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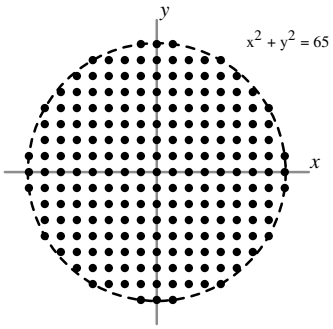
1. The angles of a triangle measure $A°$, $B°$ and $C°$, where A, B, C are positive integers with $A+B+C = 180$. For how many *ordered* triples (A, B, C) is $\sin^2 A° + \sin^2 B° + \sin^2 C° = 2$?

2. A rectangle is said to be *circumscribed* about the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ if each of its four sides lies on a line tangent to the ellipse. Circumscribed rectangles may be tilted at any angle to the coordinate axes. Let M and m be the largest and the smallest possible areas of a circumscribed rectangle. Find $M + m$.



3. How many 3-element subsets $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 24\}$ have the property that the product abc is divisible by 32?

4. Let S be the set of all points (x, y) with integer coordinates satisfying $x^2 + y^2 \leq 65$. A rubber band is stretched around all the points of S; it forms the smallest convex polygon K containing S. Find the area of K.



5. Real numbers $x_1, x_2, \ldots, x_{10}$ satisfy $x_1+x_2+\cdots+x_{10} = 1$. Set $x_0 = x_{11} = 0$ and let $E = \sum_{k=0}^{10}(x_{k+1}-x_k)^2$. The least possible value of E is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

6. Five of the twenty vertices of a regular 20-gon are chosen at random, all $\binom{20}{5}$ choices being equally likely. The five chosen vertices cut the boundary of the polygon into five arcs, and the *length* of an arc is the number of sides of the polygon it contains (so the five lengths are positive integers with sum 20). The expected length of the shortest of the five arcs is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

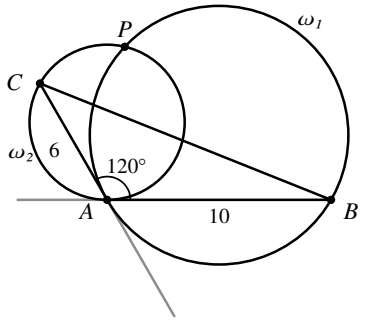
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7. Eight books numbered 1 through 8 stand in a row on a shelf, in some order. A *move* consists of taking any one book out of the row and putting it back at the right-hand end of the row. For each of the 8! possible starting orders, let m be the least number of moves needed to arrange the books as $1, 2, 3, \ldots, 8$ from left to right, and let S be the sum of these 8! values of m. Find the remainder when S is divided by 1000.

8. An infinite right circular cone (a single nappe) has apex V, and every generating line makes an angle α with the axis, where $\sin\alpha = \frac{1}{3}$. Two balls of radii 1 and 3 sit inside the cone, each touching the lateral surface of the cone along a full circle. A plane is tangent to both balls, with the two balls on opposite sides of the plane. The plane cuts the cone in an ellipse, whose area can be written as $\pi\sqrt{N}$ for a positive integer N. Find N.

9. A permutation $a_1, a_2, \ldots, a_{12}$ of $1, 2, \ldots, 12$ is chosen uniformly at random. A *descent* is an index i with $1 \le i \le 11$ and $a_i > a_{i+1}$; let D be the number of descents. The expected value of D^2 is $\frac{m}{n}$ in lowest terms. Find $m + n$.

10. Triangle ABC has $AB = 10$, $AC = 6$ and $\angle BAC = 120^\circ$. Let ω_1 be the circle that passes through B and is tangent to line AC at A, and let ω_2 be the circle that passes through C and is tangent to line AB at A. The circles ω_1 and ω_2 meet at A and at a second point P. Then $AP^2 = \frac{m}{n}$ in lowest terms. Find $m + n$.



11. How many rational numbers r with $\frac{1}{3} < r < \frac{2}{5}$ have denominator at most 100 when written in lowest terms?

12. How many permutations σ of $\{1, 2, \ldots, 10\}$ satisfy $1\cdot\sigma(1) + 2\cdot\sigma(2) + 3\cdot\sigma(3) + \cdots + 10\cdot\sigma(10) = 380$?

13. Six points are labelled 1 through 6, and each of the 15 segments joining two of them is coloured red or blue, giving 2^{15} colourings. A triangle (three of the points) is *monochromatic* if its three segments have the same colour. It is a known fact that every colouring has at least two monochromatic triangles. Let N be the number of colourings with exactly two monochromatic triangles. Find the remainder when N is divided by 1000.

14. Find the number of ordered triples (a, b, c) of positive integers with $a, b, c \le 50$ such that abc is a perfect square and each of a, b, c divides the product of the other two (that is, $a \mid bc$, $b \mid ca$ and $c \mid ab$).

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15. How many ordered pairs (P, Q) of polynomials, each of whose coefficients is 0 or 1, satisfy $P(x)\,Q(x) = 1 + x + x^2 + \cdots + x^{999}$?

Answer Key — AIME — Hard Edition Practice Contest 2

1. 267

2. 162

3. 371

4. 198

5. 111

6. 740

7. 279

8. 24

9. 97

10. 919

11. 201

12. 177

13. 760

14. 988

15. 504

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Solutions — AIME — Hard Edition Practice Contest 2

1. Answer: 267**. The idea.** In every triangle the sum of the three squared sines is 2 plus a correction, and the correction is $2\cos A\cos B\cos C$. So the equation says exactly that one angle is a right angle.

