## MathCastle

# AIME

Hard Edition

15 questions • 3 hours

**Name** _____

**School** _____

**Grade** _____

**INSTRUCTIONS**

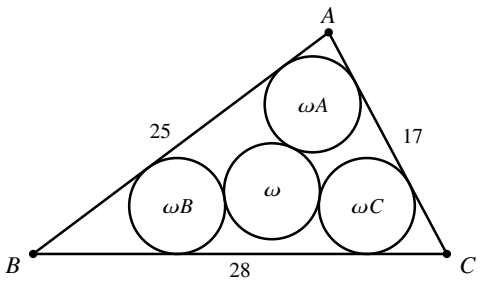
1. Do not open this booklet until you are told to begin.
2. This is a 15-question test. Every answer is an integer from 000 to 999 inclusive.
3. Record your answers as three digits, with leading zeros if needed (for example, 007).
4. 1 point for each correct answer. There is no penalty for a wrong answer.
5. Only scratch paper, graph paper, rulers, compasses and erasers are allowed. No calculators.
6. Figures are not necessarily drawn to scale.
7. You have 3 hours to finish the examination.

**DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.**

MathCastle practice paper. Original problems in the contest format; not affiliated with or endorsed by the contest organisers.

**1.** Let $N = 10!$. For a positive integer $m$, let $\varphi(m)$ be the number of integers in $\{1, 2, \ldots, m\}$ that are relatively prime to $m$, and let $\tau(m)$ be the number of positive divisors of $m$. Define $S = \displaystyle\sum_{d \mid N} \varphi(d)\, \tau\!\left(\frac{N}{d}\right)$, where the sum runs over all positive divisors $d$ of $N$. Find the remainder when $S$ is divided by 1000.

**2.** Triangle $ABC$ has $AB = 25$, $BC = 28$ and $CA = 17$. Four circles, all with the same radius $\rho$, lie inside the triangle: circle $\omega_A$ is tangent to sides $\overline{AB}$ and $\overline{AC}$, circle $\omega_B$ is tangent to sides $\overline{BA}$ and $\overline{BC}$, circle $\omega_C$ is tangent to sides $\overline{CA}$ and $\overline{CB}$, and the fourth circle $\omega$ is externally tangent to each of $\omega_A$, $\omega_B$ and $\omega_C$. Then $\rho = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.




**3.** A string of 12 letters, each letter $H$ or $T$, is written in a row. An occurrence of $HH$ is a pair of consecutive positions that both hold $H$; occurrences may overlap, so $HHH$ contains two occurrences of $HH$. Occurrences of $TT$ are defined the same way. How many of the $2^{12}$ strings contain exactly one more occurrence of $HH$ than of $TT$?

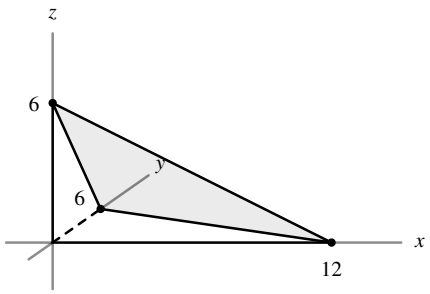
**4.** For how many ordered pairs $(a, b)$ of integers with $-30 \le a \le 30$, $-30 \le b \le 30$ and $b \neq 0$ is one root of the polynomial $x^2 + ax + b$ an integer multiple of the other root? The two roots may be real or non-real, and a repeated root counts as both roots; for example, $x^2 - 4x + 4$ (roots 2 and 2) and $x^2 - 6x + 8$ (roots 2 and 4) both qualify, while $x^2 - 5x + 6$ (roots 2 and 3) does not.

**5.** Each of the 12 edges of a cube is coloured red or blue, independently and with probability $\frac{1}{2}$ each. The probability that every one of the 8 vertices of the cube is an endpoint of an odd number of red edges is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**6.** Let $R$ be the set of complex numbers $z = x + yi$ with $1 \le x \le 2$ and $1 \le y \le 3$ (a filled rectangle in the complex plane). The set $\{z^2 : z \in R\}$ of all squares of numbers in $R$ is a region in the complex plane whose area is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**7.** In three-dimensional space, consider the four planes $x = 0$, $y = 0$, $z = 0$ and $x + 2y + 2z = 12$. A sphere of positive radius is tangent to a plane when the distance from its centre to the plane equals its radius. There are only finitely many spheres, located anywhere in space, that are tangent to all four planes. The sum of the radii of all of these spheres is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

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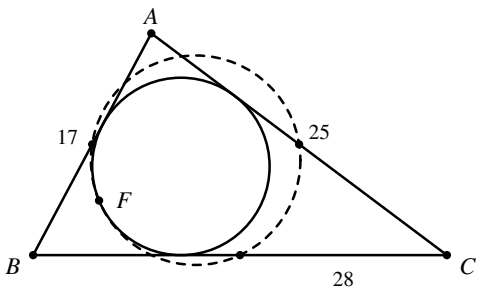

**8.** The number 101 is prime. For how many integers $n$ with $1 \le n \le 1000$ does the congruence $x^n \equiv 6 \pmod{101}$ have at least one integer solution $x$?

**9.** A bag holds 5 red balls and 5 blue balls. The balls are drawn one at a time, without replacement, until the bag is empty, all $\binom{10}{5}$ colour orders being equally likely. After each of the 10 draws we compare the number of red balls drawn so far with the number of blue balls drawn so far. The expected number of draws after which these two numbers are equal (the tenth draw included) is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**10.** The integers $1, 2, 3, \ldots, 10$ are written around a circle in some order, each exactly once. For each of the 10 pairs of numbers that are next to each other on the circle, compute their product, and let $S$ be the sum of these 10 products. Find the smallest possible value of $S$.

**11.** How many right triangles have three integer side lengths and perimeter $8! = 40320$? (Congruent triangles are counted only once.)

**12.** Triangle $ABC$ has $AB = 17$, $BC = 28$ and $CA = 25$. Its nine-point circle (the circle through the midpoints of the three sides) touches its incircle at exactly one point $F$. Then $BF^2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

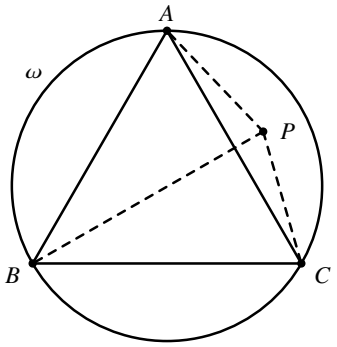



**13.** A token starts at 0 on the number line. Each second it moves one unit to the right with probability $\frac{2}{3}$, or two units to the left with probability $\frac{1}{3}$, independently of all previous moves. It stops as soon as it lands

on 4 or lands on any number less than or equal to $-5$. The probability that it stops at 4 is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

**14.** As $t$ runs over $0 \le t < 2\pi$, the point $(\sin 6t,\ \cos 11t)$ traces a closed curve. Call a point of the plane a crossing point if it equals $(\sin 6t, \cos 11t)$ for at least two different values of $t$ with $0 \le t < 2\pi$. How many crossing points lie strictly inside the circle $x^2 + y^2 = 1$?