Work it out. Use $\sin^2 x = \frac{1-\cos 2x}{2}$ and sum-to-product: $\sin^2 A+\sin^2 B = 1-\frac{1}{2}(\cos 2A+\cos 2B) = 1-\cos(A+B)\cos(A-B) = 1+\cos C\cos(A-B)$, because $A+B = 180^\circ - C$. Adding $\sin^2 C = 1-\cos^2 C$ and writing $\cos C = -\cos(A+B)$ in the last term:

$$\sin^2 A+\sin^2 B+\sin^2 C = 2+\cos C\,(\cos(A-B)+\cos(A+B)) = 2+2\cos A\cos B\cos C.$$

So the equation holds iff $\cos A\cos B\cos C = 0$, i.e. iff one of the angles is 90°. At most one angle can be 90° (two right angles would leave 0° for the third), so the cases do not overlap.

If $A = 90$, then $B + C = 90$ with $B, C \geq 1$: B can be $1, 2, \ldots, 89$, which gives 89 triples. The right angle can be in any of the 3 positions, so the count is $3 \cdot 89 = 267$.

Check. An equilateral triangle gives $3 \cdot \frac{3}{4} = \frac{9}{4} = 2 + 2 \cdot \frac{1}{8}$, matching the identity. For the triangle $(30, 60, 90)$: $\frac{1}{4} + \frac{3}{4} + 1 = 2$, as predicted. An obtuse triangle such as $(30, 30, 120)$ gives $\frac{1}{4} + \frac{1}{4} + \frac{3}{4} = \frac{5}{4} < 2$, consistent with $\cos C < 0$ there.

2. Answer: 162**. The idea.** For a direction with unit normal $(\cos\theta, \sin\theta)$, the two tangent lines perpendicular to that normal sit at distance $h(\theta)$ on either side of the centre, where h is the ellipse's *support function*. A circumscribed rectangle is determined by θ, and its sides are $2h(\theta)$ and $2h(\theta + 90^\circ)$.

Work it out. The farthest the ellipse reaches in direction $(\cos\theta, \sin\theta)$ is the maximum of $x\cos\theta + y\sin\theta$ on the ellipse. Write it as $\frac{x}{5} \cdot 5\cos\theta + \frac{y}{4} \cdot 4\sin\theta$; by Cauchy–Schwarz this is at most $\sqrt{\frac{x^2}{25} + \frac{y^2}{16}}\sqrt{25\cos^2\theta + 16\sin^2\theta}$, with equality attainable. So

$$p^2 = h(\theta)^2 = 25\cos^2\theta + 16\sin^2\theta \text{ and } q^2 = h(\theta + 90^\circ)^2 = 25\sin^2\theta + 16\cos^2\theta.$$

The rectangle has area $2p \cdot 2q = 4pq$. Put $c = \cos^2\theta$, $s = \sin^2\theta$, $c + s = 1$:

$$p^2q^2 = (25c+16s)(25s+16c) = 400(c^2+s^2)+(625+256)cs = 400(c+s)^2+81cs = 400+81cs.$$

Now $cs = \frac{1}{4}\sin^2 2\theta$ ranges over $[0, \frac{1}{4}]$. The minimum $p^2q^2 = 400$ (axis-parallel rectangle, 10×8) gives $m = 4 \cdot 20 = 80$. The maximum $p^2q^2 = 400 + \frac{81}{4} = \frac{1681}{4}$ (at $\theta = 45^\circ$) gives $pq = \frac{41}{2}$ and $M = 82$. Hence $M + m = 162$.

Check. Always $p^2 + q^2 = 41$: every vertex of a circumscribed rectangle is at distance $\sqrt{41}$ from the centre (the ellipse's director circle). A rectangle with half-diagonal $\sqrt{41}$ has area at most that of the square with that diagonal, $2 \cdot 41 = 82$, attained at $\theta = 45^\circ$ where $p = q$. The axis-parallel 10×8 box indeed has area 80.

3. Answer: 371**. The idea.** Only the powers of 2 matter. Write $v(n)$ for the exponent of 2 in n; then $32 \mid abc$ exactly when $v(a) + v(b) + v(c) \geq 5$. Sort $1, \ldots, 24$ by v and count the complement, the triples whose exponents add up to at most 4.

Work it out. The classes are: $v = 0$, the 12 odd numbers; $v = 1$: 2, 6, 10, 14, 18, 22 (6 numbers); $v = 2$: 4, 12, 20 (3); $v = 3$: 8, 24 (2); $v = 4$: 16 (1). There are $\binom{24}{3} = 2024$ subsets in all.

The multisets of exponents with sum at most 4, and the number of subsets of each kind:

$\{0,0,0\}$: $\binom{12}{3} = 220$. $\{0,0,1\}$: $\binom{12}{2}\cdot 6 = 396$. $\{0,0,2\}$: $66\cdot 3 = 198$. $\{0,0,3\}$: $66\cdot 2 = 132$. $\{0,0,4\}$: $66\cdot 1 = 66$.

$\{0,1,1\}$: $12\binom{6}{2} = 180$. $\{0,1,2\}$: $12\cdot 6\cdot 3 = 216$. $\{0,1,3\}$: $12\cdot 6\cdot 2 = 144$. $\{0,2,2\}$: $12\binom{3}{2} = 36$.

$\{1,1,1\}$: $\binom{6}{3} = 20$. $\{1,1,2\}$: $\binom{6}{2}\cdot 3 = 45$.

These are all the multisets of three exponents from $\{0,1,2,3,4\}$ with sum at most 4 (with at least one zero the other two sum to at most 4; with no zero only $\{1,1,1\}$ and $\{1,1,2\}$ fit). They total 1653, so the number of good subsets is $2024 - 1653 = 371$.

Check. Count the good subsets containing 16 directly: 16 contributes 4, so the other two numbers need v-sum at least 1, i.e. not both odd: $\binom{23}{2} - \binom{12}{2} = 253 - 66 = 187$. Without 16 we need v-sum ≥ 5 from the classes of sizes 12, 6, 3, 2: $\{0,2,3\}$: $12\cdot 3\cdot 2 = 72$; $\{0,3,3\}$: 12; $\{1,1,3\}$: $15\cdot 2 = 30$; $\{1,2,2\}$: $6\cdot 3 = 18$; $\{1,2,3\}$: 36; $\{1,3,3\}$: 6; $\{2,2,2\}$: 1; $\{2,2,3\}$: 6; $\{2,3,3\}$: 3. That is 184, and $187 + 184 = 371$.

4. Answer: 198**. The idea.** K has the full symmetry of the square (reflections in both axes and in $y = x$), so it is enough to find its boundary in the first quadrant from the topmost lattice point in each column and then keep only the corners where the boundary actually turns.

Work it out. For $x = 0, 1, \ldots, 8$ the largest y with $x^2 + y^2 \leq 65$ is $8, 8, 7, 7, 7, 6, 5, 4, 1$ (for example $x = 4$: $16 + 49 = 65$; $x = 5$: $25 + 36 = 61$ but $25 + 49 > 65$; $x = 8$: $64 + 1 = 65$). So the candidates are $(0,8)$, $(1,8)$, $(2,7)$, $(3,7)$, $(4,7)$, $(5,6)$, $(6,5)$, $(7,4)$, $(8,1)$, and by symmetry also $(8,0)$.

Walk from $(1,8)$ clockwise. The segment to $(4,7)$ has slope $-\frac{1}{3}$; at $x = 2, 3$ it is at heights $7\frac{2}{3}$ and $7\frac{1}{3}$, above $(2,7)$ and $(3,7)$, which are therefore inside. From $(4,7)$ to $(7,4)$ the slope is -1 and $(5,6)$, $(6,5)$ lie on that segment (not corners). From $(7,4)$ to $(8,1)$ the slope is -3, then the side $x = 8$ is vertical. The slopes $0, -\frac{1}{3}, -1, -3, \infty$ keep turning the same way, so the corners in this quadrant are $(1,8)$, $(4,7)$, $(7,4)$, $(8,1)$, and K has 16 vertices in all.

Cut K into triangles from the origin. In the first quadrant (using half of the triangles on the axes):

$\triangle O(8,-1)(8,1)$ has area 8, so its half is 4; $\triangle O(8,1)(7,4)$: $\frac{1}{2}|8\cdot 4 - 1\cdot 7| = \frac{25}{2}$; $\triangle O(7,4)(4,7)$: $\frac{1}{2}|49 - 16| = \frac{33}{2}$; $\triangle O(4,7)(1,8)$: $\frac{25}{2}$; half of $\triangle O(1,8)(-1,8)$: 4.

The quadrant contributes $4 + \frac{25}{2} + \frac{33}{2} + \frac{25}{2} + 4 = \frac{99}{2}$, and the area is $4\cdot\frac{99}{2} = 198$.

Check. Pick's theorem, Area $= I + \frac{B}{2} - 1$. On the boundary, each quadrant contributes $(8,0)$, $(8,1)$, $(7,4)$, $(6,5)$, $(5$ (the sides of slope -3 and $-\frac{1}{3}$ contain no other lattice points), so $B = 4\cdot 7 = 28$. Counting S column by column, $x = 0, \pm 1, \pm 2, \ldots, \pm 8$ contribute $17, 17, 15, 15, 15, 13, 11, 9, 3$ points, so $|S| = 17 + 2\cdot 98 = 213$; every lattice point of K is in S, hence $I = 213 - 28 = 185$ and the area is $185 + 14 - 1 = 198$.

5. Answer: 111**. The idea.** Rewrite everything in terms of the eleven steps $d_k = x_{k+1} - x_k$ ($k = 0, \ldots, 10$). The sum of the x's becomes a fixed linear combination of the steps, and then Cauchy–Schwarz bounds $E = \sum d_k^2$ from below.

Work it out. Since $x_0 = x_{11} = 0$, the steps satisfy $\sum_{k=0}^{10} d_k = 0$, and $x_j = d_0 + d_1 + \cdots + d_{j-1}$. The step d_k appears in $x_{k+1}, \ldots, x_{10}$, i.e. $10 - k$ times, so

$1 = \sum_{j=1}^{10} x_j = \sum_{k=0}^{10}(10-k)d_k = \sum_{k=0}^{10}(5-k)d_k + 5\sum_{k=0}^{10} d_k = \sum_{k=0}^{10}(5-k)d_k$.

By Cauchy–Schwarz, $1 = (\sum(5-k)d_k)^2 \leq \sum_{k=0}^{10}(5-k)^2 \cdot \sum d_k^2 = 110\,E$, since $\sum(5-k)^2 = 2(1 + 4 + 9 + 16 + 25) = 110$. So $E \geq \frac{1}{110}$.

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Equality needs d_k proportional to $5-k$; take $d_k = \frac{5-k}{110}$. These do sum to 0, so they come from a genuine sequence with $x_0 = x_{11} = 0$, namely $x_j = \frac{1}{110}\sum_{k=0}^{j-1}(5-k) = \frac{j(11-j)}{220}$, whose sum is $\frac{1}{220}\sum j(11-j) = \frac{220}{220} = 1$. The minimum is $\frac{1}{110}$ and $m+n = 111$.