**15.** Let $ABC$ be an equilateral triangle, and let $\omega$ be the circle through $A$, $B$ and $C$. A point $P$ is chosen uniformly at random from the interior of $\omega$ (so $P$ may lie outside the triangle). For every such point, the three lengths $PA$, $PB$, $PC$ are the side lengths of a non-degenerate triangle. The probability that this triangle is obtuse can be written as $\frac{p}{q} - \frac{r\sqrt{s}}{\pi}$, where $p, q, r, s$ are positive integers, $\gcd(p,q) = 1$, and $s$ is not divisible by the square of any prime. Find $p+q+r+s$.



# Answer Key — AIME — Hard Edition Practice Contest 1

**1.** 88

**2.** 667

**3.** 462

**4.** 216

**5.** 129

**6.** 163

**7.** 57

**8.** 720

**9.** 256

**10.** 224

**11.** 21

**12.** 449

**13.** 835

**14.** 67

**15.** 13

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# Solutions — AIME — Hard Edition Practice Contest 1

**1. Answer:** 88**. The idea.** The sum pairs each divisor $d$ with its partner $N/d$, and both $\varphi$ and $\tau$ are multiplicative, so $S$ is a multiplicative function of $N$. That reduces everything to a single prime power, where the sum turns out to be the plain sum of divisors: $S = \sigma(N)$.

**Work it out.** Write $f(N) = \sum_{d|N} \varphi(d)\tau(N/d)$. If $N = ab$ with $\gcd(a, b) = 1$, every divisor of $N$ is uniquely $d = d_1 d_2$ with $d_1 \mid a$, $d_2 \mid b$, and then $\varphi(d) = \varphi(d_1)\varphi(d_2)$, $\tau(N/d) = \tau(a/d_1)\tau(b/d_2)$. So the sum factors: $f(ab) = f(a)f(b)$.

For a prime power, $\tau(p^{e-i}) = e - i + 1$ counts the exponents $j$ with $i \le j \le e$. Hence $f(p^e) = \displaystyle\sum_{i=0}^{e} \varphi(p^i)\,\#\{j : i \le j \le e\} = \sum_{j=0}^{e}\sum_{i=0}^{j} \varphi(p^i) = \sum_{j=0}^{e} p^j$, because $\sum_{i\le j} \varphi(p^i) = \sum_{c|p^j} \varphi(c) = p^j$ (the totients telescope: $1 + (p-1) + (p^2-p) + \cdots + (p^j - p^{j-1}) = p^j$). So $f(p^e) = 1 + p + \cdots + p^e$ and $f = \sigma$, the sum-of-divisors function.

Now $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, so $S = \sigma(10!) = (2^9 - 1) \cdot \dfrac{3^5 - 1}{2} \cdot (1+5+25) \cdot (1+7) = 511 \cdot 121 \cdot 31 \cdot 8$. $511 \cdot 121 = 61831$, $61831 \cdot 31 = 1916761$, and $1916761 \cdot 8 = 15334088$. The remainder is **088**.

**Check.** Test the identity on $N = 12 = 2^2 \cdot 3$: the divisors $1, 2, 3, 4, 6, 12$ give $\varphi(d)\tau(12/d) = 1 \cdot 6 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 2 + 2 \cdot 2 + 4 \cdot 1 = 28$, and indeed $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. (In the language of Dirichlet convolution: $\varphi * \tau = \varphi * 1 * 1 = \mathrm{id} * 1 = \sigma$.)

**2. Answer:** 667**. The idea.** A circle of radius $\rho$ tangent to the two sides at $A$ has its centre $O_A$ on the angle bisector $AI$, at the fraction $\frac{\rho}{r}$ of the way from $A$ to the incenter $I$ ($r$ is the inradius). So $IO_A = \frac{r-\rho}{r}\,IA$, and the same for $B$ and $C$: the triangle $O_A O_B O_C$ is the image of $ABC$ under the homothety centred at $I$ with ratio $\frac{r-\rho}{r}$. The centre of $\omega$ is at distance $2\rho$ from all three corner centres, so it is the circumcentre of $O_A O_B O_C$, and $2\rho$ equals that triangle's circumradius $\frac{r-\rho}{r}R$.

**Work it out.** The semiperimeter is $s = 35$, so Heron gives the area $K = \sqrt{35 \cdot 7 \cdot 18 \cdot 10} = \sqrt{44100} = 210$. Then $r = \frac{K}{s} = 6$ and $R = \frac{abc}{4K} = \frac{25 \cdot 28 \cdot 17}{840} = \frac{85}{6}$.

For a given $\rho$ the three corner circles are forced (their centres lie on the bisectors, at distance $\rho$ from the sides), and then the centre of $\omega$ is forced too (the only point equidistant from $O_A$, $O_B$, $O_C$). So the single condition is $2\rho = \dfrac{85}{6} \cdot \dfrac{6-\rho}{6}$, $\qquad 72\rho = 510 - 85\rho$, $\qquad \rho = \dfrac{510}{157}$. The left side increases and the right side decreases in $\rho$, so this is the only solution. Since 157 is prime and $510 = 2 \cdot 3 \cdot 5 \cdot 17$, the fraction is in lowest terms and $m + n = 510 + 157 =$ **667**.

**Check.** Coordinates: $B = (0, 0)$, $C = (28, 0)$, $A = (20, 15)$ (indeed $20^2 + 15^2 = 25^2$ and $8^2 + 15^2 = 17^2$). The incenter is $\frac{28A+17B+25C}{70} = (18, 6)$, so $r = 6$; the circumcentre is $\left(14, \frac{13}{6}\right)$, at distance $\sqrt{196 + \frac{169}{36}} = \frac{85}{6}$ from $B$. With $\frac{\rho}{r} = \frac{85}{157}$, the corner centre at $B$ is $\frac{85}{157}(18, 6) = \left(\frac{1530}{157}, \frac{510}{157}\right)$, and the centre of $\omega$ is $I + \frac{72}{157}(O - I) = \left(\frac{2538}{157}, \frac{666}{157}\right)$. Their distance is $\frac{\sqrt{1008^2 + 156^2}}{157} = \frac{1020}{157} = 2\rho$, as required. The centre of $\omega$ is at height $\frac{666}{157} > \rho$ above $BC$ (and farther than $\rho$ from the other two sides too), so all four circles really do fit inside the triangle.

**3. Answer:** 462**. The idea.** Cut the string into runs (maximal blocks of equal letters). A run of length $\ell$ contains $\ell - 1$ occurrences of its doubled letter, so if there are $h$ heads in $r_H$ runs and $t$

tails in $r_T$ runs, then $\#HH = h - r_H$ and $\#TT = t - r_T$. Runs alternate, so $r_H - r_T \in \{-1, 0, 1\}$, and a parity argument pins down the letter counts.