Check. The optimal profile $x_j = \frac{j(11-j)}{220}$ is a discrete parabola: its second difference $x_{j+1} - 2x_j + x_{j-1} = -\frac{2}{220}$ is constant, exactly what a Lagrange-multiplier condition $2(2x_j - x_{j-1} - x_{j+1}) = \lambda$ demands. Directly, $E = \sum\left(\frac{5-k}{110}\right)^2 = \frac{110}{110^2} = \frac{1}{110}$. For contrast, the flat choice $x_j = \frac{1}{10}$ gives $E = \frac{2}{100} = \frac{1}{50}$, larger, as it must be.

6. Answer: 740**. The idea.** For a random variable L taking positive integer values, $E[L] = \sum_{g\geq 1} P(L \geq g)$. And "the shortest arc has length at least g" means *every* arc has length at least g, which is easy to count: fix a starting chosen vertex and the problem becomes a composition of 20 into five parts, each at least g.

Work it out. Let $N(g)$ be the number of 5-sets of vertices whose five arcs all have length $\geq g$. Count pairs (set, marked chosen vertex) instead. Pick the marked vertex (20 ways) and then read off the arc lengths $a_1, \ldots, a_5$ going clockwise from it: these are any ordered 5-tuple with $a_i \geq g$ and $a_1 + \cdots + a_5 = 20$, and they determine the set. Putting $b_i = a_i - (g-1) \geq 1$ with $\sum b_i = 25 - 5g$, stars and bars gives $\binom{24-5g}{4}$ tuples. Each set is counted once for each of its 5 vertices, so

$N(g) = \frac{20}{5}\binom{24-5g}{4} = 4\binom{24-5g}{4}$.

$g = 1$: $4\binom{19}{4} = 4 \cdot 3876 = 15504 = \binom{20}{5}$ (every set, as it should be). $g = 2$: $4\binom{14}{4} = 4004$. $g = 3$: $4\binom{9}{4} = 504$. $g = 4$: $4\binom{4}{4} = 4$. For $g \geq 5$ the count is 0, since five arcs of length ≥ 5 would need at least 25 sides.

Therefore $E[L] = \frac{15504+4004+504+4}{15504} = \frac{20016}{15504}$. Now $20016 = 2^4 \cdot 3^2 \cdot 139$ and $15504 = 2^4 \cdot 3 \cdot 17 \cdot 19$, so $E[L] = \frac{417}{323}$, and $m + n = 417 + 323 = 740$.

Check. The count $N(4) = 4$ is right: all five arcs at least 4 with sum 20 forces all of them to equal 4, i.e. a regular pentagon, and there are $20/5 = 4$ of those. Also the answer is sensible: $E[L]$ lies between 1 and 4, and it is a little above 1 because a random 5-set of a 20-gon usually contains two adjacent vertices (probability $1 - \frac{4004}{15504} = \frac{2875}{3876} \approx 0.74$). A full enumeration of all 15504 sets gives the same $\frac{417}{323}$.

7. Answer: 279**. The idea.** Look at the books that are never moved. They keep their original left-to-right order, and at the end they all stand to the left of every book that was moved. So the unmoved books must be $1, 2, \ldots, j$ for some j, and they must already appear in that order. This pins down m exactly, and then the sum over all orders is a short linearity computation.

Work it out. For a starting order let k be the largest j such that the books $1, 2, \ldots, j$ appear from left to right in increasing order (always $k \geq 1$). We show $m = 8 - k$.

Lower bound. Fix any sequence of moves that sorts the shelf and let U be the set of books it never moves. A moved book goes to the far right at its last move, and afterwards only other books are put to its right, so at the end every moved book is to the right of every book of U. Books of U never change their relative order. In the sorted row the books of U therefore occupy the first $|U|$ places, so $U = \{1, \ldots, |U|\}$, and these books were in increasing order at the start. Hence $|U| \leq k$ and the number of moves is at least $8 - |U| \geq 8 - k$.

Upper bound. Move books $k+1, k+2, \ldots, 8$ to the end, in that order. What remains on the left is $1, \ldots, k$ in increasing order, followed by $k+1, \ldots, 8$. That is $8 - k$ moves.

Now sum. $S = \sum(8-k) = 8\cdot 8! - \sum k$, and $\sum k = \sum_{j=1}^{8} \#\{\text{orders with } k \geq j\}$. The condition $k \geq j$ says the books $1, \ldots, j$ appear in increasing order; by symmetry this happens for exactly $\frac{1}{j!}$ of all orders, i.e. for $\frac{8!}{j!}$ of them. So

$\sum k = 40320 + 20160 + 6720 + 1680 + 336 + 56 + 8 + 1 = 69281,$

and $S = 322560 - 69281 = 253279$. The remainder is 279.

Check. With three books the rule gives $m = 0$ for 123; $m = 1$ for 132 and 312 (where $k = 2$); and $m = 2$ for 213, 231, 321 (where $k = 1$). For example $231 \to 312 \to 123$ (move 2, then 3). The total $0 + 1 + 1 + 2 + 2 + 2 = 8$ matches the formula $3 \cdot 3! - (6 + 3 + 1) = 8$. A breadth-first search over all 8! orders gives $S = 253279$.

8. Answer: 24. **The idea.** These are Dandelin spheres. The two balls touch the cutting plane exactly at the two foci F_1, F_2 of the ellipse, and for any point X of the ellipse, $XF_1 + XF_2$ equals the length of the generator of the cone between the two circles of contact. So the major axis is a length along the cone, and the focal distance is a length between the tangency points, and both can be read off from the two balls alone.

Work it out. A ball of radius r inscribed in the cone has its centre on the axis at distance $r/\sin\alpha = 3r$ from V, so the centres are at distances 3 and 9 from V, and the distance between the centres is $D = 6$. A generator touches the ball at distance $\sqrt{(3r)^2 - r^2} = r\cot\alpha = 2\sqrt{2}\,r$ from V (here $\cot\alpha = \sqrt{8}$). Take X on the ellipse; the generator through X meets the two circles of contact at T_1, T_2. Tangent lengths from X to a ball are equal, so $XF_1 = XT_1$ and $XF_2 = XT_2$, and since X lies between the circles, $XF_1 + XF_2 = T_1T_2 = 2\sqrt{2}(3-1) = 4\sqrt{2}$. Hence $2a = 4\sqrt{2}$, $a^2 = 8$.

The foci are the feet of the perpendiculars from the two centres to the plane. The centres are at distances 1 and 3 from the plane, on opposite sides, so their components perpendicular to the plane differ by $1 + 3 = 4$, and the distance between the feet is $2c = \sqrt{D^2 - 4^2} = \sqrt{20}$, so $c^2 = 5$. Then $b^2 = a^2 - c^2 = 3$ and the area is $\pi ab = \pi\sqrt{a^2b^2} = \pi\sqrt{24}$. So $N = 24$.

Check. Coordinates in the axial cross-section: the plane appears as the internal common tangent of the two circles, which meets the generators $x = \pm y\tan\alpha$ at the ends of the major axis, and these are $4\sqrt{2}$ apart. At the midpoint (x_0, y_0) of that segment the cone's horizontal circle has radius $y_0\tan\alpha$, and the half-chord perpendicular to the cross-section is $b = \sqrt{y_0^2\tan^2\alpha - x_0^2} = \sqrt{3}$, agreeing with $b^2 = 3$.

9. Answer: 97. **The idea.** Write $D = I_1 + I_2 + \cdots + I_{11}$, where $I_i = 1$ when i is a descent. Then $E[D^2] = \sum_i E[I_i] + 2\sum_{i<j} E[I_iI_j]$, and everything reduces to the probability that two given positions are both descents. The trap is to treat all pairs as independent: neighbouring descents are not.

Work it out. $E[I_i] = \frac{1}{2}$ by symmetry, so $\sum E[I_i] = \frac{11}{2}$. For $j = i+1$, both are descents iff $a_i > a_{i+1} > a_{i+2}$; the three entries are in uniformly random relative order, so the probability is $\frac{1}{6}$. There are 10 such adjacent pairs. For $j \geq i+2$ the positions $\{i, i+1\}$ and $\{j, j+1\}$ are disjoint, and the relative orders of two disjoint pairs of entries are independent, so the probability is $\frac{1}{4}$. There are $\binom{11}{2} - 10 = 45$ such pairs. Therefore $E[D^2] = \frac{11}{2} + 2\left(10\cdot\frac{1}{6} + 45\cdot\frac{1}{4}\right) = \frac{11}{2} + \frac{10}{3} + \frac{45}{2} = 28 + \frac{10}{3} = \frac{94}{3}$. So $m + n = 94 + 3 = 97$.

Check. Equivalently $E[D] = \frac{11}{2}$ and $\operatorname{Var} D = 11\cdot\frac{1}{4} + 2\cdot 10\left(\frac{1}{6} - \frac{1}{4}\right) = \frac{13}{12}$, and $\frac{121}{4} + \frac{13}{12} = \frac{376}{12} = \frac{94}{3}$. The same method for $n = 3$ gives $E[D^2] = \frac{2}{2} + 2\cdot\frac{1}{6} = \frac{4}{3}$; directly, the six permutations of $1, 2, 3$ have $0, 1, 1, 1, 1, 2$ descents, and $\frac{0+1+1+1+1+4}{6} = \frac{4}{3}$. The Eulerian numbers for $n = 12$ confirm $\frac{94}{3}$.

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10. Answer: 919**. The idea.** Each circle is tangent to a side at A, so the tangent–chord angle gives an equal inscribed angle. The two equalities make triangles PBA and PAC similar, which fixes the ratios $PB : PA : PC$ and, better still, forces $\angle BPC = 120°$. Then one law of cosines finishes.

Work it out. In ω_1, the angle between tangent AC and chord AP equals the inscribed angle on chord AP from B: $\angle PAC = \angle PBA$. In ω_2, likewise $\angle PAB = \angle PCA$. So triangles PBA and PAC have two pairs of equal angles ($B \leftrightarrow A$, $A \leftrightarrow C$) and are similar, with ratio $\frac{BA}{AC} = \frac{10}{6} = \frac{5}{3}$: $PB = \frac{5}{3}PA$ and $PA = \frac{5}{3}PC$, i.e. $PC = \frac{3}{5}PA$.

Now the angle at P: $\angle APB = 180° - \angle PAB - \angle PBA = 180° - (\angle PAB + \angle PAC) = 180° - \angle BAC = 60°$, and by the similarity $\angle APC = \angle BPA = 60°$. Hence $\angle BPC = 120°$. First, $BC^2 = 100 + 36 + 60 = 196$. The law of cosines in $\triangle BPC$ gives $BC^2 = PB^2 + PC^2 + PB \cdot PC = PA^2\left(\frac{25}{9} + \frac{9}{25} + 1\right) = PA^2 \cdot \frac{931}{225}$. So $AP^2 = \frac{196 \cdot 225}{931} = \frac{196 \cdot 225}{49 \cdot 19} = \frac{900}{19}$, and $m + n = 919$.