**Work it out.** The condition $\#HH = \#TT + 1$ reads $h - t = 1 + (r_H - r_T)$. Since $h + t = 12$, $h - t$ is even, so $r_H - r_T$ is odd: $r_H - r_T = \pm 1$. That leaves two cases.

Case 1: $r_T = r_H + 1$ and $h = t = 6$. The string starts and ends with $T$; with $r$ runs of $H$ there are $r + 1$ runs of $T$. Splitting 6 heads into $r$ nonempty runs can be done in $\binom{5}{r-1}$ ways and 6 tails into $r+1$ runs in $\binom{5}{r}$ ways. So the count is $\sum_r \binom{5}{r-1}\binom{5}{r} = \sum_r \binom{5}{r-1}\binom{5}{5-r} = \binom{10}{4} = 210$ by Vandermonde.

Case 2: $r_H = r_T + 1$ and $h - t = 2$, so $h = 7$, $t = 5$. The string starts and ends with $H$; with $r$ tail runs there are $r + 1$ head runs. The count is $\sum_r \binom{6}{r}\binom{4}{r-1} = \sum_r \binom{6}{r}\binom{4}{5-r} = \binom{10}{5} = 252$.

Total: $210 + 252 =$ **462**.

**Check.** Try length 4 by hand with the same condition. Case 1 needs $h = t = 2$, starting and ending with $T$: $THHT$ only (one $HH$, no $TT$). Case 2 needs $h = 3$, $t = 1$, starting and ending with $H$: $HTHH$ and $HHTH$ (each has one $HH$, no $TT$). Formulas: $\sum_r \binom{1}{r-1}\binom{1}{r} = 1$ and $\sum_r \binom{2}{r}\binom{0}{r-1} = 2$, total 3; listing all strings with $\#HH = \#TT + 1$ confirms exactly $THHT$, $HTHH$, $HHTH$ (for instance $HHHT$ has 2 vs 0, $HHTT$ has 1 vs 1).

**4. Answer:** 216**. The idea.** If $s = kr$ with $k \neq -1$, then $r = \frac{-a}{k+1}$ is rational, and because $k$ and $k + 1$ share no factor, $b = kr^2$ can only be an integer if $r$ is an integer. So apart from one special ratio, the polynomial must factor as $(x - r)(x - s)$ over the integers with one root dividing the other. The special ratio is $k = -1$, which is exactly $a = 0$; it works for every $b$, including $b > 0$, where the roots $\pm i\sqrt{b}$ are not real.

**Work it out.** Let the roots be $r$ and $s = kr$ with $k$ an integer. Since $rs = b \neq 0$, neither root is 0, so $k \neq 0$.

*The ratio* $-1$. If $k = -1$ then $a = -(r + s) = 0$. Conversely, if $a = 0$ the roots are $\pm\sqrt{-b}$ (real if $b < 0$, imaginary if $b > 0$), and one is $-1$ times the other. That gives all 60 pairs $(0, b)$ with $b \neq 0$.

*Any other ratio.* If $k \neq -1$, Vieta gives $-a = r + s = (k + 1)r$, so $r = \frac{-a}{k+1}$ is rational, and $b = rs = kr^2 = \frac{ka^2}{(k+1)^2}$. As $\gcd(k, (k+1)^2) = 1$, the integer $b$ forces $(k+1)^2 \mid a^2$, so $(k+1) \mid a$, so $r$ and $s = kr$ are integers. (Here $a \neq 0$, since $a = 0$ would give $k = -1$.) Conversely, any nonzero integers $r, s$ with $r + s \neq 0$ and one dividing the other give a valid pair $(a, b) = (-(r + s), rs)$, and different unordered pairs $\{r, s\}$ give different polynomials. So we count the unordered pairs $\{r, s\}$ of nonzero integers with $r + s \neq 0$, $|r|$ dividing $|s|$, $|rs| \leq 30$ and $|r + s| \leq 30$. Write $|r| = p$ and $|s| = pq$ with $p, q \geq 1$; then $|rs| = p^2 q \leq 30$, so $p \leq 5$.

*Same signs.* Here $|r + s| = p(q + 1)$. For $p = 1$: $q \leq 30$ and $q + 1 \leq 30$, so 29 choices. For $p = 2, 3, 4, 5$: $q \leq 7, 3, 1, 1$, and then $p(q+1) \leq 16$ automatically, giving $7+3+1+1 = 12$. That is 41 pairs with both roots positive, and 41 more with both negative: 82.

*Opposite signs.* Now $q = 1$ would mean $s = -r$ (already counted under $a = 0$), so $q \geq 2$, and $|r + s| = p(q - 1) < p^2 q \leq 30$ holds automatically. For $p = 1, 2, 3$: $q$ runs over 2..30, 2..7, 2..3, giving $29 + 6 + 2 = 37$; $p = 4, 5$ give none. Either root can be the negative one ($\{p, -pq\}$ or $\{-p, pq\}$), so 74 pairs.

Total: $60 + 82 + 74 =$ **216**.

**Check.** Shrink the box to $|a| \leq 3$, $1 \leq |b| \leq 3$ and list by hand. The rule predicts 6 pairs with $a = 0$, then roots $\{1, 1\}$, $\{1, 2\}$ and their negatives, then $\{1, -2\}$, $\{-1, 2\}$, $\{1, -3\}$, $\{-1, 3\}$:

$6+4+4=14$. For example, $x^2-2x-3=(x-3)(x+1)$ qualifies, while $x^2+x+1$ has roots $\omega,\omega^2$ with ratio $\omega$, not an integer, and $x^2+2x+3$ has roots $-1\pm i\sqrt{2}$, whose ratio is not even real. A direct check of all $6\cdot 7=42$ small pairs finds exactly 14, and the same brute force over the full box finds 216.

**5. Answer:** 129**. The idea.** Think of a colouring as a vector in $\mathbb{F}_2^{12}$ (red = 1) and send it to its parity pattern in $\mathbb{F}_2^8$: the red degree of each vertex mod 2. This map is linear, so every pattern it hits has exactly the same number of preimages, namely $2^{12}/|\text{image}|$. So we only need the size of the image and whether the all-odd pattern is in it.

**Work it out.** Linearity: recolouring one edge flips the parity at exactly its two endpoints, so the pattern of a sum of edge sets is the sum of the patterns.

The image: every pattern has an even number of odd vertices (the degrees add up to twice the number of red edges). Conversely, any pattern with an even number of 1's is hit: pair up the odd vertices and, for each pair, add (mod 2) the edges of a path joining them; each path flips exactly its two ends. The cube is connected, so such paths exist. So the image is the set of even-weight patterns, which has $2^7=128$ elements.

The all-odd pattern has weight 8, which is even, so it is in the image (a perfect matching, such as the four vertical edges, realises it). Therefore exactly $2^{12}/2^7=32$ colourings work, and the probability is $\frac{32}{4096}=\frac{1}{128}$. So $m+n=1+128=$ **129**.

**Check.** Count the colourings directly from one of them. Two colourings with the same pattern differ by an edge set with all degrees even, i.e. a union of edge-disjoint cycles. The cube has $V-1=7$ edges in a spanning tree and $12-7=5$ independent cycles (for instance the 5 faces other than the top), giving $2^5=32$ even edge sets, matching the 32 above.