Check. The similarity gives $PA^2 = PB \cdot PC$ and $PB : PC = AB^2 : AC^2$, the signature of the A-symmedian: P is the midpoint of the symmedian chord, and $AP = \frac{bc}{2m_a}$ with median $m_a^2 = \frac{2 \cdot 36 + 2 \cdot 100 - 196}{4} = 19$. So $AP^2 = \frac{3600}{4 \cdot 19} = \frac{900}{19}$ again. An exact coordinate construction of both circles agrees.

11. Answer: 201**. The idea.** The endpoints are Farey neighbours: $2 \cdot 3 - 1 \cdot 5 = 1$. For such a pair every fraction strictly between them is a unique "weighted mediant" $\frac{a \cdot 1 + b \cdot 2}{a \cdot 3 + b \cdot 5}$ with positive integers a, b, and it is in lowest terms exactly when $\gcd(a, b) = 1$. So the question becomes a lattice-point count: coprime pairs $a, b \geq 1$ with $3a + 5b \leq 100$.

Work it out. Given $\frac{1}{3} < \frac{p}{q} < \frac{2}{5}$ in lowest terms, solve $p = a + 2b$, $q = 3a + 5b$: $a = 2q - 5p$ and $b = 3p - q$. These are integers, and positive because $\frac{p}{q} < \frac{2}{5}$ and $\frac{p}{q} > \frac{1}{3}$. Conversely any positive a, b give a fraction strictly between, since $\frac{a+2b}{3a+5b}$ is a mediant-type average of $\frac{1}{3}$ and $\frac{2}{5}$. The substitution has determinant $1 \cdot 5 - 2 \cdot 3 = -1$, so it is invertible over the integers and $\gcd(p, q) = \gcd(a, b)$. Hence the answer is the number of coprime pairs (a, b) of positive integers with $3a + 5b \leq 100$.

Let $L(M)$ be the number of all pairs $a, b \geq 1$ with $3a + 5b \leq M$, i.e. $L(M) = \sum_{b \geq 1} \left\lfloor \frac{M - 5b}{3} \right\rfloor$ over $5b \leq M - 3$. Pairs with $d \mid \gcd(a, b)$ are d times pairs with $3a' + 5b' \leq \lfloor 100/d \rfloor$, so by Möbius inclusion–exclusion the count is $\sum_d \mu(d)\, L(\lfloor 100/d \rfloor)$, and $L(M) = 0$ for $M < 8$, so only $d \leq 12$ matter:

$L(100) = 31 + 30 + 28 + 26 + 25 + 23 + 21 + 20 + 18 + 16 + 15 + 13 + 11 + 10 + 8 + 6 + 5 + 3 + 1 = 310$, $L(50) = 15 + 13 + 11 + 10 + 8 + 6 + 5 + 3 + 1 = 72$, $L(33) = 9 + 7 + 6 + 4 + 2 + 1 = 29$, $L(20) = 5 + 3 + 1 = 9$, $L(16) = 3 + 2 = 5$, $L(14) = 3 + 1 = 4$, $L(10) = L(9) = 1$.

With $\mu(2) = \mu(3) = \mu(5) = \mu(7) = \mu(11) = -1$, $\mu(6) = \mu(10) = +1$ and $\mu(4) = \mu(8) = \mu(9) = \mu(12) = 0$: $310 - 72 - 29 - 9 + 5 - 4 + 1 - 1 = 201$.

Check. Smallest case: with denominators at most 8 the only fraction strictly between $\frac{1}{3} \approx 0.333$ and $\frac{2}{5} = 0.4$ is $\frac{3}{8}$, and indeed the only pair with $3a + 5b \leq 8$ is $a = b = 1$, giving $\frac{1+2}{3+5} = \frac{3}{8}$. For size: coprime pairs have density $6/\pi^2$ and the triangle $3a + 5b \leq 100$ has area $\frac{100^2}{30} \approx 333$, predicting about $0.61 \cdot 333 \approx 203$. A direct computer list of all p/q with $q \leq 100$ in the interval has 201 distinct values.

12. Answer: 177**. The idea.** The sum is largest (385, the identity) and the shortfall measures how far σ moves things: since $\sum \sigma(i)^2 = \sum i^2$, we get $\sum_i \big(i - \sigma(i)\big)^2 = 2\sum i^2 - 2\sum i\sigma(i)$, so $\sum i\sigma(i) =$

$385 - \frac{1}{2}E$ with the "energy" $E = \sum(i - \sigma(i))^2$. The condition is $E = 10$. Energy adds over the cycles of σ, and only a handful of cycles are cheap.

Work it out. Energy of a cycle is the sum of the squares of its jumps. A transposition $(a\ b)$ costs $2(b-a)^2$: 2 if adjacent, 8 if $b = a+2$, at least 18 otherwise. A 3-cycle on $a < b < c$ with $x = b - a$, $y = c - b$ costs $x^2 + y^2 + (x+y)^2$: 6 for three consecutive numbers (two directions), at least 14 otherwise. For a cycle of length $L \geq 4$ spanning from its least element m to its largest M, the jumps go up and back down, so they total at least $2(M - m) \geq 2(L-1)$. For $L = 4$: four positive jumps summing to at least 6 have squares summing to at least $1 + 1 + 4 + 4 = 10$, with equality only for four consecutive numbers visited as $i \to i{+}1 \to i{+}3 \to i{+}2 \to i$ or its reverse. For $L \geq 5$ the minimum is at least $1 + 1 + 4 + 4 + 4 = 14$.

So the cycles of cost at most 10 are: adjacent swap (2), consecutive 3-cycle (6), swap at distance 2 (8), special 4-cycle (10). Ways to total 10 with disjoint cycles:

(i) five adjacent swaps: they must cover all ten numbers, so $(1\,2)(3\,4)(5\,6)(7\,8)(9\,10)$: 1 way.

(ii) a consecutive 3-cycle plus two adjacent swaps: place one block of length 3, two blocks of length 2 and three fixed points in a row: $\frac{6!}{1!\,2!\,3!} = 60$ orders, times 2 directions of the 3-cycle: 120.

(iii) a swap $(i\ i{+}2)$ plus an adjacent swap: $i+1$ cannot be in the adjacent swap (its neighbours are taken), so treat $\{i, i{+}1, i{+}2\}$ as a block of length 3; one block of 3, one of 2, five fixed points: $7 \cdot 6 = 42$.

(iv) one special 4-cycle: 7 positions times 2 directions: 14.

Total $1 + 120 + 42 + 14 = 177$.

Check. For $\{1, 2, 3\}$ the same method predicts $E = 2$ (sum 13) for the two adjacent swaps, and indeed $1 \cdot 2 + 2 \cdot 1 + 3 \cdot 3 = 13 = 14 - 1$. A computer count over all 10! permutations gives 177.

13. Answer: 760**. The idea.** Count the non-monochromatic triangles through their "two-coloured corners". A non-monochromatic triangle has exactly two vertices where a red side meets a blue side. At a point with r_v red and b_v blue segments there are $r_v b_v$ red–blue pairs of sides, each opening one triangle. This turns "exactly two" into a condition on degrees.

Work it out. Counting (vertex, two-coloured corner) pairs gives $\sum_v r_v b_v = 2$ (non-mono), so the number of monochromatic triangles is $\binom{6}{3} - \frac{1}{2}\sum_v r_v b_v$. Since $r_v + b_v = 5$, $r_v b_v \leq 2 \cdot 3 = 6$, so there are at least $20 - 18 = 2$ monochromatic triangles, with equality exactly when every point has red degree 2 or 3. So N is the number of labelled graphs (the red segments) on 6 vertices with every degree in $\{2, 3\}$. The number of degree-3 vertices is even.

None of degree 3 (2-regular): a hexagon, $\frac{5!}{2} = 60$, or two triangles, $\frac{1}{2}\binom{6}{3} = 10$: 70. *All six of degree 3*: complements of these, 70.

Exactly two of degree 3, say u, v (15 choices), the other four W of degree 2; there are $\frac{6+8}{2} = 7$ edges. If $u \sim v$: u and v each pick 2 neighbours in W (sets S, T), leaving 2 edges inside W. If $S = T$, the other two points of W need degree 2 from each other alone: impossible. If S, T are disjoint (6 ways), each point of W needs one more edge: a perfect matching of W, 3 ways, giving 18. If $|S \cap T| = 1$, $S = \{x, y\}$, $T = \{x, z\}$ ($4 \cdot 3 \cdot 2 = 24$ ways), the fourth point w must join both y and z, which is forced: 24. If $u \nsim v$: each has 3 neighbours in W, leaving 1 edge inside W. $S = T$ leaves the fourth point stranded; otherwise $S = W \setminus \{b\}$, $T = W \setminus \{a\}$ with $a \neq b$ (12 ordered choices) and the edge ab is forced: 12. So each pair gives $18 + 24 + 12 = 54$ graphs, and there are $15 \cdot 54 = 810$. *Exactly four of degree 3*: complements of the previous case, 810.

So $N = 70 + 70 + 810 + 810 = 1760$, and the remainder is 760.

Check. The bound is sharp and the count is symmetric under swapping red and blue (complements), as it must be. A computer check of all 2^{15} colourings finds minimum 2, attained by exactly 1760 colourings.

14. Answer: 988**. The idea.** The conditions say precisely that $a = BC$, $b = CA$, $c = AB$ for positive integers A, B, C. So we are counting triples (A, B, C) whose three pairwise products are at most 50, which is a manageable lattice-point count.

Work it out. *The parametrisation.* If $a = BC$, $b = CA$, $c = AB$, then $abc = (ABC)^2$ is a square and $bc = A^2 \cdot BC = A^2 a$ is a multiple of a; likewise for b and c. Conversely, suppose the conditions hold and let $s = \sqrt{abc}$, an integer. Put $A = s/a$. Then $A^2 = \frac{abc}{a^2} = \frac{bc}{a}$ is an integer, and a rational number whose square is an integer is itself an integer, so A is a positive integer; similarly $B = s/b$ and $C = s/c$. Then $BC = \frac{s^2}{bc} = a$, $CA = b$, $AB = c$. Since $A = \sqrt{bc/a}$ is determined by (a, b, c), this is a bijection between good triples and triples (A, B, C) of positive integers with $BC, CA, AB \le 50$.

The count. Fix C and put $M = \lfloor 50/C \rfloor$. Then we need $A, B \le M$ and $AB \le 50$; call the number of such pairs $h(M) = \sum_{A=1}^{M} \min(M, \lfloor 50/A \rfloor)$.

For $C \ge 8$ we have $M \le 6$ and $AB \le 36 \le 50$ automatically, so $h(M) = M^2$: $C = 8$ gives 36; $C = 9, 10$ give 25 each; $C = 11, 12$ give 16 each; $C = 13$–16 give 9 each; $C = 17$–25 give 4 each; $C = 26$–50 give 1 each. Sum: $36 + 50 + 32 + 36 + 36 + 25 = 215$.