**6. Answer:** 163**. The idea.** Squaring doubles arguments and squares moduli, so it is one-to-one on the first quadrant: different points of the quadrant have different squares. Hence areas of images can be added and subtracted exactly like areas of the original sets. Build $R$ out of rectangles with a corner at 0 by inclusion–exclusion; the image of such a rectangle lies between two parabolas, and Archimedes' $\frac{2}{3}$ rule gives its area.

**Work it out.** Let $Q(a,b)$ be the rectangle $0\le x\le a$, $0\le y\le b$, and write $w=z^2=u+vi$, so $u=x^2-y^2$ and $v=2xy$. The side $x=a$ goes to $u=a^2-\frac{v^2}{4a^2}$, the side $y=b$ goes to $u=\frac{v^2}{4b^2}-b^2$, and the two sides on the axes go to the segment $-b^2\le u\le a^2$ of the real axis. Both parabolic arcs end at $w=(a+bi)^2$, at height $v=2ab$. So the image of $Q(a,b)$ is the set of points with $0\le v\le 2ab$ lying between the two parabolas, and at height $v$ its width is $\left(a^2-\frac{v^2}{4a^2}\right)-\left(\frac{v^2}{4b^2}-b^2\right)=(a^2+b^2)\left(1-\frac{v^2}{4a^2b^2}\right)$. A region whose width at height $v$ is $W\left(1-\frac{v^2}{V^2}\right)$ for $0\le v\le V$ has the same area (Cavalieri) as half of a parabolic segment of width $W$ and height $V$, which by Archimedes is $\frac{2}{3}WV$. So the image of $Q(a,b)$ has area $A(a,b)=\frac{2}{3}(a^2+b^2)(2ab)=\frac{4}{3}ab(a^2+b^2)$.

Now $R=Q(2,3)-Q(1,3)-Q(2,1)+Q(1,1)$ in the sense of inclusion–exclusion (up to boundary segments of zero area), and since squaring is one-to-one on the quadrant the images obey the same relation. With $A(2,3)=\frac{4}{3}\cdot 6\cdot 13=104$, $A(1,3)=\frac{4}{3}\cdot 3\cdot 10=40$, $A(2,1)=\frac{4}{3}\cdot 2\cdot 5=\frac{40}{3}$ and $A(1,1)=\frac{4}{3}\cdot 1\cdot 2=\frac{8}{3}$, the area is $104-40-\dfrac{40}{3}+\dfrac{8}{3}=64-\dfrac{32}{3}=\dfrac{160}{3}$, so $m+n=160+3=$ **163**.

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**Check.** The formula passes a sanity test: squaring stretches areas near $z$ by $|2z|^2 = 4(x^2+y^2)$, and $A(a,b) = \frac{4}{3}ab(a^2+b^2)$ is exactly 4 times the integral of $x^2+y^2$ over $Q(a,b)$, namely $4\left(\frac{a^3b}{3} + \frac{ab^3}{3}\right)$. For $R$ itself this gives $4\left(\frac{8-1}{3}\cdot 2 + 1\cdot\frac{27-1}{3}\right) = 4\cdot\frac{14+26}{3} = \frac{160}{3}$, matching. A trap: using only the four corners $(1+i)^2 = 2i$, $(2+i)^2 = 3+4i$, $(2+3i)^2 = -5+12i$, $(1+3i)^2 = -8+6i$ as the vertices of a quadrilateral gives area 50 by the shoelace formula, which is wrong because the sides of the image are parabolic arcs, not segments.

**7. Answer:** 57**. The idea.** Tangency to the three coordinate planes forces the centre to be $(\pm r, \pm r, \pm r)$: one sign pattern per octant. Tangency to the last plane is then a linear equation in $r$ with an absolute value, which each octant solves in at most two ways. The answer is a careful sign bookkeeping, not a picture of the tetrahedron.

**Work it out.** The centre is $(s_1 r, s_2 r, s_3 r)$ with each $s_i = \pm 1$. The distance to $x + 2y + 2z = 12$ is $\frac{|rt-12|}{3}$, where $t = s_1 + 2s_2 + 2s_3$ and $\sqrt{1+4+4} = 3$. Tangency means $|rt - 12| = 3r$, so $r(t-3) = 12$ or $r(t+3) = 12$, and we need $r > 0$.

The eight octants give $t = 5$ for $(+,+,+)$; $t = 3$ for $(-,+,+)$; $t = 1$ for $(+,+,-)$ and $(+,-,+)$; $t = -1$ for $(-,+,-)$ and $(-,-,+)$; $t = -3$ for $(+,-,-)$; $t = -5$ for $(-,-,-)$.

$t = 5$: $r = 6$ or $r = \frac{12}{8} = \frac{3}{2}$ (two spheres). $t = 3$: $t - 3 = 0$ fails, $r = \frac{12}{6} = 2$. $t = 1$: only $r = \frac{12}{4} = 3$, in each of the two octants. $t = -1$: only $r = \frac{12}{2} = 6$, in each of the two octants. $t = -3$ and $t = -5$: both $t-3$ and $t+3$ are $\le 0$, no sphere. That is seven spheres, with radii summing to $\frac{3}{2} + 6 + 2 + 3 + 3 + 6 + 6 = \frac{55}{2}$, so $m + n = 55 + 2 = \mathbf{057}$.

**Check.** The planes bound the tetrahedron with vertices $O$, $(12,0,0)$, $(0,6,0)$, $(0,0,6)$: volume $V = 72$, face areas 36 (in $z=0$), 36 (in $y=0$), 18 (in $x=0$) and 54 (the slanted face), total 144. The tangent spheres have radius $\frac{3V}{\pm F_1 \pm F_2 \pm F_3 \pm F_4}$ whenever that signed area sum is positive: $\frac{216}{144} = \frac{3}{2}$ (insphere); $\frac{216}{144-108} = 6$, $\frac{216}{144-72} = 3$ (twice), $\frac{216}{144-36} = 2$ (exspheres); and for two faces against two: $|54 + 18 - 36 - 36| = 0$ (no sphere) and $|54 + 36 - 36 - 18| = 36$ twice, giving 6 and 6. The same seven radii.

**8. Answer:** 720**. The idea.** Modulo the prime 101 the nonzero residues form a cyclic group of order 100: there is a primitive root $g$, and every nonzero residue is $g^k$ for a unique $k$ modulo 100. Raising to the $n$-th power is then multiplication of exponents by $n$, so its image is easy to describe. Whether 6 lies in that image depends only on the order of 6 and on $\gcd(n, 100)$.

**Work it out.** First find the order of 6. We have $6^2 = 36$ and $6^5 = 7776 = 77 \cdot 101 - 1$, so $6^5 \equiv -1$ and $6^{10} \equiv 1$. Since $6^2 \not\equiv 1$ and $6^5 \not\equiv 1$, the order of 6 is exactly 10.

Write $x = g^j$. Then $x^n = g^{nj}$, and as $j$ runs over all residues mod 100, $nj$ runs over exactly the multiples of $d = \gcd(n, 100)$. So the $n$-th powers are the $100/d$ elements $g^{dm}$, which form the unique subgroup of order $100/d$, namely the elements $y$ with $y^{100/d} \equiv 1$.

Hence 6 is an $n$-th power exactly when its order 10 divides $100/d$, i.e. when $d \mid 10$. Now $d = \gcd(n, 100) = 2^{\min(v_2(n),2)}\, 5^{\min(v_5(n),2)}$, and $d \mid 10$ means neither 4 nor 25 divides $d$: that is, $4 \nmid n$ and $25 \nmid n$.

Count by inclusion–exclusion among $1, \ldots, 1000$: there are 250 multiples of 4, 40 multiples of 25 and 10 multiples of 100, so the answer is $1000 - 250 - 40 + 10 = 720$.

**Check.** Small cases agree with the criterion. For $n = 4$: the fourth powers form the subgroup of order 25, and $6^{25} = (6^5)^5 \equiv -1$, so 6 is not a fourth power. For $n = 2$: squares are the subgroup

of order 50, and $6^{50} = (6^{10})^5 \equiv 1$, so 6 is a square mod 101. Indeed 6 is a quadratic residue mod 101 because $\left(\frac{2}{101}\right) = -1$ and $\left(\frac{3}{101}\right) = \left(\frac{101}{3}\right) = \left(\frac{2}{3}\right) = -1$, whose product is +1. The answer is **720**.

**9. Answer:** 256**. The idea.** By linearity of expectation, the answer is the sum over the possible tie times of the probability of a tie at that time. A tie can only happen after an even number $2k$ of draws, and each probability is a ratio of binomial coefficients. The sum then collapses by the classical identity $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.

**Work it out.** A tie after $2k$ draws means exactly $k$ of the first $2k$ balls are red. The colour order is a uniformly random choice of which 5 of the 10 positions are red, so $P(\text{tie at } 2k) = \dfrac{\binom{2k}{k}\binom{10-2k}{5-k}}{\binom{10}{5}}$ for $k = 1, 2, 3, 4, 5$.

The $k = 0$ term of the same formula equals 1. So $E = \dfrac{1}{252}\left(\sum_{k=0}^{5} \binom{2k}{k}\binom{10-2k}{5-k}\right) - 1$.

The identity: $\sum_{k\geq 0} \binom{2k}{k} x^k = (1-4x)^{-1/2}$, and squaring gives $(1-4x)^{-1} = \sum 4^n x^n$. Comparing the coefficients of $x^5$ gives $\sum_{k=0}^{5} \binom{2k}{k}\binom{10-2k}{5-k} = 4^5 = 1024$. Directly: $252 + 70 \cdot 2 + 20 \cdot 6 + 6 \cdot 20 + 2 \cdot 70 + 252 = 1024$.

Hence $E = \dfrac{1024}{252} - 1 = \dfrac{256}{63} - 1 = \dfrac{193}{63}$. Since 193 is prime, the fraction is in lowest terms, and $m + n = 193 + 63 = 256$.

**Check.** The individual terms are $\frac{140}{252}, \frac{120}{252}, \frac{120}{252}, \frac{140}{252}, \frac{252}{252}$ for $2k = 2, 4, 6, 8, 10$. They sum to $\frac{772}{252} = \frac{193}{63} \approx 3.06$, which is plausible: the final draw is always a tie, and the first two draws are split with probability $\frac{5}{9}$. The answer is **256**.

**10. Answer:** 224**. The idea.** Call $6, 7, 8, 9, 10$ big and $1, 2, 3, 4, 5$ small. Reversing an arc of the circle changes just two products. This shows that an optimal circle never puts two big numbers side by side, so the numbers alternate big, small, big, small. For an alternating circle, $S$ splits into layers that can each be bounded separately.

**Work it out.** *Step 1 (exchange).* Suppose the circle reads $p, q, \ldots, r, s, \ldots$ around, with $pq$ and $rs$ two neighbouring pairs that share no number. Reversing the arc from $q$ to $r$ gives the circle $p, r, \ldots, q, s, \ldots$. Only the products $pq + rs$ change, and they become $pr + qs$. The change in $S$ is $pr + qs - pq - rs = (p - s)(r - q)$.

*Step 2 (alternation).* Each number has two neighbours. Counting the neighbour slots of the five big numbers gives $10 = 2(\#\text{big–big pairs}) + \#\text{mixed pairs}$, and the same count for the small numbers gives $10 = 2(\#\text{small–small pairs}) + \#\text{mixed pairs}$. So there are as many big–big pairs as small–small pairs. If a big–big pair $pq$ exists, take a small–small pair $rs$, labelled so the circle reads $p, q, \ldots, r, s$. The change $(p - s)(r - q)$ is (positive)(negative) $< 0$, so $S$ was not minimal. Hence a minimal circle alternates big and small.

*Step 3 (layers).* In an alternating circle every pair is big times small. Let $t_b$ be the sum of the two (small) neighbours of the big number $b$. Every small number has two big neighbours, so $\sum_b t_b = 2(1 + 2 + 3 + 4 + 5) = 30$. Writing $b = 5 + \sum_{k=1}^{5} [b \geq 5 + k]$ gives $S = \sum_b b\, t_b = 5 \cdot 30 + T_1 + T_2 + T_3 + T_4 + T_5$, where $T_k$ is the total of the small neighbours of the big numbers $\geq 5 + k$. Bound each layer.

$T_1 = 30$.

$T_2 = 30 - t_6 \geq 30 - (4 + 5) = 21$.

$T_3$ adds 6 small neighbour slots of $8, 9, 10$. Each small number fills at most two slots, so $T_3 \geq 1 + 1 + 2 + 2 + 3 + 3 = 12$. Equality would force $1, 2, 3$ to have both neighbours among $8, 9, 10$, closing a 6-cycle, which is impossible on a single circle of 10. So $T_3 \geq 13$.

$T_4$ adds 4 slots of $9, 10$, so $T_4 \geq 1 + 1 + 2 + 2 = 6$. Equality would close the 4-cycle $9, 1, 10, 2$, so $T_4 \geq 7$.

$T_5 = t_{10} \geq 1 + 2 = 3$.

So $S \geq 150 + 30 + 21 + 13 + 7 + 3 = 224$.

*Step 4 (equality).* The circle $1, 9, 3, 7, 5, 6, 4, 8, 2, 10$ has $t_{10} = 3$, $t_9 = 4$, $t_8 = 6$, $t_7 = 8$, $t_6 = 9$, meeting every bound. Its products are $9 + 27 + 21 + 35 + 30 + 24 + 32 + 16 + 20 + 10 = 224$.

**Check.** A computer search over all $9!/2$ circular orders confirms that 224 is the minimum. For comparison, the natural order $1, 2, \ldots, 10$ gives $2 + 6 + 12 + \cdots + 90 + 10 = 340$. The answer is **224**.