For $C \le 7$ use the values $\lfloor 50/A \rfloor = 50, 25, 16, 12, 10, 8, 7, 6, 5, 5, 4, 4, 3, 3, 3, 3$, then 2 for $A = 17$–25 and 1 for $A = 26$–50:

$C = 1$, $M = 50$: $h = 50 + 25 + 16 + 12 + 10 + 8 + 7 + 6 + 5 + 5 + 4 + 4 + 4 \cdot 3 + 9 \cdot 2 + 25 = 207$.

$C = 2$, $M = 25$: the first term drops from 50 to 25 and the 25 ones (for $A = 26$–50) disappear: $207 - 25 - 25 = 157$.

$C = 3$, $M = 16$: $16 + 16 + 16 + 12 + 10 + 8 + 7 + 6 + 5 + 5 + 4 + 4 + 3 + 3 + 3 + 3 = 121$.

$C = 4$, $M = 12$: $4 \cdot 12 + 10 + 8 + 7 + 6 + 5 + 5 + 4 + 4 = 97$.

$C = 5$, $M = 10$: $5 \cdot 10 + 8 + 7 + 6 + 5 + 5 = 81$.

$C = 6$, $M = 8$: $6 \cdot 8 + 7 + 6 = 61$. $C = 7$, $M = 7$: $7 \cdot 7 = 49$.

These give $207 + 157 + 121 + 97 + 81 + 61 + 49 = 773$, and the answer is $773 + 215 = 988$.

Check. An example: $(a, b, c) = (6, 10, 15)$ has $abc = 900 = 30^2$ and $6 \mid 150$, $10 \mid 90$, $15 \mid 60$; it comes from $A = \sqrt{150/6} = 5$, $B = \sqrt{90/10} = 3$, $C = \sqrt{60/15} = 2$, and indeed $BC = 6$, $CA = 10$, $AB = 15$. The square condition matters: $(2, 2, 2)$ meets all three divisibilities, but $abc = 8$ is not a square, and correspondingly $\sqrt{bc/a} = \sqrt{2}$ is not an integer. A computer search over all 50^3 triples finds exactly 988 (without the square condition there would be 1682).

15. Answer: 504**. The idea.** Let A and B be the sets of exponents that appear in P and Q. The coefficient of x^n in PQ counts the pairs $(a, b) \in A \times B$ with $a + b = n$, so the equation says: every n with $0 \le n \le 999$ is $a + b$ in exactly one way, and no other sums occur. Such splittings have a rigid "mixed radix" shape, peeled off one level at a time, and counting them turns into counting ordered factorisations of 1000.

Work it out. Both constant terms are 1, so $0 \in A \cap B$, and 1 lies in exactly one of A, B; say $1 \in A$. If $B = \{0\}$ then $P = 1 + \cdots + x^{999}$, $Q = 1$. Otherwise let m be the least positive element of B; every $k < m$ can only be $k + 0$, so $\{0, \ldots, m - 1\} \subseteq A$. *Claim:* B consists of multiples of m, and

A is a union of whole blocks $\{jm, \dots, jm+m-1\}$. Induct on blocks: suppose it holds below tm, and write $tm = a + b$. If $b = tm$, then $tm + i = i + tm$ for $0 \le i < m$, and none of these numbers can also lie in A or B (that would be a second representation). If $0 < b < tm$, then b is a multiple of m, so $a = tm - b$ starts a block of A and the block $a, \dots, a + m - 1$ covers $tm, \dots, tm + m - 1$ with this b; again none of them can join A or B. If $b = 0$, then $tm \in A$; if some $tm + i$ ($0 < i < m$, least) were not in A, it would be $a' + b'$ with $b' > 0$: either $b' = rm < tm$, and then $(t - r)m \in A$ gives tm the second representation $(t - r)m + rm$; or $b' = tm + j$ with $0 \le j < i$, and then $tm + j$ lies in both A and B, two representations. So the block of tm is in A, and B gets no new element inside it. This proves the claim, and since the blocks tile $\{0, \dots, 999\}$, $m \mid 1000$.

So $P(x) = (1 + x + \cdots + x^{m-1})\, P'(x^m)$ and $Q(x) = Q'(x^m)$, where $Q'P'$ is a splitting of $1 + y + \cdots + y^{1000/m-1}$ with the linear term now in Q', and conversely each such choice works. Let $f(N)$ count splittings of $1 + \cdots + x^{N-1}$ with $1 \in A$ (and $f(1) = 1$). Then $f(N) = \sum_{m \mid N,\, m>1} f(N/m)$, so $f(N)$ is the number of ordered factorisations of N into factors greater than 1, and the answer is $2f(1000)$ (the linear term can be in P or in Q).

For $N = 2^a 5^b$ write $g(a, b)$; the recursion says $g(a, b)$ is the sum of all $g(i, j)$ with $(i, j) \le (a, b)$, $(i, j) \ne (a, b)$. Row by row: $g(0, \cdot) = 1, 1, 2, 4$; $g(1, \cdot) = 1, 3, 8, 20$; $g(2, \cdot) = 2, 8, 26, 76$; $g(3, \cdot) = 4, 20, 76, 252$. So $f(1000) = 252$ and there are 504 ordered pairs.

Check. For $N = 4$ the rule predicts $2 \cdot 2 = 4$ pairs: $(1, 1{+}x{+}x^2{+}x^3)$, $(1{+}x, 1{+}x^2)$ and their swaps, and indeed $1 + x + x^2 + x^3$ has no other $0/1$ splitting. For $N = 6$ it predicts $2 \cdot 3 = 6$: the trivial pair, $(1{+}x)(1{+}x^2{+}x^4)$ and $(1{+}x{+}x^2)(1{+}x^3)$, each in both orders. A computer search that builds A and B one number at a time, with no assumption about their shape, finds exactly 504 splittings for $N = 1000$.

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