**11. Answer:** 21**. The idea.** Every right triangle with integer sides is a multiple of a primitive one, and Euclid's formula describes the primitive ones by two parameters $m > n$. The perimeter then factors as $2km(m+n)$, so the problem becomes a divisor count of $\frac{8!}{2}$ with one inequality and two side conditions (coprimality and parity) that are easy to forget.

**Work it out.** *Parametrisation.* If $a^2 + b^2 = c^2$ with $\gcd(a, b, c) = k$, then $(a/k, b/k, c/k)$ is primitive, so exactly one leg is even; call the odd leg $a$. From $\left(\frac{c-a}{2}\right)\left(\frac{c+a}{2}\right) = \left(\frac{b}{2}\right)^2$ with the two factors coprime (after dividing by $k$), both are squares, $n^2$ and $m^2$. So every triangle is exactly one $(k, m, n)$ with $m > n \geq 1$, $\gcd(m, n) = 1$, $m - n$ odd, and sides $k(m^2 - n^2)$, $2kmn$, $k(m^2 + n^2)$. Its perimeter is $2km(m + n)$.

*Reformulate.* Put $s = m + n$. The conditions become: $s$ is odd, $\gcd(m, s) = 1$, and $m < s < 2m$ (from $1 \leq n < m$). The perimeter condition is $k\,m\,s = \frac{8!}{2} = 20160 = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. So we count pairs $(m, s)$: $s$ is an odd divisor of 315 with $s > 1$, and $m$ lies strictly between $\frac{s}{2}$ and $s$, is coprime to $s$, and divides $20160/s$ ($k$ is then forced).

*Count, divisor by divisor.* $s = 3$: $m = 2$. (1)

$s = 5$: $m = 3, 4$. (2)

$s = 7$: $m = 4, 5, 6$. (3)

$s = 9$: $m$ from $5, \ldots, 8$, prime to 3, dividing $2240 = 2^6 \cdot 5 \cdot 7$: $m = 5, 7, 8$. (3)

$s = 15$: $m$ from $8, \ldots, 14$ built from 2 and 7 only: $8, 14$. (2)

$s = 21$: $m$ from $11, \ldots, 20$ built from 2 and 5: $16, 20$. (2)

$s = 35$: $m$ from $18, \ldots, 34$ dividing $576 = 2^6 \cdot 3^2$: $18, 24, 32$. (3)

$s = 45$: $m$ from $23, \ldots, 44$ dividing $448 = 2^6 \cdot 7$: $28, 32$. (2)

$s = 63$: $m$ from $32, \ldots, 62$ dividing $320 = 2^6 \cdot 5$: $32, 40$. (2)

$s = 105$: $m$ from $53, \ldots, 104$ built from 2 only (it must divide $192 = 2^6 \cdot 3$ and be prime to 3): 64. (1)

$s = 315$: $m$ would have to be a divisor of 64 above 157: none.

Total: $1 + 2 + 3 + 3 + 2 + 2 + 3 + 2 + 2 + 1 = 21$.

**Check.** Two examples. $s = 3$, $m = 2$ gives $n = 1$ and $k = 20160/6 = 3360$: the 3-4-5 triangle scaled by 3360, perimeter $12 \cdot 3360 = 40320$. $s = 105$, $m = 64$ gives $n = 41$, $k = 3$: sides

$3(64^2-41^2)=7245$, $3\cdot 2\cdot 64\cdot 41=15744$, $3(64^2+41^2)=17331$, and $7245+15744+17331=40320$. Forgetting that $s$ must be odd, or that $\gcd(m,s)=1$, would count some triangles twice (for instance $s=15$, $m=12$ passes every other test, but $(m,n)=(12,3)$ is $3\cdot(4,1)$, the case $s=5$, $m=4$ again). A direct computer search over all legs confirms 21. The answer is **021**.

**12. Answer:** 449**. The idea.** Two circles that touch meet on the line through their centres. Here the incircle lies inside the nine-point circle, so $F$ is the point of the incircle farthest from the nine-point centre $N$: $F=I+r\cdot\frac{I-N}{|I-N|}$. So we only need coordinates for the incentre $I$ and for $N$, the midpoint of the circumcentre $O$ and the orthocentre $H$. This triangle has very clean coordinates.

**Work it out.** Put $B=(0,0)$ and $C=(28,0)$. Then $A=(x,y)$ with $x^2+y^2=289$ and $(x-28)^2+y^2=625$. Subtracting gives $-56x+784=336$, so $x=8$ and $y=15$: $A=(8,15)$.

*Incentre.* The area is $\frac{1}{2}\cdot 28\cdot 15=210$ and $s=35$, so $r=6$. With side lengths $a=28$, $b=25$, $c=17$ as weights, $I=\frac{28A+25B+17C}{70}=\left(\frac{224+476}{70},\frac{420}{70}\right)=(10,6)$.

*Circumcentre.* $O=(14,k)$ with $14^2+k^2=6^2+(15-k)^2$, so $30k=65$ and $k=\frac{13}{6}$. Then $R^2=196+\frac{169}{36}=\frac{7225}{36}$, so $R=\frac{85}{6}$.

*Orthocentre and N.* $H=A+B+C-2O=(8,15-\frac{13}{3})=(8,\frac{32}{3})$. Then $N=\frac{O+H}{2}=\left(11,\frac{77}{12}\right)$, and the nine-point radius is $\frac{R}{2}=\frac{85}{12}$.

*The touching point.* $I-N=\left(-1,-\frac{5}{12}\right)$ has length $\frac{13}{12}$, which is exactly $\frac{85}{12}-6$: the incircle touches the nine-point circle from inside. So $F=(10,6)+6\cdot\frac{12}{13}\left(-1,-\frac{5}{12}\right)=\left(10-\frac{72}{13},\,6-\frac{30}{13}\right)=\left(\frac{58}{13},\frac{48}{13}\right)$.

Finally $BF^2=\frac{58^2+48^2}{169}=\frac{3364+2304}{169}=\frac{5668}{169}=\frac{436}{13}$, and $m+n=436+13=449$.

**Check.** $|F-N|=\frac{13}{12}+6=\frac{85}{12}$, so $F$ is on the nine-point circle, and $|F-I|=6$, so it is on the incircle. As a further check, the midpoint of $BC$, $(14,0)$, is at distance $\sqrt{9+(77/12)^2}=\sqrt{\frac{7225}{144}}=\frac{85}{12}$ from $N$, as it should be. The answer is **449**.

**13. Answer:** 835**. The idea.** Let $p(k)$ be the probability of stopping at 4 from position $k$. It satisfies a linear recurrence whose characteristic polynomial has a double root at 1, because the walk has zero drift: $\frac{2}{3}\cdot 1+\frac{1}{3}\cdot(-2)=0$. So $p$ has a three-parameter closed form. The subtle part is the boundary: a jump of $-2$ can overshoot, so the walk can stop at $-5$ *or* at $-6$, and both give conditions.

**Work it out.** For $-4\le k\le 3$, conditioning on the first move gives $p(k)=\frac{2}{3}p(k+1)+\frac{1}{3}p(k-2)$. The boundary values are $p(4)=1$ and $p(-5)=p(-6)=0$. (From $-3$ or $-4$ a left jump lands on $-5$ or $-6$, and 4 can only be reached exactly, by a right step.)

Try $p(k)=x^k$: then $3x^2=2x^3+1$, i.e. $2x^3-3x^2+1=(x-1)^2(2x+1)=0$. So $p(k)=\alpha+\beta k+\gamma\left(-\frac{1}{2}\right)^k$ on $-6\le k\le 4$. This form satisfies the order-3 recurrence and has three free constants, which is exactly the number needed.

The conditions: $p(-6)=\alpha-6\beta+64\gamma=0$, $p(-5)=\alpha-5\beta-32\gamma=0$, $p(4)=\alpha+4\beta+\frac{1}{16}\gamma=1$.

Subtracting the first two gives $\beta=96\gamma$, and then $\alpha=6\beta-64\gamma=512\gamma$. The third becomes $\left(512+384+\frac{1}{16}\right)\gamma=1$, so $\gamma=\frac{16}{14337}$.

Then $p(0) = \alpha + \gamma = 513\gamma = \frac{8208}{14337}$. Factor: $8208 = 2^4 \cdot 3^3 \cdot 19$ and $14337 = 3^5 \cdot 59$, so $p(0) = \frac{304}{531}$ and $m + n = 835$.

**Check.** Test the recurrence at $k = 3$, i.e. $p(3) = \frac{2}{3}p(4) + \frac{1}{3}p(1)$. From the formula, $p(3) = (512 + 288 - \frac{1}{8})\gamma = \frac{6399}{8}\gamma$, $p(1) = (512 + 96 - \frac{1}{2})\gamma = \frac{1215}{2}\gamma$ and $p(4) = 1 = \frac{14337}{16}\gamma$. The right side is $\left(\frac{14337}{24} + \frac{4860}{24}\right)\gamma = \frac{19197}{24}\gamma = \frac{6399}{8}\gamma$, as required. Also, $p(0) \approx 0.573$ is a bit more than $\frac{5}{9}$, the value a ruin problem with no overshoot (targets 4 and $-5$) would give; overshooting to $-6$ helps the right side slightly. The answer is **835**.

**14. Answer:** 67**. The idea.** Two parameters $t \neq s$ give the same point exactly when $\sin 6s = \sin 6t$ and $\cos 11s = \cos 11t$. Each equation says that either $s - t$ or $s + t$ lies in a known lattice, and a parity argument removes one of the four combinations. Each of the remaining two combinations lists its crossing points explicitly, with coordinates $\pm\sin$ or $\pm\cos$ of rational multiples of $\pi$. Being inside the circle then compares two sines, which reduces to comparing distances to the nearest integer.

**Work it out.** Write $\|u\|$ for the distance from $u$ to the nearest integer. For real $u, v$, $\sin^2 \pi u < \sin^2 \pi v$ exactly when $\|u\| < \|v\|$, and $|\cos \pi u| = |\sin \pi(u + \frac{1}{2})|$.

*When do two parameters meet?* Mod $2\pi$, $\sin 6s = \sin 6t$ means $6s \equiv 6t$ or $6s \equiv \pi - 6t$, that is $s - t \in \frac{\pi}{3}\mathbb{Z}$ or $s + t \in \frac{\pi}{6} + \frac{\pi}{3}\mathbb{Z}$. Also, $\cos 11s = \cos 11t$ means $s - t \in \frac{2\pi}{11}\mathbb{Z}$ or $s + t \in \frac{2\pi}{11}\mathbb{Z}$. Of the four combinations:

(i) $s - t \in \frac{\pi}{3}\mathbb{Z} \cap \frac{2\pi}{11}\mathbb{Z} = 2\pi\mathbb{Z}$ gives $s = t$.

(ii) $s + t \equiv \frac{(2j+1)\pi}{6} \equiv \frac{2\pi k}{11} \pmod{2\pi}$ would give $11(2j + 1) - 12k \in 132\mathbb{Z}$, i.e. odd = even. Impossible.

(A) $s - t = \frac{\pi j}{3}$ with $1 \leq j \leq 5$ and $s + t \equiv \frac{2\pi k}{11}$ with $0 \leq k \leq 10$.

(B) $s + t \equiv \frac{(2j+1)\pi}{6}$ with $0 \leq j \leq 5$ and $s - t \equiv \frac{2\pi k}{11}$ with $1 \leq k \leq 10$.

In each case $2t$ is fixed mod $2\pi$, so $t$ has two values, $\pi$ apart, and $s$ follows. That gives $5 \cdot 11 \cdot 2 = 110$ ordered pairs $(t, s)$ of type A and $6 \cdot 10 \cdot 2 = 120$ of type B.

*No point is hit three times.* Type A pairs have $s - t \in \frac{\pi}{3}\mathbb{Z}$ and type B pairs have $s - t \in \frac{2\pi}{11}\mathbb{Z}$. Suppose $t_1, t_2, t_3$ are distinct and hit the same point. If $(t_1, t_2)$ and $(t_1, t_3)$ are both type A, then $t_2 + t_1$ and $t_3 + t_1$ are both in $\frac{2\pi}{11}\mathbb{Z}$, so $t_2 - t_3$ lies in $\frac{2\pi}{11}\mathbb{Z} \cap \frac{\pi}{3}\mathbb{Z} = 2\pi\mathbb{Z}$. That forces $t_2 = t_3$, a contradiction. Two type B pairs fail the same way. Mixed types fail through the pair $(t_2, t_3)$: if $(t_1, t_2)$ is type A and $(t_1, t_3)$ is type B, then $(t_2, t_3)$ is either type A, which puts $t_3 - t_1$ in both lattices, or type B, which does the same for $t_2 - t_1$. So each crossing point comes from exactly one unordered pair. There are $\frac{110+120}{2} = 115$ crossing points, and we count those inside the circle as unordered pairs.

*Type A.* With $t = \frac{\pi k}{11} - \frac{\pi j}{6} + \pi h$ $(h = 0, 1)$, we get $6t = \frac{6\pi k}{11} - \pi j + 6\pi h$ and $11t = \pi k - \frac{11\pi j}{6} + 11\pi h$. So $x = \pm\sin\frac{6\pi k}{11}$ and $y = \pm\cos\frac{11\pi j}{6}$. Then $x^2 + y^2 < 1$ exactly when $\sin^2\frac{6\pi k}{11} < \sin^2\frac{11\pi j}{6}$, i.e. $\left\|\frac{6k}{11}\right\| < \left\|\frac{11j}{6}\right\|$. As $k$ runs over $0, \ldots, 10$, $6k$ runs over all residues mod 11, so the left side takes the value 0 once and each of $\frac{m}{11}$ $(m = 1, \ldots, 5)$ twice. As $j$ runs over $1, \ldots, 5$, the right side takes the values $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$. We need $\frac{m}{11} < \frac{1}{6}$ ($m \leq 1$: 3 values of $k$), $\frac{m}{11} < \frac{1}{3}$ ($m \leq 3$: 7 values), and $\frac{m}{11} < \frac{1}{2}$ (all 11). That is $3 + 7 + 11 + 7 + 3 = 31$ choices of $(j, k)$. Each choice gives 2 ordered pairs (from $h$), i.e. one unordered pair, so type A contributes 31 points.

*Type B.* Now $t = \frac{(2j+1)\pi}{12} - \frac{\pi k}{11} + \pi h$. So $6t = \frac{(2j+1)\pi}{2} - \frac{6\pi k}{11} + 6\pi h$ and $11t = \frac{11(2j+1)\pi}{12} - \pi k + 11\pi h$, giving $x = \pm\cos\frac{6\pi k}{11}$ and $y = \pm\cos\frac{11(2j+1)\pi}{12}$. Inside means $\cos^2\frac{6\pi k}{11} < \sin^2\frac{11(2j+1)\pi}{12}$, i.e.

$\left\|\frac{12k+11}{22}\right\| < \left\|\frac{11(2j+1)}{12}\right\|$. For $k = 1, \ldots, 10$, $12k + 11$ runs over the odd residues mod 22 other than 11. So the left side is $\frac{u}{22}$, with each $u \in \{1, 3, 5, 7, 9\}$ occurring twice. For $j = 0, \ldots, 5$, $11(2j + 1)$ runs over the odd residues mod 12, so the right side is $\frac{v}{12}$, with each $v \in \{1, 3, 5\}$ occurring twice. The condition $\frac{u}{22} < \frac{v}{12}$, i.e. $u < \frac{11v}{6}$, allows 1, 3 or 5 values of $u$ for $v = 1, 3, 5$. So type B contributes $2 \cdot 2 \cdot (1 + 3 + 5) = 36$ points.

No crossing point lies on the circle: that would need $\frac{m}{11} = \frac{l}{6}$ with $0 < \frac{l}{6} \le \frac{1}{2}$, or $\frac{u}{22} = \frac{v}{12}$, i.e. $6u = 11v$ with $u \le 9$. Neither can happen. The answer is $31 + 36 =$ **067**.

**Check.** The same method for the smaller curve $(\sin 2t, \cos 3t)$ predicts $1 \cdot 3 + 2 \cdot 2 = 7$ crossing points. Type A gives $\left(\pm \sin \frac{2\pi k}{3}, 0\right)$, i.e. $(0, 0)$ and $\left(\pm\frac{\sqrt{3}}{2}, 0\right)$. Type B gives $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)$. All 7 are inside the unit circle, matching a picture of that curve. For the real curve, a computer traced $(\sin 6t, \cos 11t)$ as a polygon with 24000 sides and found exactly 115 self-crossings, 67 of them inside the circle; none was within 0.005 of it.

**15. Answer:** 13**. The idea.** The triangle with sides $PA, PB, PC$ is obtuse, with the obtuse angle opposite $PA$, exactly when $PA^2 > PB^2 + PC^2$. The combination $PB^2 + PC^2 - PA^2$ has coefficients summing to 1, so as a function of $P$ it is $|P - X|^2$ plus a constant for one fixed point $X$, and this holds everywhere in the plane, not just inside the triangle. So each obtuse region is the part of the disc of $\omega$ inside a second circle: a lens made of two circular segments.

**Work it out.** *Why a triangle exists.* Ptolemy's inequality gives $PA \cdot BC \le PB \cdot CA + PC \cdot AB$, i.e. $PA \le PB + PC$, with equality only when $P$ is on the arc $BC$ of $\omega$. So inside $\omega$ all three triangle inequalities are strict.

*The circle.* Take side 1, put the origin at $A$, and let $\mathbf{b}, \mathbf{c}$ be the position vectors of $B, C$, so $|\mathbf{b}| = |\mathbf{c}| = 1$ and $\mathbf{b} \cdot \mathbf{c} = \frac{1}{2}$. For $P = \mathbf{p}$, $PB^2 + PC^2 - PA^2 = |\mathbf{p}|^2 - 2\mathbf{p} \cdot (\mathbf{b} + \mathbf{c}) + 2 = |P - A'|^2 - 1$, where $A' = \mathbf{b} + \mathbf{c}$ is the reflection of $A$ in $BC$ and $|\mathbf{b} + \mathbf{c}|^2 = 3$. So the angle opposite $PA$ is obtuse exactly when $P$ is inside the circle $\gamma$ with centre $A'$ and radius 1. It passes through $B$ and $C$, with $\angle BA'C = 60°$.

*The lens.* Both $\gamma$ and $\omega$ pass through $B$ and $C$; describe each disc by the angle $\angle BPC$. On $A$'s side of $BC$: $P$ is inside $\omega$ iff $\angle BPC > 60°$, and inside $\gamma$ iff $\angle BPC > 180° - 30° = 150°$. So there the region is the segment of $\gamma$ cut off by $BC$ (it lies inside the triangle). On the far side of $BC$: $P$ is inside $\omega$ iff $\angle BPC > 180° - 60° = 120°$, and inside $\gamma$ iff $\angle BPC > 30°$. So there the region is the whole segment of $\omega$ beyond $BC$.

*Areas.* The segment of $\gamma$ is a 60° sector of radius 1 minus the equilateral triangle $A'BC$: $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. The circumradius is $R = \frac{1}{\sqrt{3}}$ and $\angle BOC = 120°$, so the segment of $\omega$ beyond $BC$ is $\frac{\pi R^2}{3} - \frac{1}{2}R^2 \sin 120° = \frac{\pi}{9} - \frac{\sqrt{3}}{12}$. One lens has area $\frac{5\pi}{18} - \frac{\sqrt{3}}{3}$.

The same holds for the obtuse angle opposite $PB$ and opposite $PC$. The three lenses do not overlap (a triangle has at most one obtuse angle). The disc has area $\pi R^2 = \frac{\pi}{3}$, so probability $= \frac{3\left(\frac{5\pi}{18} - \frac{\sqrt{3}}{3}\right)}{\frac{\pi}{3}} = \frac{5}{2} - \frac{3\sqrt{3}}{\pi}$.

So $p = 5$, $q = 2$, $r = 3$, $s = 3$, and $p + q + r + s = 13$.

**Check.** Numerically $\frac{5}{2} - \frac{3\sqrt{3}}{\pi} \approx 2.5 - 1.6540 = 0.8460$. Inside the triangle alone, the three segments of the $\gamma$ circles cover the fraction $\frac{2\pi}{\sqrt{3}} - 3 \approx 0.628$ of it. Every point of $\omega$'s disc beyond a side counts, and those caps are a large share of the disc, so the overall probability is higher. At the centre $PA = PB = PC$, an acute triangle. Near the midpoint of arc $BC$ we have $PA \approx 2R$

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and $PB = PC \approx R$, so $PA^2 \approx 4R^2 > 2R^2$: obtuse, as the lens predicts. A grid check of the raw condition over the disc agrees. The answer is **013**.